Optimal Time-Consistent Monetary Policy in the New Keynesian Model with Repeated Simultaneous Play

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Summary

- There are two definitions of “discretion” in the literature

- These definitions differ in terms of within-period timing of play

- Within-period timing has *major* equilibrium implications

- In the New Keynesian model with repeated Stackelberg play, there are multiple equilibria (King-Wolman, 2004)

- In the New Keynesian model with repeated simultaneous play, there is a unique equilibrium (this paper)

- Empirical relevance: Will the 1970s repeat itself?
Background and Motivation

Time-consistent (discretionary) policy: Kydland and Prescott (1977)

There are multiple equilibria under discretion:
  - Barro and Gordon (1983)
  - Chari, Christiano, Eichenbaum (1998)

Critiques of the Barro-Gordon/CEE result:
  - enormous number, range of equilibria make theory impossible to test or reject
  - equilibria require fantastic sophistication, coordination across continuum of atomistic agents
Background and Motivation

Literature has thus changed focus to *Markov perfect equilibria*:
- King and Wolman (2004)

King and Wolman (2004):
- standard New Keynesian model
- assume repeated Stackelberg within-period play
- there are two Markov perfect equilibria

But recall LQ literature:
- Pearlman (1994)
- assume repeated simultaneous within-period play
Repeated Stackelberg Play

Period $t$

- Policymaker precommits to policy (money supply or interest rate)
- Private agents take actions, markets clear

Repeated Stackelberg like “within-period commitment”?  
But policymakers’ actions are much more restricted  
Our results suggest policymaker actually has greater control with repeated simultaneous timing assumption
Comparison: Fiscal Policy

- two definitions of discretion in the tax literature
- Klein, Krusell, Rios-Rull (2004): repeated Stackelberg
- different timing assumption lead to different equilibria, welfare

In this paper:
- defining repeated simultaneous play is more subtle: Walras
- timing assumption changes not just payoffs, welfare, but *multiplicity* of equilibria
The Game $\Gamma_0$

Discretion is a game between private sector and central bank

For clarity, begin definition of game without central bank:

- assume interest rate process $\{r_t\}$ is i.i.d.
- call this game $\Gamma_0$

Game $\Gamma_0$:

- time is discrete, continues forever
- $\Gamma_0$ begins at $t_0$, but inherits history $h^{t_0}$
- define:
  - players
  - payoffs
  - information sets
  - action spaces
Game $\Gamma_0$: Players and Payoffs

1. Firms indexed by $i \in [0, 1]$: produce differentiated products; face Dixit-Stiglitz demand curves; have production function $y_t(i) = l_t(i)$; hire labor at wage rate $w_t$; payoff each period is profit:

$$\Pi_t(i) = p_t(i)y_t(i) - w_t l_t(i)$$

2. Households indexed by $j \in [0, 1]$: supply labor $L_t(j)$; consume final good $C_t(j)$; borrow or lend a one-period nominal bond $B_t(j)$; payoff each period is utility flow:

$$u(C_s(j), L_s(j)) = \frac{C_s(j)^{1-\varphi} - 1}{1 - \varphi} - \chi_0 \frac{L_s(j)^{1+\chi}}{1 + \chi}$$

Note: there is a final good aggregator that is not a player of $\Gamma_0$
Individual households and firms are anonymous:

- only aggregate variables and aggregate outcomes are publicly observed

Information set of each firm $i$ at time $t$ is thus:

- history of aggregate outcomes: $\{C_s, L_s, P_s, r_s, w_s, \Pi_s\}$, $s < t$
- history of firm $i$’s own actions

Information set of each household $j$ at time $t$ is thus:

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- history of household $j$’s own actions
Aggregate Resource Constraints

In games of industry competition:
- Bertrand
- Cournot
- Stackelberg

Action spaces are just real numbers: e.g., price, quantity

In a macroeconomic game, there are aggregate resource constraints that must be respected, e.g.:
- total labor supplied by households must equal total labor demanded by firms
- total output supplied by firms must equal total consumption demanded by households
- money supplied by central bank must equal total money demanded by households (in game $\Gamma_1$)
Walrasian Auctioneer

To ensure that aggregate resource constraints are respected, we introduce a Walrasian auctioneer

