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Stochastic asymmetric Blotto games: Some new results

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HIGHLIGHTS

- We study Col. Blotto games with asymmetric values using a lottery success function.
- We characterize equilibrium for total expected payoff and majority rule objectives.
- The game with a majority rule objective is a version of the U.S. Electoral College.
- Allocations under majority rule generally differ from the Banzhaf power index.

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ABSTRACT

We develop some new theoretical results for stochastic asymmetric Blotto games.

1. Introduction

The Colonel Blotto game (Borel, 1921), is a two-player non-cooperative game in which players decide how to allocate their given resources across n battlefields. In Borel's original version of this game, the player who allocates the most resources to any given battlefield wins that battlefield with certainty. The players' objective function is either to maximize the sum of the value of the battlefields won, or to win a majority value of the n battlefields. In this paper we study stochastic asymmetric versions of the Blotto game under both of these objective functions. In an "asymmetric Blotto" game, the values of the n battlefields may differ from one another though these different values are common to all players. In the "stochastic asymmetric" version of the Blotto game, the deterministic rule for determining which player wins each battlefield is replaced by a lottery contest success function where the chances of winning a given battlefield are increasing with the amount of resources devoted to that battlefield. This stochastic lottery specification makes the payoff function continuous; as a result, if a Nash equilibrium exists, it is unique and in pure strategies, as opposed to the multiplicity of (typically mixed strategy) equilibria that arise in deterministic versions of the Blotto game.

There are two main theoretical papers about stochastic asymmetric Blotto games.[1] The first one, Friedman (1958), considers

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[1] See Kovenock and Roberson (2012) for a broader survey of the Blotto game literature.

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two players who seek to maximize their expected total payoff. We show that Friedman's result can be extended to any number of players. The second paper, Lake (1979), was the first to study the stochastic asymmetric "majority rule" Blotto game.[2] This version of the game is particularly relevant to understanding electoral competitions in two party systems, e.g., the electoral college system for electing the US president. Lake studied only the case of equal budget constraints. We show that if players' budgets are the same (as in Lake) or if they are sufficiently similar and the number of items (battlefields) is not too large, then resource allocation under the majority rule version of the stochastic, asymmetric Blotto game is proportional to the Banzhaf power index for each item, while more generally, resource allocation for a particular item will *not* be proportional to each item's Banzhaf power index. Our findings thus generalize those of Lake (1979).

2. Stochastic asymmetric Blotto games

There are $L \geq 2$ players and a set of n items. Item i has common value $W_i > 0$. Each player l has a budget $X(l) > 0$ and competes for all n items by allocating her budget across all n items. All players allocate their budgets simultaneously.

A pure strategy of player l is a nonnegative n-dimensional vector $(x_1(l), \ldots, x_n(l))$, such that $\sum_{j=1}^{n} x_j(l) = X(l)$ and $x_i(l)$ is player l's spending on item i. Each item is allocated by means of a lottery in which player l obtains item i with probability $\frac{x_i(l)}{\sum_{j=1}^{L} x_i(j)}$.[3,4] Denote the total value of all n items by:

$$W = \sum_{i=1}^{n} W_i.$$

2.1. Plurality: maximizing the expected value

Suppose that all players seek to maximize their expected item values:

$$\max_{x_1(l),\ldots,x_n(l)} \sum_{i=1}^{n} \frac{x_i(l)}{\sum_{j=1}^{L} x_i(j)} W_i,$$

$$s.t. \sum_{i=1}^{n} x_i(l) = X(l) \quad \text{and} \quad x_i(l) \geq 0 \quad \forall l.$$

Then,

Theorem 1. *The stochastic Blotto game has a unique Nash equilibrium. In this Nash equilibrium,*

$$(x_1(l), \ldots, x_n(l)) = \left(\frac{W_1}{W}, \ldots, \frac{W_n}{W}\right) X(l) \quad \text{for } l \in \{1, \ldots, L\}.$$

The expected equilibrium payoff of player l is $\frac{X(l)}{\sum_{j=1}^{L} X(j)} W$.

Friedman (1958) presents Theorem 1 for the case of $L = 2$.[5] The proof of Theorem 1 is similar to Friedman's proof and is available on request. There are several corollaries to Theorem 1. First, note that the Nash equilibrium described is unique. Second, both players compete for all items in the Nash equilibrium of this version of the Blotto game. Third, the unique Nash equilibrium has a monotonic property: the player with the *greater budget* has a greater chance to win *each item*.

