The Pure Theory of Elevators

A. Wuffle

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The Pure Theory of Elevators

A WUFFLE
School of Social Sciences
University of California, Irvine
Irvine, CA 92717

Let us consider three puzzles which will turn out to be mathematically very similar.

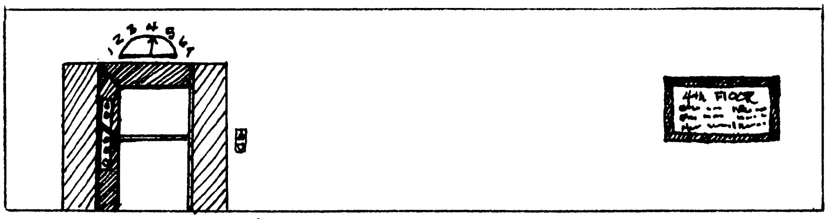
PUZZLE 1. It has often been noted that, unless you are waiting at either the bottom floor or at the top floor of a building, the next elevator is almost always going in the *wrong* direction. (See e.g., [**1**], 10–11.) On the other hand it seems reasonable, since what goes down must come up (the author is indebted to Miss Shari Shreiber for pointing this out to him), and conversely, that it ought to be just as easy (i.e., take just as long) to get an elevator going up as one going down. How can both of these things be true?

PUZZLE 2. A New Yorker has two girlfriends, one who lives uptown and one downtown, each of whom he likes equally. He goes (as the whim takes him) to the subway station and takes whichever train (uptown or downtown) comes along first. Trains run equally often in both directions. Despite his equal attachment to the two women he ends up seeing one considerably more often than the other. How can this be? (I learned of this puzzle through my colleague Louis Narens. See [**2**].)

PUZZLE 3. ([**1**], 59–60)

> In a small midwestern town there lived a retired railroad engineer named William Johnson. The main line on which he had worked for so many years passed through the town. Mr. Johnson suffered from insomnia and would often wake up at any odd hour of the night and be unable to fall asleep again. He found it helpful, in such cases, to take a walk along the deserted streets of the town, and his way always led him to the railroad crossing. He would stand there thoughtfully watching the track until a train thundered by through the dead of night. The sight always cheered the old railroad man, and he would walk back home with a good chance of falling asleep.
>
> After a while he made a curious observation; it seemed to him that most of the trains he saw at the crossing were traveling eastward, and only a few were going west. Knowing very well that this line was carrying equal numbers of eastbound and westbound trains, and that they alternated regularly, he decided at first that he must have been mistaken in this reckoning. To make sure, he got a little notebook, and began putting down "E" or "W", depending on which way the first train to pass was traveling. At the end of a week, there were five "E's" and only two "W's" and the observations of the next week gave essentially the same proportion. Could it be that he always woke up at the same hour of night, mostly before the passage of eastbound trains?
>
> Being puzzled by this situation, he decided to undertake a rigorous statistical study of the problem, extending it also to the daytime. He asked a friend to make a long list of arbitrary times such as 9:35 a.m., 12:00 noon, 3:07 p.m., and so on, and he went to the railroad crossing punctually at these times to see which train would come first. However, the result was the same as before. Out of one hundred trains he met, about seventy-five were going east and only twenty-five west. In despair, he called the depot in the nearest big city to find whether some of the westbound trains had been rerouted through another line, but this was not the case. He was, in fact, assured that the trains were running exactly on schedule, and that equal numbers of trains daily were going each way. This mystery brought him to such despair that he became completely unable to sleep and was a very sick man.