- Instead of playing a price $p_t$, firms now play a price schedule $p_t(X_t)$, where $X_t$ denotes aggregate variables realized at $t$
- this is just the usual NK assumption that firms take wages, interest rate, aggregates at time $t$ as given
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- Instead of playing a consumption-labor pair $(C_t, L_t)$, households play a joint schedule $(C_t(X_t), L_t(X_t))$
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Walrasian auctioneer then determines the equilibrium $X_t$ that satisfies aggregate resource constraints
1. Firms
   - set prices for two periods in Taylor contracts; must supply whatever output is demanded at posted price
   - firms in $[0, 1/2)$:
     - for $t$ odd, action space is set of measurable functions $p_t(X_t)$
     - for $t$ even, action space is trivial
   - firms in $[1/2, 1)$:
     - for $t$ even, action space is set of measurable functions $p_t(X_t)$
     - for $t$ odd, action space is trivial

2. Households
   - in each period, action space is set of measurable functions $(C_t(X_t), L_t(X_t))$
Game $\Gamma_0$: Action Spaces

Note:

- all firms $i$ and households $j$ play simultaneously in each period $t$
- Walrasian auctioneer clears markets, aggregate resource constraints

Also, do not confuse *action spaces* here with *strategies*:

- a *strategy* is a mapping from history $h^t$ to the action space
- here, action spaces are functions of aggregate variables realized at $t$
- but strategies are unrestricted, may depend on arbitrary history of aggregate variables (until we impose Markovian restriction)
Game $\Gamma_0$: Firm Optimality Conditions

Each firm that resets price faces a standard NK optimal pricing condition:

$$p_t^*(i) = (1 + \theta) \frac{E_{it} P_t^{(1+\theta)/\theta} Y_t w_t + E_{it} Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{E_{it} P_t^{(1+\theta)/\theta} Y_t + E_{it} Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}},$$

$$= (1 + \theta) \frac{P_t^{(1+\theta)/\theta} Y_t w_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}}.$$ 

$E_{it} \to E_t$ because firm can play functions of variables dated $t$
Game $\Gamma_0$: Household Optimality Conditions

Each household $j$ faces a standard dynamic programming problem with initial bond holdings $B_{t-1}(j)$.

Optimality conditions are standard:

\[
C_t^*(j)^{-\varphi} = E_{jt} \beta (1 + r_t) \frac{P_t}{P_{t+1}} C_{t+1}^*(j)^{-\varphi},
\]

\[
\chi_0 L_t^*(j)^\chi = E_{jt} \frac{w_t}{P_t} C_t^*(j)^{-\varphi},
\]

\[
E_{jt} \sum_{T=t}^{\infty} R_{t,T} P_T C_T^*(j) = B_{t-1}(j) + E_{jt} \sum_{T=t}^{\infty} R_{t,T} [w_T L_T^*(j) + \Pi_T],
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$$E_{jt} \sum_{T=t}^{\infty} R_{t,T} P_T C^*_T(j) = B_{t-1}(j) + E_{jt} \sum_{T=t}^{\infty} R_{t,T} [w_T L^*_T(j) + \Pi_T],$$

Note: $E_{jt} \rightarrow E_t$ once we establish symmetry across households
Private Sector Equilibrium and Markov Perfect Equilibrium

- Private Sector Equilibrium
- State Variables of the Game $\Gamma_0$
- Markov Perfect Equilibrium in the Game $\Gamma_0$
- Markov Perfect Equilibrium Conditions
Game $\Gamma_0$: Private Sector Equilibrium

Definition 1: Given the i.i.d. stochastic process for $\{r_t\}$ and initial conditions $p_{t_0-1}(i)$ and $B_{t_0-1}(j)$ for all firms $i$ and households $j$, we define a Private Sector Equilibrium (PSE) to be a subgame perfect equilibrium of the game $\Gamma_0$.

In particular, a PSE implies a collection of stochastic processes for $\{L_t, r_t, P_t, w_t, Y_t, \Pi_t, Q_{t,t+1}, p_t(i), y_t(i), l_t(i), C_t(j), L_t(j), B_t(j)\}$ for $t \geq t_0$ and for all $i, j$ that satisfy: (i) the price optimality condition (14) of the firm’s maximization problem; (ii) the consumption and labor optimality conditions (15)–(17) of the household’s maximization problem; (iii) the Dixit-Stiglitz aggregation and demand conditions (7)–(9) of the competitive goods aggregator; and (iv) the aggregate resource constraints (10)–(12) imposed by the Walrasian auctioneer.
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In particular, a PSE implies a collection of stochastic processes for \( \{ L_t, r_t, P_t, w_t, Y_t, \Pi_t, Q_{t,t+1}, p_t(i), y_t(i), l_t(i), C_t(j), L_t(j), B_t(j) \} \) for \( t \geq t_0 \) and for all \( i, j \) that satisfy: (i) the price optimality condition (14) of the firm’s maximization problem; (ii) the consumption and labor optimality conditions (15)–(17) of the household’s maximization problem; (iii) the Dixit-Stiglitz aggregation and demand conditions (7)–(9) of the competitive goods aggregator; and (iv) the aggregate resource constraints (10)–(12) imposed by the Walrasian auctioneer.
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Game $\Gamma_0$: State Variables

