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A DYNAMIC MODEL OF PROTOCOALITION FORMATION IN IDEOLOGICAL *N*-SPACE[1]

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Axelrod's (1970) notion of "connected coalition" is generalized to the *N*-dimensional case, and a simple but powerful model of coalition dynamics is put forth which generates connected coalitions with certainty in the unidimensional case and which usually, but not necessarily, gives rise to minimal winning coalitions. Political decision making is discussed at the levels of the group, organization, society, and supranational system. Unlike most other models in the coalition literature, the model presented: (a) is based on notions of ideological policy proximity rather than on notions such as least resources or zero-sum conflict; (b) posits a dynamic process of protocoalition formation which permits two actors to join in a (proto)coalition only when each is the other's *most* preferred partner; and (c) for sufficient information about the policy preferences/ideological views of the political actors, yields unique predictions as to which coalition can be expected to form.

KEY WORDS: social system, group, organization, society, supranational system, political decision making, coalition, association, spatial model, cluster analysis.

INTRODUCTION

IN THE past two decades, a number of models of political coalition formation have been proposed, most arising out of a game-theoretic tradition. (See e.g., Gamson, 1961, 1962; Riker, 1962; Rohde, 1972a, 1972b; Rohde & Spaeth, 1976; Hinckley, 1972; Li & Hinckley, 1976; Koehler, 1972; Kelley, 1968; Leiserson, 1970b; Axelrod, 1970; Brams, 1972; DeSwaan, 1970; Brams & Riker, 1972; Dodd, 1976; Schofield, 1976; Fiorina & Plott, 1978; Hoffman & Plott, 1980.) With only a handful of exceptions (e.g., Brams, 1972; Brams & Riker, 1972; Zais & Kessel, 1973; Brams & Garriga-Pico, 1975; Riker, 1962; Straffin, 1977; Hoffman & Packel, 1980) these coalition models have been essentially static, "attempting prediction solely in terms of final outcomes rather than of the processes by which coalitions are achieved" (Zais & Kessel, 1973; pp. 140–141; also cf. Leiserson, 1970a). Moreover, all of the dynamic coalition models of which we are aware have been limited to two-candidate competitions in which each candidate is seeking to win converts to his cause.

We shall offer a general dynamic model of protocoalition formation in an ideological space based on what is in effect an algorithm for pairwise cluster formation. Our model is similar in spirit to the work of Rosenthal (1969); Leiserson (1970b); DeSwaan (1970, 1973); Axelrod (1970); Zais and Kessel (1973); Flanagan (1973); Morgan (1976); Budge and Fairlie (1978); Shaffer, Yarnell, and Kessel (1978). We believe it to be potentially applicable to coalition processes in such diverse areas as cabinet formation, Supreme Court opinion coalitions, legislative policy making, and trade route formation among networks of spatially separated potential trading partners. In general, it is potentially applicable to groups, organizations, and supranational systems.

Consider a decision body of finite size whose members' most preferred positions (policies) may be represented as points in

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some N-dimensional space. Assume that each actor evaluates the relative desirability of alternatives in terms of their distance from that most preferred point, and that alternatives are always sufficiently distinct so that ties never arise. For simplicity, let us assume the utility function to be the Euclidean distance metric. The simplest case of interest would be policies which could be located along a unidimensional continuum such as a liberal-conservative dimension (MacRae, 1970; Rohde & Spaeth, 1976). This assumption of Euclidean distance is not crucial to the proofs we offer. Alternative assumptions as to distance metric could readily be accommodated. (See discussion of spatial modeling in Riker & Ordeshook, 1973.) Another assumption made below, that of coalition formation as a pairwise process, could also in principle be modified, e.g., to ascertain ideologically compatible triads. We shall not, however, pursue such complications here.

We assume that each actor is a potential member/focus of a protocoalition, and that each actor/protocoalition seeks to attract others into a coalition with him/it so as to eventually form a winning coalition. Actors need not all be of equal weight. We posit a multistage process of protocoalition formation. At Stage 1 each actor looks to form a protocoalition of himself and the actor nearest to him in N-space. Nearness is defined in terms of proximity based on weighted distance; i.e., the proximity of actor i with weight w_i to actor j with weight w_j, when these two are separated by distance d, is given by $dw_j/(w_i + w_j)$. Hence, although distance is, of course, symmetric, proximity is not symmetric, except for the special case where $w_i = w_j$. *If and only if* actor i is the actor closest to actor j *and* actor j is also the actor closest to actor i, where closeness is defined as above in terms of *weighted* distance, do the two join together in a protocoalition. (Recall that we have assumed that distances are sufficiently "finely" measured so as to eliminate the possibility of ties.) Our reason for using a proximity measure which is a function of both distance and (relative) weight is simple. We believe that in a coalition between a strong (high weight) actor and a weak (low weight) actor the position in N-space which the protocoalition will adopt will reflect the relative weights of the actors and thus will be closer to the ideal point of the stronger actor than to the ideal point of the weaker. Our proximity measure captures the notion that if weak actor i joins strong actor j, i must move further from his ideal point than j does from his ideal point; i.e., the proximity of i to j is a measure of the distance actor i would "travel" if he were to join a protocoalition with actor j.