Let us look at Puzzle 1 first. Consider a building with N floors and r elevators. To simplify, let us begin with the special case $r=1$. At any moment the elevator can be in any one of $2(N-1)$ states, which we may denote $1\uparrow, 2\uparrow, 2\downarrow, 3\uparrow, 3\downarrow, \ldots, (N-1)\uparrow, (N-1)\downarrow, N\downarrow$. The arrow indicates the direction in which the elevator is going after it stops on the ith floor. Of course, at the top floor [bottom floor] the elevator must always next go down [up]. Let the time between two floors



be t. We shall assume that t incorporates some fixed (negligible) waiting time per floor. Consider a high traffic case where, for simplicity, we assume the elevator stops at each floor. (An anonymous referee has pointed out that orthodox Jews, who can't "operate" machinery on the Sabbath and other religious holidays, will sometimes set elevators so that they will automatically stop at every floor for a fixed amount of time. These are called *Shabbos elevators*.) If you are at floor k, what is the probability that the next elevator will be going up? Let us denote this probability as $p(k\uparrow)$. For $1<k<N$, the next elevator will be going up whenever the elevator is in one of the states $1\uparrow, 2\uparrow, \ldots, (k-1)\uparrow$ or in one of the states $2\downarrow, 3\downarrow, \ldots, k\downarrow$. There are $k-1+k-1$ $(=2k-2)$ such states. We may hypothesize that all $2(N-1)$ possible states are equally likely. For $1<k<N$,

$$p(k\uparrow)=\frac{2(k-1)}{2(N-1)}=\frac{k-1}{N-1} \tag{1}$$

and

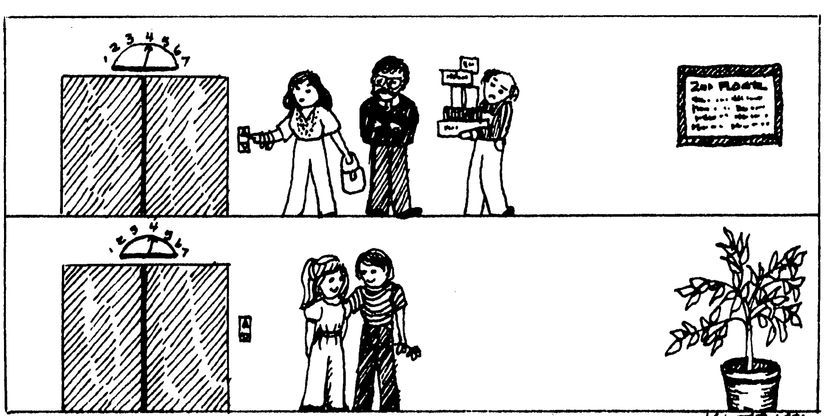
$$p(k\downarrow)=\frac{N-k}{N-1}=1-p(k\uparrow). \tag{2}$$

Of course $p(1\uparrow)=1$ and $p(N\downarrow)=1$. *Thus, for example, on the second floor of Social Sciences Tower at the University of California, Irvine* ($N=7$), *when one of the two elevators is (as happens quite often) out of order, the probability that the next elevator is going up is only* $1/6$.

Now, however, let us look at a different though related question. If we are standing on floor k, how long must we wait, on average, till the next up elevator? Consider the following chain:

$$k\uparrow, (k+1)\uparrow, (k+2)\uparrow, \ldots, (N-1)\uparrow, N, (N-1)\downarrow,$$
$$(N-2)\downarrow, (N-3)\downarrow, \ldots, 1\uparrow, 2\uparrow, 3\uparrow, \ldots, (k-1)\uparrow.$$

There will, of course, be exactly $2(N-1)$ states in this chain. If we are standing on floor k, the waiting time for an up elevator if the elevator is in state one of this chain is either 0 or $(2N-2)t$. We shall assume it is $(2N-2)t$, i.e., you just missed the elevator. The waiting time for an up elevator if the elevator is in state two of this chain is $(2N-3)t$. The waiting time for an up elevator if the elevator is in state j of this chain, is simply $(2N-j-1)t$. Thus, if we are standing on floor k $(1<k<N)$ in the one-elevator case under our simplifying assumptions, the expected waiting time, T_1, till the next elevator is given by the equation



$$T_1 = \sum_{j=1}^{2N-2} \frac{2N-j-1}{2N-2} t. \tag{3}$$

By symmetry, this may be written as

$$T_1 = t \sum_{j=1}^{2N-2} \frac{j}{2N-2}. \tag{4}$$

Using the well-known identity $\Sigma_{i=1}^{n} i = n(n+1)/2$, the equation in (4) can be simplified to