There are two sets of state variables for the game $\Gamma_0$ (and also $\Gamma_1$):

1. Distribution of household bond holdings, $B_{t-1}(j)$, $j \in [0, 1]$
2. Two measures of the distribution of inherited prices:
   - $\int p_{t-1}(i) - 1/\theta \, di$
   - $\int p_{t-1}(i) - (1 + \theta)/\theta \, di$
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\[
\int p_{t-1}(i)^{-1/\theta} \, di
\]

and

\[
\int p_{t-1}(i)^{-(1+\theta)/\theta} \, di
\]
Proposition 1: Suppose that $B_{t-1}(j)$ is the same for all households $j \in [0, 1]$ except possibly a set $S$ of measure zero. Then the optimal action $(C^*_t(j), L^*_t(j)) \in L(\Omega, \mathbb{R}^2_+)$ is the same for every household $j \notin S$. We denote this optimal action by $(C^*_t, L^*_t)$. 
Proposition 1: Suppose that $B_{t-1}(j)$ is the same for all households $j \in [0, 1]$ except possibly a set $S$ of measure zero. Then the optimal action $(C_t^*(j), L_t^*(j)) \in L(\Omega, \mathbb{R}^2_+)$ is the same for every household $j \notin S$. We denote this optimal action by $(C_t^*, L_t^*)$.

Proof: The household optimality conditions:

$$C_t^*(j)^{-\varphi} = E_{jt} \beta(1 + r_t) \frac{P_t}{P_{t+1}} C_{t+1}^*(j)^{-\varphi},$$

$$\chi_0 L_t^*(j)^\chi = E_{jt} \frac{w_t}{P_t} C_t^*(j)^{-\varphi},$$

$$E_{jt} \sum_{T=t}^{\infty} R_{t,T} P_T C_T^*(j) = B_{t-1}(j) + E_{jt} \sum_{T=t}^{\infty} R_{t,T} [w_T L_T^*(j) + \Pi_T],$$

for households $j_1$ and $j_2$ are identical if $B_{t-1}(j_1) = B_{t-1}(j_2)$. 
Proposition 2: The optimal choice of price schedule $p^*_t(i) \in L(\Omega, \mathbb{R}_+)$ is the same for all firms $i$ that reset price in period $t$. We denote this optimal price schedule, given by (14), by $p_t^*$. 
Proposition 2: The optimal choice of price schedule $p_t^*(i) \in L(\Omega, \mathbb{R}_+)$ is the same for all firms $i$ that reset price in period $t$. We denote this optimal price schedule, given by (14), by $p_t^*$.

Proof: The right-hand side of firm optimality condition:

$$p_t^*(i) = (1 + \theta) \frac{P_t^{(1+\theta)/\theta} Y_t w_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1} w_{t+1}}{P_t^{(1+\theta)/\theta} Y_t + E_t Q_{t,t+1} P_{t+1}^{(1+\theta)/\theta} Y_{t+1}},$$

is identical for all firms $i$. 
Game $\Gamma_0$: State Variables

Starting from symmetric initial conditions in period $t_0$:

- Propositions 1 and 2 show that the distributions $B_{t-1}(\cdot)$ and $\rho_{t-1}(\cdot)$ are degenerate for all times $t \geq t_0$ along the equilibrium path in any subgame perfect equilibrium of $\Gamma_0$.

- We henceforth restrict definition of game $\Gamma_0$ to case of symmetric initial conditions in period $t_0$.

Note: we will not write out how play evolves off of the equilibrium path (if a positive measure of firms or households were to deviate), but simply assert that agents will continue to play according to their optimality conditions (Phelan-Stachetti, 2001).
**Game \( \Gamma_0 \): Markov Perfect Equilibrium**

*Definition 2:* A Markov Perfect Equilibrium (MPE) of the game \( \Gamma_0 \) is a set of strategies for households and firms that, at each date \( t \), depend only on the state variables of \( \Gamma_0 \) at time \( t \), and yield a Nash equilibrium in every proper subgame of \( \Gamma_0 \).
Game $\Gamma_0$: Markov Perfect Equilibrium

Definition 2: A Markov Perfect Equilibrium (MPE) of the game $\Gamma_0$ is a set of strategies for households and firms that, at each date $t$, depend only on the state variables of $\Gamma_0$ at time $t$, and yield a Nash equilibrium in every proper subgame of $\Gamma_0$.