2.2. Weighted majority: maximizing the probability of winning a majority

We now assume that $L = 2$ and each player wants to maximize her probability to win a majority of all items' values as in the US electoral college example. The game we study involves two players x and y, and n items. Player x has a given budget of size X and player y has a given budget of size Y.

We begin by noting that each possible subset of items $\{W_1, \ldots, W_n\}$ can be represented by a binary, n-dimensional characteristic vector $\mathbf{t} = (t_1, \ldots, t_n)$, where $t_i \in \{0, 1\}$ for any $i = 1, \ldots, n$. If $t_i = 1$, then item i belongs to the subset, and if $t_i = 0$, then item i does not belong to the subset. We will use the corresponding characteristic vector to represent subsets in the rest of the paper. There are 2^n such subsets.

Denote by $\mathbf{V}$ the set of winning subsets under the win-a-majority-of-item-values objective. Then a subset $\mathbf{t} \in \mathbf{V}$, if

$$\sum_{j=1}^{n} t_j W_j > \frac{W}{2}.$$

A player wins the stochastic majority Blotto game if she gets a winning subset. Without loss of generality, a player receives a payoff of 1 from winning the game and a payoff of 0 from losing the game. Player x maximizes her chance to get a winning subset by solving the following maximization problem:

$$\max_{x_1,\ldots,x_n} \sum_{\mathbf{t}\in\mathbf{V}} \prod_{j:t_j=1} \frac{x_j}{x_j + y_j} \prod_{k:t_k=0} \frac{y_k}{x_k + y_k}, \tag{1}$$

$$s.t. \sum_{j=1}^{n} x_j = X, \tag{2}$$

where $\prod_{j:t_j=1} \frac{x_j}{x_j+y_j}$ is the probability of winning all the items that belong to subset $\mathbf{t}$ and $\prod_{k:t_k=0} \frac{y_k}{x_k+y_k}$ is the probability of losing all the items that do not belong to subset $\mathbf{t}$.

Similarly, player y solves the following maximization problem:

$$\max_{y_1,\ldots,y_n} \sum_{\mathbf{t}\in\mathbf{V}} \prod_{j:t_j=1} \frac{y_j}{x_j + y_j} \prod_{k:t_k=0} \frac{x_k}{x_k + y_k}, \tag{3}$$

$$s.t. \sum_{j=1}^{n} y_j = Y. \tag{4}$$

We next make a technical assumption that guarantees a unique majority winner in all realizations of individual lotteries.[6]

Assumption 1.

$$\sum_{j=1}^{n} t_j W_j \neq \frac{W}{2} \quad \text{for any subset } \mathbf{t}. \tag{5}$$

We will need the following definition.

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2 We discovered Lake's (1979) paper only after we had completed our analysis. His proofs are different from ours, but his main result coincides with our prediction for the case of equal budgets. We thank Steve Brams for providing this reference.

3 We assume that if $x_i(1) = \cdots = x_i(L) = 0$, then each player has $1/L$ probability to win item i.

4 We assume that all lotteries are statistically independent.

5 Osorio (2013) generalizes Friedman's result to the case where battlefield valuations are both asymmetric and heterogeneous across the two players.

6 A stronger version of Assumption 1 is typical in the literature. Usually in this literature, all values are the same, $W_1 = \cdots = W_n$, in which case Assumption 1 becomes $n = 2k + 1,\ k = 1, 2, \ldots$.

Definition 1. An item i is *pivotal* in subset $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$ if $(t_i = 1, \mathbf{t}_{-i})$ is a winning subset but $(t_i = 0, \mathbf{t}_{-i})$ is a losing subset.

Denote by $\mathbf{V}_i$ a set of winning subsets where item i is pivotal and by $Piv\,(i)$ the number of winning subsets in which item i is pivotal, or

$$Piv\,(i) = \|\mathbf{V}_i\| = \sum_{\mathbf{t}\in\mathbf{V}_i} 1. \tag{6}$$

We now introduce the Banzhaf Power Index[7] for item i in the following way:

$$BPI(i) = \frac{Piv(i)}{Piv(1) + \cdots + Piv(N)} = \frac{\sum_{t\in\mathbf{V}_i} 1}{\sum_{\mathbf{t}\in\mathbf{V}_1} 1 + \cdots + \sum_{\mathbf{t}\in\mathbf{V}_n} 1}. \tag{7}$$