$$T_1 = \frac{(2N-1)(2N-2)t}{2(2N-2)} = \frac{(2N-1)t}{2}. \tag{5}$$

It is clear from equation (5) for T_1 that no matter what floor we are on (other than the top and bottom floors), the waiting time to an up elevator is constant. It is also apparent that (except at the top and bottom floors) the waiting time for a down elevator is the same as for an up elevator. Nonetheless, as we demonstrated above, the probability that the *next* elevator to come by will be going up depends on what floor we happen to be on (and how many floors are in the building), as specified in expressions (1) and (2). (If we assume you will always catch an elevator when it is waiting on your floor, the expression becomes

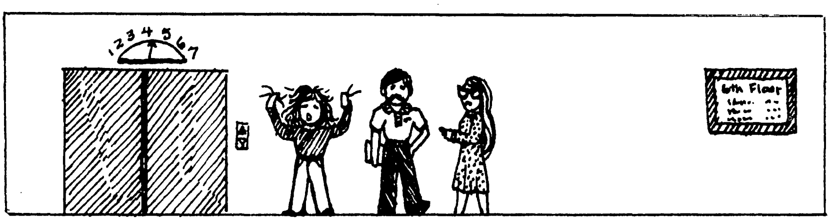
$$T_1 = \frac{2N-3}{2} t.$$

We can, of course, modify the above expression to take into account some probability, say one-half, of missing an elevator which is going in the desired direction waiting on your floor. We shall, however, neglect such complications.)

Now let us look at Puzzle 2. An analysis similar to that given above can shed light on the seeming paradox. Imagine that trains run every hour. The uptown comes at ten minutes to the hour, the downtown on the hour exactly. It's easy to see that our Lothario is five times more likely to go uptown than downtown—since only if he arrives in the ten minutes between when the uptown has just left and the downtown has not yet arrived will he end up going downtown. Of course, the expected waiting time will be one-half hour for both the uptown and downtown trains.

The solution to Puzzle 3 is essentially identical to that for Puzzle 2. Assume trains from each terminus depart on a fixed schedule (say, one every twelve hours). Our midwestern train buff is located at a distance from the western rail terminus (L.A.) and the eastern rail terminus (Chicago) such that the first train he sees is far more likely to be an eastbound train than a westbound train.

Now that the puzzles have been explained, let us continue our exploration of the behavior of elevators and of the people who wait for them. (This discussion may serve as an inquiry into a phenomenon which we term "elevator madness," by those who wait (and wait...) for an elevator.)



Let the conditional probability that a person who wants an elevator wants to go up given that he is on the kth floor be denoted by $p_k(U)$, and similarly let $p_k(D)$ equal the probability that someone on the kth floor who wants an elevator wants to go down. Let us initially assume that, except at rush hour, the relative attractiveness of all floors of the building is equal, i.e.,

$$p_k(U) = \frac{N-k}{N-1}$$

and

$$p_k(D)=\frac{k-1}{N-1}.$$

Such an assumption is not unreasonable for a single-company-owned office building.

We may define a frustration index f_k for someone on the kth floor waiting for an elevator as the likelihood that the next elevator is going in the wrong direction. For $0<k<N$, we have

$$f_k=p_k(U)p(k\downarrow)+p_k(D)p(k\uparrow) \tag{6}$$
$$=\left(\frac{N-k}{N-1}\right)\left(\frac{N-k}{N-1}\right)+\left(\frac{k-1}{N-1}\right)\left(\frac{k-1}{N-1}\right),\ 1<k<N.$$

After some straightforward algebra, we may rewrite f_k in the form

$$f_k=1-\frac{2(N-k)(k-1)}{(N-1)^2}. \tag{7}$$

It is clear from (7) that

$$f_k=f_{N-k+1}.$$

Since $(N-k)(k-1)>(N-k-1)k$ for $k>N/2$, it is easy to see from equation (7) that for N odd, f_k is minimal for $k=(N+1)/2$, $1<k<N$. *Indeed, the closer you are to either the top or bottom floor of a one-elevator building (as long as you're not actually on the top or bottom floor), the more you can be expected to suffer from elevator madness.* Furthermore, from (7) it is straightforward to show that $f_{(N+1)/2}=1/2$.