Note:

- state variables of general game correspond to coarsest partition of original game tree into equivalence classes that preserve payoffs and action spaces (Fudenberg-Tirole, 1993)
- for $\Gamma_0$, can define action spaces, payoffs in real terms
- normalize $\Gamma_0$ by $p_{t-1}$
Game $\Gamma_0$: Markov Perfect Equilibrium Conditions

Now consolidate and simplify necessary conditions for an MPE.
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Define:

$$x_t \equiv \frac{p_t}{p_{t-1}}.$$
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First:

$$P_t = \left[ \int_0^1 p_t(i)^{-1/\theta} \, di \right]^{-\theta} \iff \frac{p_t}{P_t} = 2^{-\theta} \left(1 + x_t^{1/\theta}\right)^{\theta}.$$
Then, consolidating necessary conditions yields:

\[ \int_0^1 l_t(i) di = L_t \iff \frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}}, \]

firm optimality \iff \quad 2^{-\theta} (1 + x_t^{1/\theta})^\theta = (1 + \theta) \frac{\chi_0 [Y_t L_t^x + \beta (1 + x_t^{1/\theta})^{1+\theta} h_{1t}]}{Y_t^{1-\varphi} + \beta (1 + x_t^{1/\theta}) h_{2t}},

Euler \iff \quad Y_t^{-\varphi} (1 + x_t^{1/\theta}) = \beta (1 + r_t) h_{3t}, \]
Game $\Gamma_0$: Markov Perfect Equilibrium Conditions

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firm optimality $\iff 2^{-\theta}(1 + x_t^{1/\theta})^{\theta} = (1 + \theta)\frac{\chi_0[Y_tL_t^\chi + \beta(1 + x_t^{1/\theta})^{1+\theta}h_{1t}]}{Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t}},$

Euler $\iff Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_{3t},$

$$h_{1t} \equiv E_t \frac{Y_{t+1}L_{t+1}^\chi}{(1 + x_{t+1}^{-1/\theta})^{1+\theta}},$$

$$h_{2t} \equiv E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}},$$

$$h_{3t} \equiv E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}).$$
**Game $\Gamma_0$: Markov Perfect Equilibrium Conditions**

Proposition 4: Given the i.i.d. stochastic process for $\{r_t\}$ and symmetric initial conditions $p_{t_0-1}(i) = p_{t_0-1} \in \mathbb{R}_+$ and $B_{t_0-1}(j) = 0$ for all firms $i$ and households $j$, necessary conditions for an equilibrium path of a Markov Perfect Equilibrium (MPE) of the game $\Gamma_0$ are that, for all $t \geq t_0$: (i) $(L_t, x_t, Y_t)$ satisfy households’ and firms’ optimality conditions (19)-(21), taking $r_t$ and $(h_{1t}, h_{2t}, h_{3t})$ as given; (ii) $(h_{1t}, h_{2t}, h_{3t})$ satisfy conditions (22)-(24) for rational expectations; and (iii) households’ and firms’ strategies along the equilibrium path are independent of history and independent of time.
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**Game \( \Gamma_0 \): Markov Perfect Equilibrium Conditions**

*Proposition 5:* Along the equilibrium path of a Markov Perfect Equilibrium of the game \( \Gamma_0 \), there exist positive real numbers \( h_1, h_2, \) and \( h_3 \) such that \((h_{1t}, h_{2t}, h_{3t}) = (h_1, h_2, h_3)\) for all times \( t \).
Proposition 5: Along the equilibrium path of a Markov Perfect Equilibrium of the game $\Gamma_0$, there exist positive real numbers $h_1$, $h_2$, and $h_3$ such that $(h_{1t}, h_{2t}, h_{3t}) = (h_1, h_2, h_3)$ for all times $t$.

Proof:

- $h_{1t}, h_{2t}, h_{3t}$ are conditional expectations of variables in $t + 1$
- Variables in $t + 1$ depend only on variables dated $t + 1$ or later
- $r_t$ is i.i.d. over time
- No sunspots or time-dependence (Markov)
- $\implies h_{1t}, h_{2t}, h_{3t}$ are the same in every period $t$
Game $\Gamma_0$: Markov Perfect Equilibrium Conditions

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- $h_{1t}, h_{2t}, h_{3t}$ are conditional expectations of variables in $t + 1$
- variables in $t + 1$ depend only on variables dated $t + 1$ or later
- $r_t$ is i.i.d. over time
- no sunspots or time-dependence (Markov)
- $\implies h_{1t}, h_{2t}, h_{3t}$ are the same in every period $t$

Note that this does not rule out the possibility of multiple MPE:
- there may be multiple sets of $(h_1, h_2, h_3)$ each of which can support an MPE
- any given $(h_1, h_2, h_3)$ may be able to support multiple MPE
Game $\Gamma_0$: Markov Perfect Equilibrium Conditions

Proposition 6: Let $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1})$ and $(L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$ lie on the equilibrium path of an MPE of $\Gamma_0$. Then $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1}) = (L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$.