Lake (1979) considers the special case where both players have equal budget constraints:

$$X = Y. \tag{8}$$

In that case,

Theorem 2 (*Lake, 1979*)**.** *Suppose that conditions (5) and (8) hold. Then, there exists a unique Nash equilibrium in which*

$$x_i = BPI\,(i)\,X \quad \text{and} \quad y_i = BPI\,(i)\,Y, \quad \text{for } i = 1, 2, \ldots, n.$$

Denote by

$$K_i \equiv \sum_{\mathbf{t}\in\mathbf{V}_i} \left(\prod_{j\neq i: t_j=1} X \prod_{k\neq i: t_k=0} Y \right). \tag{9}$$

Note that from (9) we have that $K_i = 0$ if item i is never pivotal and $K_i > 0$ otherwise. Moreover, if players have equal budgets, i.e., if condition (8) holds, then from (9)

$$K_i = X^{n-1} \sum_{\mathbf{t}\in\mathbf{V}_i} 1 \quad \text{and}$$

$$\frac{K_i}{K_1 + \cdots + K_n} = BPI\,(i) \quad \text{for } i = 1, \ldots, n.$$

Therefore, if condition (8) holds, then in the unique Nash equilibrium:

$$x_i = \frac{K_i}{K_1 + \cdots + K_n} X, \quad \text{for } i = 1, .., n,$$

and

$$y_i = \frac{K_i}{K_1 + \cdots + K_n} Y, \quad \text{for } i = 1, .., n.$$

We can now state the following result.

Theorem 3. *Suppose that there exists a pure-strategy Nash equilibrium $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$. Then, the equilibrium is unique and*

$$x_i = \frac{K_i}{K_1 + \cdots + K_n} X \quad \text{and}$$
$$y_i = \frac{K_i}{K_1 + \cdots + K_n} Y, \quad \text{for } i = 1, 2, \ldots, n, \tag{10}$$

where K_i is defined in (9).

[7] See Banzhaf (1965) for discussion.

Proof. Suppose that there exists a pure-strategy Nash equilibrium $(x_1, \ldots, x_n), (y_1, \ldots, y_n)$. Obviously, a player allocates some resources for item i in maximization problem (1)–(2) or (3)–(4) only if item i is pivotal in some winning subset.

Suppose that item i is pivotal in the subset $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$. Then, item i is also pivotal in the subset $(\overline{t_1}, \ldots, \overline{t_{i-1}}, 1, \overline{t_{i+1}}, \ldots, \overline{t_n})$ where $t_j + \overline{t_j} = 1$ for all $j = 1, \ldots, n$. Therefore, item i is pivotal in *pairs* of subsets. Consider such a pair: subsets $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$ and $\left(\overline{t_1}, \ldots, \overline{t_{i-1}}, 1, \overline{t_{i+1}}, \ldots, \overline{t_n}\right)$. The corresponding terms in the maximization problem are:

$$\frac{x_i}{x_i + y_i} \prod_{j\neq i: t_j=1} \frac{x_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{y_k}{x_k + y_k} + \frac{x_i}{x_i + y_i} \times \prod_{j\neq i: t_j=1} \frac{y_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{x_k}{x_k + y_k}.$$

It follows that the first order condition for the maximization problem (1)–(2) in variable x_i (for item i) has to have the following pair of terms:

$$\frac{y_i}{(x_i + y_i)^2} \prod_{j\neq i: t_j=1} \frac{x_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{y_k}{x_k + y_k} + \frac{y_i}{(x_i + y_i)^2} \times \prod_{j\neq i: t_j=1} \frac{y_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{x_k}{x_k + y_k}$$
$$= \frac{y_i}{(x_i + y_i)^2} \left(\prod_{j\neq i: t_j=1} \frac{x_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{y_k}{x_k + y_k} + \prod_{j\neq i: t_j=1} \frac{y_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{x_k}{x_k + y_k} \right).$$

Analogously, the first order condition for the maximization problem (3)–(4) in variable y_i (for item i) has to have the following pair of terms:

$$\frac{x_i}{(x_i + y_i)^2} \prod_{j\neq i: t_j=1} \frac{x_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{y_k}{x_k + y_k} + \frac{x_i}{(x_i + y_i)^2} \times \prod_{j\neq i: t_j=1} \frac{y_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{x_k}{x_k + y_k}$$
$$= \frac{x_i}{(x_i + y_i)^2} \left(\prod_{j\neq i: t_j=1} \frac{x_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{y_k}{x_k + y_k} + \prod_{j\neq i: t_j=1} \frac{y_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{x_k}{x_k + y_k} \right).$$