At the top or bottom floors of a building, the next elevator is always going in the desired direction, hence we may let $f_1=f_N=0$.

Under the above assumptions, average expected frustration, for the one-elevator case, which we shall denote by F_1, is given by

$$F_1=\sum_{k=1}^{N} f_k\cdot p(k), \tag{8}$$

where $p(k)$ is the proportion of elevator seekers on the kth floor. If we assume that $p(k)=1/N$ and continue assuming that all floors of the building are equally attractive, then using (7) and standard summation formulas, we may write F_1 as

$$F_1=\frac{1}{N}\sum_{k=2}^{N-1}\frac{1}{(N-1)^2}\left[2k^2-2k(N+1)+N^2+1\right]$$
$$=\frac{(N-2)(2N-3)}{3N(N-1)}.$$

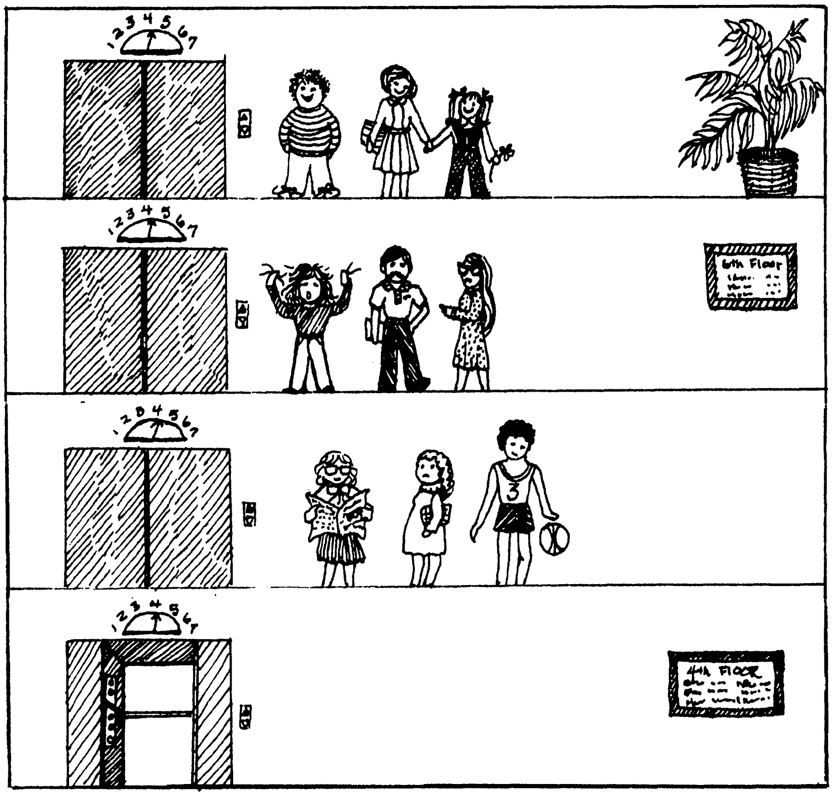
Although for small N ($N<10$), F_1 is less than $1/2$, clearly as N gets larger, F_1 approaches $2/3$.