That is, along the equilibrium path, any MPE of $\Gamma_0$ must be constant over time.
Game $\Gamma_0$: Markov Perfect Equilibrium Conditions

Proposition 6: Let $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1})$ and $(L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$ lie on the equilibrium path of an MPE of $\Gamma_0$. Then $(L_{t_1}, x_{t_1}, Y_{t_1}, h_{1t_1}, h_{2t_1}, h_{3t_1}, r_{t_1}) = (L_{t_2}, x_{t_2}, Y_{t_2}, h_{1t_2}, h_{2t_2}, h_{3t_2}, r_{t_2})$.

That is, along the equilibrium path, any MPE of $\Gamma_0$ must be constant over time.

Proof:
- household, firm strategies are independent of history, time
- $h_1$, $h_2$, and $h_3$ are independent of time (Prop. 5)
- $\implies$ any MPE is independent of time.
Now, extend the game $\Gamma_0$ to include an optimizing central bank:
- interest rate $r_t$ is set by central bank each period
- call this game $\Gamma_1$

First two sets of players (firms and households) are defined exactly as in $\Gamma_0$
3. Central bank:

sets one-period nominal interest rate $r_t$; payoff each period is given by average household welfare:

$$\int \frac{C_s(j)^{1-\varphi} - 1}{1 - \varphi} - \chi_0 \frac{L_s(j)^{1+\chi}}{1 + \chi} \, dj$$

Central bank’s information set is the history of aggregate outcomes:

$$\{C_s, L_s, P_s, r_s, w_s, \Pi_s\}, \ s < t$$

Note:

- central bank has no ability to commit to future actions (discretion)
- central bank is monolithic, while private sector is atomistic
Within-Period Timing of Play

Repeated Stackelberg play:
- each period divided into two halves
- first, central bank precommits to a value for $r_t$ (or $m_t$)
- second, firms and households play simultaneously
- Walrasian auctioneer determines equilibrium

Repeated simultaneous play:
- firms, households, and central bank all play simultaneously
- Walrasian auctioneer determines equilibrium
Simultaneous Play: Example

Linearized New Keynesian model:

\[ y_t = E_t y_{t+1} - \alpha r_t \]
\[ \pi_t = \beta E_t \pi_{t+1} + \gamma y_t \]

Under repeated simultaneous play, a Taylor rule is valid:

\[ r_t = a \pi_t + b y_t \]
Simultaneous Play: Example

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Under repeated Stackelberg play, corresponding rule would be:

\[ r_t = a E_{t-1} \pi_t + b E_{t-1} y_t \]
Simultaneous Play: Example

Linearized New Keynesian model:

\[
\begin{align*}
y_t &= E_t y_{t+1} - \alpha r_t \\
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\end{align*}
\]

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although note that this rule is not Markov
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\]

Under repeated Stackelberg play, corresponding rule would be:

\[
r_t = a E_{t-1} \pi_t + b E_{t-1} y_t
\]

although note that this rule is not Markov (model has no state variables).
Why Assume Simultaneous Play?

Practical considerations/realism:

- Makes no difference whether monetary instrument is $r_t$ or $m_t$
- Central banks monitor economic conditions continuously, adjust policy as needed

Theoretical considerations:

- Why treat central bank, private sector so asymmetrically?
- LQ literature (Svensson-Woodford 2003, 2004, Woodford 2003, Pearlman 1994, etc.) assumes simultaneous play
- Investigate sensitivity of multiple equilibria to within-period timing
In defining the game $\Gamma_1$, we assume repeated simultaneous play:

- firms $i$, households $j$, and central bank all play simultaneously in each period $t$
- action spaces of firms, households are same as in $\Gamma_0$
- for central bank, action space each period is set of measurable functions $r_t(X_t)$ (simultaneous play)
- Walrasian auctioneer clears markets, aggregate resource constraints

Again, do not confuse action spaces with strategies:

- strategies are unrestricted, may depend on arbitrary history of aggregate variables (until we impose Markovian restriction)
Policymaker Bellman Equation

\[ V_t = \max \left\{ r_t \right\} \left\{ \frac{Y_t^{1-\phi}}{1-\phi} - \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \]
Policymaker Bellman Equation

\[
V_t = \max_{\{r_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\}
\]

subject to:

\[
\frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}},
\]

\[
Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_{1t},
\]

\[
2^{-\theta}(1+x_t^{1/\theta})^\theta \left[ Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_{2t} \right] = (1+\theta)\chi_0 \left[ Y_t L_t^{\chi} + \beta(1+x_t^{1/\theta})^{1+\theta} h_{3t} \right].
\]
Policymaker Bellman Equation

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\]

where expectations of next period variables are given functions of this period’s economic state: \(h_{1t}, h_{2t}, h_{3t}\) (discretion)
Markov Perfect Equilibria of the Game $\Gamma_1$

Along the equilibrium path of any Markov Perfect Equilibrium of $\Gamma_1$, state variables are degenerate (only operative off equilibrium path)

As a result, along the equilibrium path:

$$h_{1t} = E_t Y_{t+1}^{-\varphi} (1 + x_{t+1}^{-1/\theta}) = h_1$$

$$h_{2t} = E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{-1/\theta}} = h_2$$

$$h_{3t} = E_t \frac{Y_{t+1} L_{t+1}^\chi}{(1 + x_{t+1}^{-1/\theta})^{1+\theta}} = h_3$$

Note: we will not write out how play evolves off of the equilibrium path, but simply assert that it agents will continue to play optimally (Phelan-Stachetti, 2001)
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  h_{3t} &= E_t \frac{Y_{t+1} L_{t+1}^x}{(1 + x_{t+1}^{-1/\theta})^{1+\theta}} = h_3
\end{align*}
\]

Note: we will not write out how play evolves off of the equilibrium path, but simply assert that it agents will continue to play optimally (Phelan-Stachetti, 2001)
Solving for Markov Perfect Equilibria

Solve: \[ V_t = \max_{\{r_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \]

subject to:

\[
\frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}},
\]

\[ Y_t^{1-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_1, \]

\[ 2^{-\theta}(1+x_t^{1/\theta})^\theta \left[ Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_2 \right] = (1+\theta)x_0 \left[ Y_t L_t^{\chi} + \beta(1+x_t^{1/\theta})^{1+\theta} h_3 \right]. \]

where \( h_1, h_2, h_3 \) are exogenous constants.
Solving for Markov Perfect Equilibria

Solve: \[ V_t = \max_{\{r_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \]

subject to:

\[ \frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}}, \]

\[ Y_t^{-\varphi}(1 + x_t^{1/\theta}) = \beta(1 + r_t)h_1, \]

\[ 2^{-\theta}(1+x_t^{1/\theta})^\theta \left[ Y_t^{1-\varphi} + \beta(1 + x_t^{1/\theta})h_2 \right] = (1+\theta)\chi_0 \left[ Y_t L_t^\chi + \beta(1 + x_t^{1/\theta})^{1+\theta} h_3 \right]. \]

where \( h_1, h_2, h_3 \) are exogenous constants.

Finally, impose equilibrium conditions:

\[ h_1 = E_t Y_{t+1}^{-\varphi}(1 + x_{t+1}^{-1/\theta}), \quad h_2 = E_t \frac{Y_{t+1}^{1-\varphi}}{1+x_{t+1}^{-1/\theta}}, \quad h_3 = E_t \frac{Y_{t+1}L_{t+1}^\chi}{(1+x_{t+1}^{-1/\theta})^{1+\theta}}. \]
Solving for Markov Perfect Equilibria

Solve: \[ V_t = \max_{\{r_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} \left( \frac{L_t^{1+\chi}}{1+\chi} \right) - \chi_0 \frac{L_t}{1+\chi} + \beta E_t V_{t+1} \right\} \]

subject to:

\[ \frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}}, \]

\[ Y_t^{1-\varphi} (1 + x_t^{1/\theta}) = \beta (1 + r_t) h_1, \]

\[ 2^{-\theta} (1 + x_t^{1/\theta})^\theta \left[ Y_t^{1-\varphi} + \beta (1 + x_t^{1/\theta}) h_2 \right] = (1 + \theta) \chi_0 \left[ Y_t L_t^\chi + \beta (1 + x_t^{1/\theta})^{1+\theta} h_3 \right]. \]

where \( h_1, h_2, h_3 \) are exogenous constants.