Then, the first order condition for the maximization problem (1)–(2) in variable x_i is:

$$\lambda_x = \frac{y_i}{(x_i + y_i)^2} \sum_{\mathbf{t}\in\mathbf{V}_i} \left(\prod_{j\neq i: t_j=1} \frac{x_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{y_k}{x_k + y_k} + \prod_{j\neq i: t_j=1} \frac{y_j}{x_j + y_j} \prod_{k\neq i: t_k=0} \frac{x_k}{x_k + y_k} \right), \tag{11}$$

and the first order condition for the maximization problem (3)–(4) in variable y_i is:

$$\lambda_y = \frac{x_i}{(x_i+y_i)^2}\sum_{\mathbf{t}\in\mathbf{V}_i}\left(\prod_{j\neq i:t_j=1}\frac{x_j}{x_j+y_j}\prod_{k\neq i:t_k=0}\frac{y_k}{x_k+y_k} + \prod_{j\neq i:t_j=1}\frac{y_j}{x_j+y_j}\prod_{k\neq i:t_k=0}\frac{x_k}{x_k+y_k}\right), \qquad (12)$$

where the number of terms in the brackets of the right-hand side of Eqs. (11) and (12) is exactly equal to the number of subsets when item i is pivotal. Dividing expression (11) by expression (12) gives the **proportional property** for any item that is pivotal in at least one subset:

$$\frac{\lambda_x}{\lambda_y} = \frac{y_1}{x_1} = \cdots = \frac{y_n}{x_n} = \frac{Y}{X}. \qquad (13)$$

Adding expression (11) and expression (12) gives:

$$\lambda_x + \lambda_y = \frac{1}{(x_i+y_i)}\sum_{\mathbf{t}\in\mathbf{V}_i}\prod_{j\neq i:t_j=1}\frac{x_j}{x_j+y_j}\prod_{k\neq i:t_k=0}\frac{y_k}{x_k+y_k}. \qquad (14)$$

Then, from (13) and (14) we obtain:

$$\frac{1}{x_1}\sum_{\mathbf{t}\in\mathbf{V}_1}\left(\prod_{j\neq 1:t_j=1}X\prod_{k\neq 1:t_k=0}Y\right) = \cdots = \frac{1}{x_n}\sum_{\mathbf{t}\in\mathbf{V}_n}\left(\prod_{j\neq n:t_j=1}X\prod_{k\neq n:t_k=0}Y\right). \qquad (15)$$

Eqs. (15) together with the budget constraint (2) uniquely determine the budget allocation $(x_1, \ldots, x_n)$. The proportional property (13), together with the budget allocation of player y, uniquely determine the budget allocation $(y_1, \ldots, y_n)$. Note that from (9), for all pivotal items, Eqs. (15) become:

$$\frac{K_1}{x_1} = \cdots = \frac{K_n}{x_n},$$

which together with the budget constraint (2) gives:

$$x_i = \frac{K_i}{K_1+\cdots+K_n}X, \quad \text{for all } i = 1, .., n.$$

Therefore, if a Nash equilibrium exists, then we have just described it. ■

Theorem 3 suggests that the equilibrium budget allocation for pivotal items can be found from expressions (10). Suppose that item i is pivotal, then the sum $\sum_{\mathbf{t}\in\mathbf{V}_i}\left(\prod_{j\neq i:t_j=1}X\prod_{k\neq i:t_k=0}Y\right)$ in (9) contains a number of terms that is exactly equal to the number of times item i is pivotal, $Piv(i)$, in (6). The following result establishes a connection between K_i and $BPI(i)$ for a small number of items.

Proposition 1. *Suppose that condition (5) holds and $n \leq 4$. Then*

$$\frac{K_i}{K_1+\cdots+K_n} = BPI(i) \quad \text{for } i = 1, 2, \ldots, n,$$

where K_i is defined in (9).

Proof. Straightforward calculations for $n = 2$, $n = 3$, and $n = 4$ give the result. ■

Since K_i is proportional to the Banzhaf Power Index for item i in the case of 2, 3, and 4 items, a natural question is whether this observation holds for any number of items. The following example illustrates that this is *not* the case.