Let us now consider (still for the case $r=1$) what happens during rush hours. During morning rush hour, every person is going up and during afternoon rush hours every person is going down. At morning rush hour, everyone begins on floor one, hence frustration (as we are measuring it) is zero. Afternoon rush hours are another story. At afternoon rush hour, if we assume as before that people are evenly spread through the building (i.e., $p(k)=1/N$), we have, for $k<N$,

$$f_k=\frac{k-1}{N-1} \tag{9}$$

and

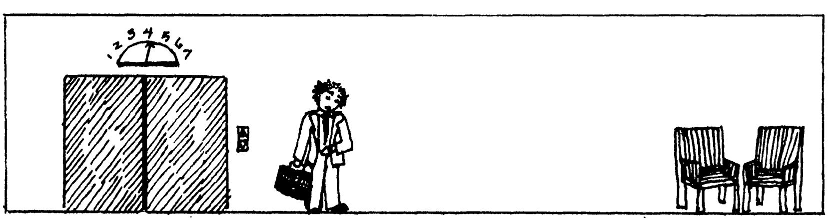
$$F_1=\sum_{k=1}^{N-1}\frac{k-1}{N(N-1)}=\frac{(N-1)(N-2)}{2N(N-1)}=\frac{(N-2)}{2N}. \tag{10}$$



Some simple algebra suffices to show that if $k > (N+1)/2N$, then the "normal" frustration of waiting for an elevator (given by expression (6)) is less than the frustration in rush hour (given by expression (9)), while the reverse is true for $k < (N+1)/2N$. *Thus (in the single-elevator case) on the upper floors of a building, elevator madness should reach its peak in the late afternoon.* (We neglect complicating factors of tiredness, anxiousness to go home, etc., which may also be assumed to be maximal in the later afternoon. To the extent such factors exist, they merely strengthen our conclusion.) However, while F_1 approaches $2/3$ for "normal" traffic, it is apparent from expression (10) that F_1 approaches $1/2$ for rush-hour traffic. (We might also note that $\int_0^1 y\,dy = 1/2$.) Hence, on average, rush-hour traffic (under our simplifying assumptions) is less frustrating (in terms of the next elevator going the wrong way) than is "normal" traffic.

Most large buildings have more than one elevator, so that we should investigate what happens when there are $r \geqslant 2$ elevators (which serve all floors).

For $r = 2$, for an individual on some specified floor k, let $q_1^{(k)} =$ the number of floors away elevator one is, in terms of its next appearance as a down [up] elevator; and define $q_2^{(k)}$ similarly.



It is clear that the number of floors away the *next* down [up] elevator is, is simply $\min(q_1^{(k)}, q_2^{(k)})$.

It is apparent that the probability that $\min(q_1^{(k)}, q_2^{(k)})$ equals h is independent of k, and thus we drop the superscript and assert that

$$\begin{aligned} p(\min(q_1,q_2)=h) = & \, p(q_1=h|q_2>h)p(q_2>h) \\ & + p(q_1>h|q_2=h)p(q_2=h) \\ & + p(q_1=h|q_2=h)p(q_2=h). \end{aligned}$$

This expression above may be rewritten as

$$\begin{aligned} p(\min(q_1,q_2)=h) &= \frac{1}{2(N-1)}\left[\frac{(2N-2-2h)}{2(N-1)} + \frac{(2N-2-h)}{2(N-1)} + \frac{1}{2(N-1)}\right] \\ &= \frac{4N-3-2h}{4(N-1)^2}. \end{aligned}$$

When it takes t minutes for the elevator to travel between consecutive floors (and waiting time is neglected), it is clear that T_2, the expected waiting time for the next down [up] elevator in the case where there are two elevators, is given by

$$T_2 = \sum_{h=1}^{2N-2}\left(\frac{4N-3-2h}{4(N-1)^2}\right)ht = \frac{(2N-1)(4N-3)t}{12(N-1)}. \tag{11}$$

A much simpler way to get the same result is to recognize that

$$p(\min(q_1,q_2)=h) = \frac{h}{2N-2}$$

and hence

$$T_2 = \sum_{h=1}^{2N-2}\frac{h^2}{4(N-1)^2} = \frac{(2N-1)(4N-3)t}{12(N-1)}.$$

More generally, when there are r elevators,

$$p(\min(q_1,q_2,\ldots,q_r)=h) = \sum_{h=1}^{2N-2} h^r$$

and hence

$$T_r = \sum_{h=1}^{2N-2}\frac{h^r}{[2(N-1)]^r}.$$

While there are several formulae to obtain Σh^r (see [**3**]) it is much simpler to assume N large and obtain the very useful result that

$$T_r \approx Nt\int_0^2\left(\frac{x^r}{2}\right)dx = \frac{2Nt}{r+1}. \tag{12}$$