Finally, impose equilibrium conditions:

\[ h_1 = E_t Y_{t+1}^{1-\varphi} (1 + x_{t+1}^{1/\theta}), \quad h_2 = E_t \frac{Y_{t+1}^{1-\varphi}}{1 + x_{t+1}^{1/\theta}}, \quad h_3 = E_t \frac{Y_{t+1} L_{t+1}^\chi}{\left(1 + x_{t+1}^{1/\theta}\right)^{1+\theta}}. \]

Note: there can still be multiplicity here, e.g. if \( h_1, h_2, h_3 \) are “bad”
Solving for Markov Perfect Equilibria

Solve policymaker’s problem via Lagrangean, yielding:

\[
\begin{align*}
\lambda_t^{\text{Euler}} &= 0 \\
\chi_0 L_t^{1+\chi} &= \lambda_t^Y \frac{L_t}{Y_t} - \lambda_t^X (1 + \theta) \chi_0 Y_t X L_t^X \\
\lambda_t^Y \frac{L_t}{Y_t} &= Y_t^{1-\varphi} + \lambda_t^X [(1 - \varphi) 2^{-\theta} (1 + x_t^{1/\theta})^\theta Y_t^{1-\varphi} - (1 + \theta) \chi_0 Y_t X L_t^X] \\
\lambda_t^X 2^\theta \frac{1 + \theta}{\theta} \frac{x_t - 1}{(1 + x_t^{1/\theta})^2 (1 + \theta)} &= \lambda_t^X \left\{ 2^{-\theta} \left[ \frac{Y_t^{1-\varphi}}{1 + x_t^{1/\theta}} + \frac{1 + \theta}{\theta} \beta h_2 \right] - \chi_0 \beta \frac{(1 + \theta)^2}{\theta} h_1 \right\}
\end{align*}
\]

Combine these first-order conditions with private sector optimality constraints
Proposition 7: The inflation rate $\pi$ in any Markov Perfect Equilibrium of the game $\Gamma_1$ must satisfy the condition:

\[
\frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \times \frac{1 + \pi^{1/\theta}}{1 + \pi^{(1+\theta)/\theta}} \times \\
\left\{ 1 - \frac{(\pi - 1) \left[ 1 + \chi - (1 - \varphi) \frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \right]}{(\pi - 1) \left[ 1 - (1 - \varphi) \frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \right] + (1 + \pi^{(1+\theta)/\theta}) \left[ 1 - \frac{1}{1 + \theta} \frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \right]} \right\} = \frac{1}{1 + \theta} \quad (*)
\]
Proposition 7: The inflation rate $\pi$ in any Markov Perfect Equilibrium of the game $\Gamma_1$ must satisfy the condition:

$$
\frac{1 + \beta \pi (1 + \theta)/\theta}{1 + \beta \pi^{1/\theta}} \quad \frac{1 + \pi^{1/\theta}}{1 + \pi (1 + \theta)/\theta} \times \\
\left\{1 - \frac{(\pi - 1) \left[1 + \chi - (1 - \varphi) \frac{1 + \beta \pi (1 + \theta)/\theta}{1 + \beta \pi^{1/\theta}}\right]}{(\pi - 1) \left[1 - (1 - \varphi) \frac{1 + \beta \pi (1 + \theta)/\theta}{1 + \beta \pi^{1/\theta}}\right] + (1 + \pi (1 + \theta)/\theta) \left[1 - \frac{1}{1 + \theta} \frac{1 + \beta \pi (1 + \theta)/\theta}{1 + \beta \pi^{1/\theta}}\right]} \right\} = \frac{1}{1 + \theta} \quad (\ast)
$$

Proposition 8: Let $\varphi = 1$, $\chi = 0$, and $\beta > \max\{1/2, 1/(1 + 2\theta)\}$. Then there is precisely one value of $\pi$ that satisfies equation (\ast).
Proposition 7: The inflation rate $\pi$ in any Markov Perfect Equilibrium of the game $\Gamma_1$ must satisfy the condition:

\[
\frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \times \frac{1 + \pi^{1/\theta}}{1 + \pi^{(1+\theta)/\theta}} \times \\
\left\{ \frac{1 - (\pi - 1) \left[ 1 + \chi - (1 - \varphi) \frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \right]}{(\pi - 1) \left[ 1 - (1 - \varphi) \frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \right] + (1 + \pi^{(1+\theta)/\theta}) \left[ 1 - \frac{1}{1+\theta} \frac{1 + \beta \pi^{(1+\theta)/\theta}}{1 + \beta \pi^{1/\theta}} \right]} \right\} = \frac{1}{1 + \theta}
\]

(\ast)

Proposition 8: Let $\varphi = 1$, $\chi = 0$, and $\beta > \max\{1/2, 1/(1 + 2\theta)\}$. Then there is precisely one value of $\pi$ that satisfies equation (\ast).