Example 1. Suppose that $n = 5$ and

$W_1 = 3, W_2 = W_3 = W_4 = W_5 = 1.$

Then there are $2^4 = 16$ winning subsets. It is easy to see that item 1 is pivotal in 14 winning subsets and items 2, 3, 4 and 5 are substitutes and pivotal in 2 winning subsets. Hence,

$$BPI(1) = \frac{7}{11}, BPI(2) = BPI(3) = BPI(4) = BPI(5) = \frac{1}{11}. \qquad (16)$$

Note that Eqs. (9) become

$$K_1 = \left(4X^3Y + 6X^2Y^2 + 4XY^3\right)$$

and

$$K_2 = K_3 = K_4 = K_5 = \left(X^3Y + XY^3\right).$$

Hence,

$$\frac{K_i}{K_1+K_2+K_3+K_4+K_5} = \frac{\left(X^3Y+XY^3\right)}{8X^3Y+6X^2Y^2+8XY^3} = \frac{X^2+Y^2}{8X^2+6XY+8Y^2}, \qquad (17)$$

for $i = 2, \ldots, 5$.

$$\frac{K_1}{K_1+K_2+K_3+K_4+K_5} = \frac{2X^2+3XY+2Y^2}{4X^2+3XY+4Y^2}. \qquad (18)$$

In general, if $X \neq Y$, then expressions (17)–(18) are different from (16). ■

Let us now address the question of the existence of an equilibrium. Denote by $\mu = Y/X > 0$ the relative endowment difference. It turns out that in the case of a few items, based on Theorem 3 and Proposition 1, we obtain the following existence and uniqueness result.

Proposition 2. *Suppose that condition (5) holds. If $n = 2$, or if $n = 3$ and $\mu \in [1/3, 3]$, or if $n = 4$ and $\mu \in [9/11, 11/9]$, then, there exists a unique Nash equilibrium in which*

$$x_i = BPI(i)\,X \quad \text{and} \quad y_i = BPI(i)\,Y, \quad \text{for } i = 1, \ldots, n. \qquad (19)$$

Proof. The case $n = 2$ is straightforward. The case $n = 3$ is similar to the proof of case $n = 4$ and is therefore is omitted.

Suppose that $n = 4$. Since assumption (5) holds, there are three cases:

- (i) $W_1 > W_2 + W_3 + W_4$,
- (ii) $W_1 + W_4 > W_2 + W_3$,
- (iii) $W_1 + W_4 < W_2 + W_3$.

Case (i) is obvious.

Consider case (ii). As there are $2^3 = 8$ winning subsets: (1, 1, 1, 1), (1, 1, 1, 0), (1, 1, 0, 1), (1, 0, 1, 1), (0, 1, 1, 1), (1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), the winning majority function is

$$F(x_1,x_2,x_3,x_4) = \frac{x_1}{x_1+y_1}\frac{x_2}{x_2+y_2}\frac{x_3}{x_3+y_3}\frac{x_4}{x_4+y_4} + \frac{x_1}{x_1+y_1}\frac{x_2}{x_2+y_2}\frac{x_3}{x_3+y_3}\frac{y_4}{x_4+y_4} + \frac{x_1}{x_1+y_1}\frac{x_2}{x_2+y_2}\frac{y_3}{x_3+y_3}\frac{x_4}{x_4+y_4}$$

$$+\frac{x_1}{x_1+y_1}\frac{y_2}{x_2+y_2}\frac{x_3}{x_3+y_3}\frac{x_4}{x_4+y_4}$$
$$+\frac{y_1}{x_1+y_1}\frac{x_2}{x_2+y_2}\frac{x_3}{x_3+y_3}\frac{x_4}{x_4+y_4}$$
$$+\frac{x_1}{x_1+y_1}\frac{x_2}{x_2+y_2}\frac{y_3}{x_3+y_3}\frac{y_4}{x_4+y_4}$$
$$+\frac{x_1}{x_1+y_1}\frac{y_2}{x_2+y_2}\frac{x_3}{x_3+y_3}\frac{y_4}{x_4+y_4}$$
$$+\frac{x_1}{x_1+y_1}\frac{y_2}{x_2+y_2}\frac{y_3}{x_3+y_3}\frac{x_4}{x_4+y_4}.$$

We have to check the signs of the three leading principle minors of $D^2F(x)$ at the values from expression (19). Note that the Hessian of F at $x_1 = 3x_2 = 3x_3 = 3x_4 = 3x = X/2$ and $y_1 = 3y_2 = 3y_3 = 3y_4 = 3y = Y/2$:

$$D^2F(x) = \frac{xxyy}{(x+y)^6} \times \begin{pmatrix} \frac{-2}{3}(1+\mu) & (\mu^2-1) & (\mu^2-1) & (\mu^2-1) \\ (\mu^2-1) & \frac{-2}{3}(1+\mu) & (\mu^2-1) & (\mu^2-1) \\ (\mu^2-1) & (\mu^2-1) & \frac{-2}{3}(1+\mu) & (\mu^2-1) \\ (\mu^2-1) & (\mu^2-1) & (\mu^2-1) & \frac{-2}{3}(1+\mu) \end{pmatrix},$$

where $\mu = y/x$.