Note that for $r=1$, the approximation in (12) gives $T_1 \approx Nt$, which compares favorably with our discrete value expression, $T_1 = (N-\frac{1}{2})t$. For $r=2$, (12) gives $T_2 \approx (2Nt)/3$, the same result obtained by taking the limit of expression (11) as $N\to\infty$. As we would expect, the more elevators there are in operation, the less is the expected waiting time to the next up [down] elevator.

For $r=2$, we may obtain expressions $p_2(k\uparrow)$ and $p_2(k\downarrow)$ for the probability that the next elevator to come to floor k will be an up or a down elevator, respectively. Because there are two elevators, we also now have the probability of a tie, i.e., the up elevator and the down elevator arriving simultaneously, which we shall denote $p_2(k^{\uparrow}_{\downarrow})$. For $r=2$, we may set up the problem in terms of a decision tree. (See FIGURE 1.)

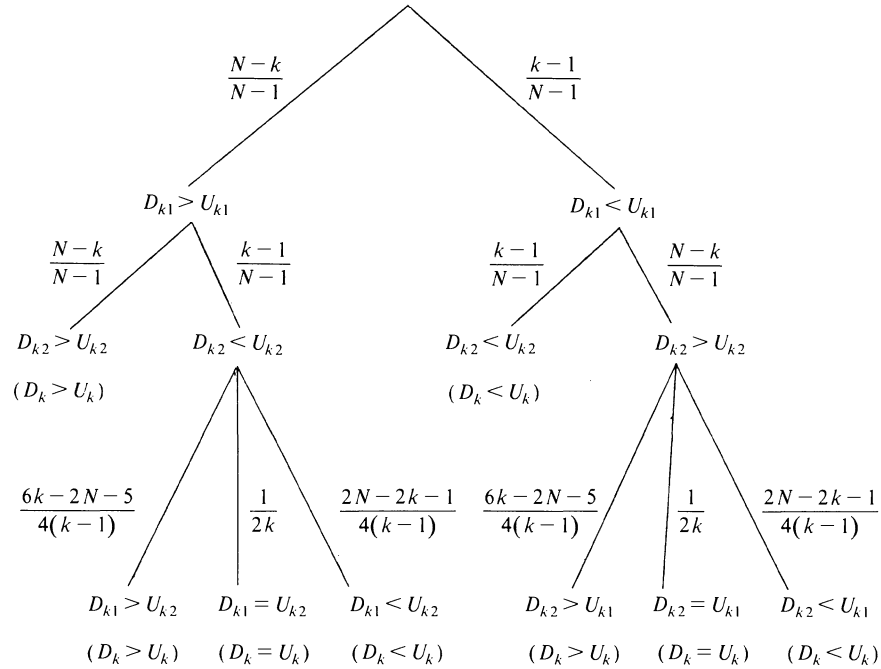


FIGURE 1. **Probability that the next elevator will be going up/down, if you're on Floor k and there are two elevators in the building ("$>$" denotes "arrives before"). Probabilities shown in bottom portion of FIGURE 1 are for $k \geqslant (N+1)/2$.**