Note:
- $\varphi = 1$, $\chi = 0$ are not special, but simplify algebra in proofs
- there is a unique equilibrium for wide range of parameters
- confirmed by extensive numerical simulation in Matlab
Repeated Stackelberg Play, with Money

Given money supply $m_t$, expectations $h_1$, $h_2$, $h_3$, and private sector optimality conditions:

\[
\frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{(1 + x_t^{1/\theta})^{1+\theta}},
\]

\[
Y_t^{-\varphi} (1 + x_t^{1/\theta}) = \beta (1 + r_t) h_1,
\]

\[
2^{-\theta} (1 + x_t^{1/\theta})^\theta \left[ Y_t^{1-\varphi} + \beta (1 + x_t^{1/\theta}) h_2 \right] = (1+\theta) x_0 \left[ Y_t L_t^x + \beta (1 + x_t^{1/\theta})^{1+\theta} h_3 \right],
\]

\[
m_t = Y_t \frac{2^\theta x_t}{(1 + x_t^{1/\theta})^\theta}
\]

Solve for:

\[
Y_t = Y(m_t), \quad x_t = x(m_t), \quad L_t = L(m_t), \quad r_t = r(m_t).
\]
Repeated Stackelberg Play, with Money

Then solve: $V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1 - \varphi} - \chi_0 \frac{L_t^{1+\chi}}{1 + \chi} + \beta E_t V_{t+1} \right\}$ subject to:

$Y_t = Y(m_t), \quad x_t = x(m_t), \quad L_t = L(m_t), \quad r_t = r(m_t)$. 
Repeated Stackelberg Play, with Money

Then solve:  
\[ V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \]  
subject to:  
\[ Y_t = Y(m_t), \quad x_t = x(m_t), \quad L_t = L(m_t), \quad r_t = r(m_t). \]

King and Wolman (2004): There are “good” and “bad” expectations \( h_1, h_2, h_3 \), which result in “good” and “bad” private sector equilibria  
\[ Y_t = Y(m_t), \quad x_t = x(m_t), \quad L_t = L(m_t), \quad r_t = r(m_t). \]
Repeated Simultaneous Play, with Money

Solve: \[ V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\phi}}{1-\phi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \]

subject to:

\[ \frac{L_t}{Y_t} = 2^\theta \frac{1 + x_t^{(1+\theta)/\theta}}{\left(1 + x_t^{1/\theta}\right)^{1+\theta}} , \]

\[ Y_t^{-\phi} (1 + x_t^{1/\theta}) = \beta (1 + r_t) h_1 , \]

\[ 2^{-\theta} (1 + x_t^{1/\theta})^\theta \left[ Y_t^{1-\phi} + \beta (1 + x_t^{1/\theta}) h_2 \right] = (1+\theta)\chi_0 \left[ Y_t L_t^\chi + \beta (1 + x_t^{1/\theta})^{1+\theta} h_3 \right] , \]

\[ m_t = Y_t \frac{2^\theta x_t}{(1 + x_t^{1/\theta})^\theta} \]
Repeated Simultaneous Play, with Money

Solve: \[ V_t = \max_{\{m_t\}} \left\{ \frac{Y_t^{1-\varphi}}{1-\varphi} - \chi_0 \frac{L_t^{1+\chi}}{1+\chi} + \beta E_t V_{t+1} \right\} \] subject to:

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\[ Y_t^{-\varphi} (1 + x_t^{1/\theta}) = \beta (1 + r_t) h_1, \]

\[ 2^{-\theta} (1 + x_t^{1/\theta})^\theta \left[ Y_t^{1-\varphi} + \beta (1 + x_t^{1/\theta}) h_2 \right] = (1+\theta) \chi_0 \left[ Y_t L_t^\chi + \beta (1 + x_t^{1/\theta})^{1+\theta} h_3 \right], \]

\[ m_t = Y_t \frac{2^\theta x_t}{(1 + x_t^{1/\theta})^\theta} \]

But first-order condition with respect to \( m_t \):

\[ \lambda^m_t = 0 \]
Conclusions

- There are two definitions of “discretion” in the literature
- These definitions differ in terms of within-period timing of play
- Within-period timing has *major* equilibrium implications
- In the New Keynesian model with repeated Stackelberg play, there are multiple equilibria (King-Wolman, 2004)
- In the New Keynesian model with repeated simultaneous play, there is a unique equilibrium (this paper)
- Open questions: other NK models, models with a (nondegenerate) state variable