Note that four leading principle minors are

$$|A_1| = \frac{-2}{3}(1+\mu)\frac{xxyy}{(x+y)^6} < 0;$$

$$|A_2| = \left(\frac{xxyy}{(x+y)^6}\right)^2 \left(\frac{4}{9}(1+\mu)^2 - \left(\mu^2-1\right)^2\right) > 0,$$

if and only if

$$\frac{1}{3} < \mu < \frac{5}{3}. \tag{20}$$

Note that

$$|A_3| < 0,$$

if and only if

$$\frac{1}{3} < \mu < \frac{4}{3}. \tag{21}$$

Finally,

$$|A_4| > 0,$$

if and only if

$$\begin{vmatrix} \frac{-2}{3}(1+\mu) & (\mu^2-1) & (\mu^2-1) & (\mu^2-1) \\ (\mu^2-1) & \frac{-2}{3}(1+\mu) & (\mu^2-1) & (\mu^2-1) \\ (\mu^2-1) & (\mu^2-1) & \frac{-2}{3}(1+\mu) & (\mu^2-1) \\ (\mu^2-1) & (\mu^2-1) & (\mu^2-1) & \frac{-2}{3}(1+\mu) \end{vmatrix}$$
$$= -3\mu^8 - \frac{16}{3}\mu^7 + 4\mu^6 + \frac{32}{3}\mu^5 + \frac{70}{81}\mu^4 - \frac{368}{81}\mu^3 - \frac{4}{27}\mu^2 + \frac{64}{81}\mu - \frac{11}{81} > 0,$$

or

$$\frac{1}{3} < \mu < \frac{11}{9}. \tag{22}$$

Since all three inequalities (20)–(22) have to hold for both players, we get the following restriction for the existence of equilibria:

$$\frac{9}{11} < \mu < \frac{11}{9}.$$

Consider case (iii): $W_1 + W_4 < W_2 + W_3$. In this case there are $2^3 = 8$ winning subsets: $(1, 1, 1, 1)$, $(1, 1, 1, 0)$, $(1, 1, 0, 1)$, $(1, 0, 1, 1)$, $(0, 1, 1, 1)$, $(1, 1, 0, 0)$, $(1, 0, 1, 0)$, $(0, 1, 1, 0)$. It is easy to see now that item 4 is never pivotal and items 1, 2, and 3 are pivotal in 4 winning subsets. As we have already seen in the case of $n = 3$, the result holds if $\mu \in [1/3, 3]$.

Note that in all the cases we find a unique critical point of the maximization function which is also a local maximum. Therefore, this maximum is also global and we have proved the proposition. ■

Proposition 2 demonstrates that the Banzhaf Power Index does not help to characterize a unique Nash equilibrium in general (for $n \geq 5$).

References

Banzhaf, J., 1965. Weighted voting doesn't work: A mathematical analysis. Rutgers Law Rev. 19, 317–343.
Borel, E., 1921. La théorie du jeu et les équations intégrales à noyan symétrique. C. R. Acad. Sci. 173, 1304–1308. English translation by L. Savage, "The theory of play and integral equations with skew symmetric kernels". Econometrica 21, (1953), 97–100.
Friedman, L., 1958. Game-theory models in the allocation of advertising expenditures. Oper. Res. 6 (5), 699–709.
Kovenock, D., Roberson, B., 2012. Conflicts with multiple battlefields. In: Garfinkel, M.R., Skaperdas, S. (Eds.), Oxford Handbook of the Economics of Peace and Conflict. Oxford University Press, Oxford, pp. 503–530.
Lake, M., 1979. A new campaign resource allocation model. In: Brams, S.J., Schotter, A., Schwodiauer, G. (Eds.), Applied Game Theory. Physica-Verlag, Wurzburg, West Germany, pp. 118–132.
Osorio, A., 2013. The lottery Blotto game. Econom. Lett. 120, 164–166.