In FIGURE 1, D_{ki} (respectively U_{ki}) denotes the ith elevator arrives on the kth floor, going down (respectively, up), and we use the symbol $>$ to denote "arrives before." $D_{ki} > U_{ki}$ denotes the event: when elevator i reaches floor k, it is going down rather than up. $D_{ki} < U_{ki}$ and $D_{ki} = U_{kj}$ are similarly defined. The outcomes are shown on the terminal nodes of the decision tree. For example, if $D_{k1} > U_{k1}$ and $D_{k2} > U_{k2}$, then the next elevator to reach floor k must be a down elevator, an outcome which we have denoted $D_k > U_k$. The probabilities shown in this tree can be straightforwardly derived. The only interesting cases are those where one elevator reaches the kth floor first coming down while the other elevator reaches it first going up. We need to calculate various conditional probabilities, e.g., $p(D_{k2} > U_{k1} \mid D_{k1} < U_{k1} \wedge D_{k2} > U_{k2})$. If $D_{ki} < U_{ki}$, then the maximum distance (in floors) to the arrival of elevator i on floor k as a down elevator is $2(N-k)$. If that were not so then elevator i would have come to floor k first as a down elevator. (Hint: try to visualize why this must be so.) Analogously, if $D_{ki} > U_{ki}$, then the maximum distance (in floors) to the arrival of elevator i on floor k as an up elevator is $2(k-1)$. Thus for $k \geqslant (N+1)/2$

$$\begin{aligned} p(D_{k2} > U_{k1} \mid D_{k1} < U_{k1} \wedge D_{k2} > U_{k2}) &= \sum_{h=1}^{2(N-k)} p(q_2 > h \mid q_1 = h) p(q_1 = h) \\ &= \sum_{h=1}^{2(N-k)} \left[\left(\frac{2(k-1)-h}{2(k-1)} \right) \left(\frac{1}{2(N-k)} \right) \right] \\ &= \frac{6k-2N-5}{4(k-1)}. \end{aligned}$$

The other probability values specified in FIGURE 1 are calculated in a similar fashion. Performing the necessary algebra, we eventually obtain, for $k \leqslant (N+1)/2$

$$p_2(k\downarrow)=\frac{(N-k)(4k-5)}{2(N-1)^2} \tag{13}$$

$$p_2(k\uparrow)=\frac{2(k-1)^2+(N-k)(2N-2k-1)}{2(N-1)^2}. \tag{14}$$

The probability of a tie is given by

$$p_2\left(k_\downarrow^\uparrow\right)=\frac{N-k}{(N-1)^2}.$$

For $k>(N+1)/2$, we reverse equations (13) and (14). Of course for $k=(N+1)/2$ those two equations give identical values.

Once again, we may define a frustration index f_k:

$$\begin{aligned} f_k &= p_k(U)p_2(k\downarrow)+p_k(D)p_2(k\uparrow) \\ &= \frac{(N-k)^2(4k-5)+(k-1)\left[2(k-1)^2+(N-k)(2N-2k-1)\right]}{2(N-1)^3}. \end{aligned}$$

Comparing equations (13) and (14) with equations (1) and (2) we observe that for $k<(N+1)/2$, $p_2(k\uparrow)<p(k\uparrow)$, while $p_2(k\downarrow)>p(k\downarrow)$; while the reverse is true for $k>(N+1)/2$. What this means is that the addition of a second elevator has only reduced the likelihood that the next elevator is up [down] for half of the floors (the lower floors). *Additional elevators tend to "even out" the probabilities that the next elevator to reach a floor will be going up as opposed to going down.*

For r and N large, useful approximations are

$$p(k\uparrow)=p(k\downarrow)\approx\frac{(2N-3)}{4(N-1)}\approx\frac{1}{2}$$

$$p\left(k_\downarrow^\uparrow\right)=\frac{1}{2(N-1)}.$$

Hence, for r and N large

$$f_k\approx\frac{1}{2}$$

$$F_r\approx\frac{1}{2}.$$

Elevators, trains, and subways are closed-loop systems. The puzzles we have discussed in this paper illuminate the difference between "frequency" and "phase" [**1**]. The reader might try to think of other examples of closed-loop systems and the mathematical problems they pose. For example, subways in New York City have both express trains and local trains (the former are faster because they make fewer stops), and you might try to formulate conditions in which it makes sense to take a train going in the wrong direction. (Hint: do you have to be at a station at which only local trains stop?)

I wish to acknowledge a considerable debt of gratitude to an anonymous referee, who provided a clear formulation of the solution to the elevator problem when the number of floors is large and who corrected a number of errors in the original statement of the equations in this paper, and to my colleague Louis Narens, who has convinced me that madness and mathematics can happily coexist.

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