If no winning coalition is formed at Stage 1, we move to Stage 2. Once a protocoalition is formed, it is assumed to act as a single actor, which is located at the center of gravity of the protocoalition. The "weight" of a protocoalition in determining the (weighted) center of gravity of subsequent larger protocoalitions it may enter is based on the weight of the actors in it. Actors who do not form pairs remain as "isolate" protocoalitions. At Stage 2, the process we have discussed for Stage 1 continues at the protocoalition level; i.e., each protocoalition seeks to merge with exactly one other protocoalition. Protocoalition I joins protocoalition J if and only if the (weighted) center of gravity of the protocoalition $I + J$ is the protocoalition whose (weighted) center of gravity in the ideological space is closest to the (weighted) center of gravity of protocoalition J *and* the (weighted) center of gravity of the protocoalition $I + J$ is the protocoalition whose (weighted) center of gravity in the ideological space is closest to the (weighted) center of gravity of protocoalition I. (Henceforth we shall drop the adjective "weighted" wherever its presence can be taken for granted.) If no winning coalition is formed, we move to Stage 3. At Stage 3 and subsequent stages, the process we have described above continues until a winning coalition is formed. While in the empirical test described later in this paper, we have followed the custom of taking a winning coalition to be one with at least a voting majority, this definition of winning coalition is not compelled by our model. In some legislative situations, e.g., where there are sharply polarized parties of left and right opposition, a centrist bloc may be able to win a vote of confidence even though only a minority government (Ian Budge, personal communication, July 20, 1981; see also Budge & Herman, 1978).

We may illustrate this model of protocoalition formation with a simple example in unidimensional space. Let N (the number of actors) be 5, and let W (the resources needed for winning coalition) be a simple majority, where all actors begin with equal weight.

A	BC	D	E

Consider the actors arrayed in ideological space as above (D and E are shown as further apart than B and C):

Stage 1. B and C join together as do D and E. A is left "isolate," since his "natural" protocoalition partner B prefers to join a protocoalition with C.

Stage 2. The protocoalitions of $B + C$ and $D + E$ coalesce to form a winning coalition. A remains isolated since its "natural" protocoalition partner $B + C$ prefers to merge with $D + E$.

This example leads us to assert two interesting results:

Result 1. The process of protocoalition formation posited above need not lead to a minimal winning coalition.

Of course, if the center of gravity of the coalition $A + B + C$ was closer to the center of gravity of $B + C$ than was the center of gravity of the coalition $(B + C) + (D + E)$, a minimal winning coalition $(A + B + C)$ would form. Such a coalition would form if the distance between A and the midpoint of $B + C$ was less than 3/2 the distance between the midpoint of $B + C$ and the midpoint of $D + E$. Because $B + C$ is a two-person protocoalition, the protocoalition $B + C$ is twice as important (has twice the weight) as the protocoalition A in determining the center of gravity of the coalition $A + (B + C)$.

Result 2. The process of protocoalition formation posited above need not lead to a winning coalition whose center of gravity is the median voter in the overall space, C; nor even a winning coalition where the median voter is the actor closest to the coalition's center of gravity.

Indeed, in the example we have given, *none* of the possible winning coalitions have the median voter, C, at their center of gravity. Furthermore, in the coalition $A + (B + C)$, B is closer to the coalition's center of gravity than is C; and in coalition $(B + C) + (D + E)$, D is closer to the coalition's center of gravity than is C. Of course, other spatial arrays, e.g.,

A	BCD	E

could lead to a centrist coalition. It is important to demonstrate that the protocoalition formation process we have posited must ultimately lead to a winning coalition and can never led to deadlock.

Result 3. The process of protocoalition formation posited above eventually leads to the formation of a winning coalition.

Proof: To prove this result we show that at each stage of protocoalition formation at least one new protocoalition merger must occur; hence, the process will eventually lead to a coalition of the whole which must be winning. The proof is by contradiction. We may pick any protocoalition at random; label that protocoalition A_1. Now consider the protocoalition A_1 wishes to join with. Label that protocoalition A_2. Now consider the protocoalition that A_2 wishes to join. If A_2 wishes to join A_1, we are done; so assume, on the contrary, that A_2 wishes to join some other coalition, say A_3. Continue this process. Since there are, by assumption, only a finite number of actors, eventually we must either have a reciprocated choice or have some protocoalition, say A_k, prefer to join with one of the protocoalitions which have previously been enumerated. In other words, if there is no reciprocated choice, then there must be a *cycle* of protocoalitions, each of which prefers the next. The proof will be established if we can show that cycles of length ≥ 3 are impossible; for if that is true, then there must be a cycle of length 2, i.e., two protocoalitions which reciprocate each other's choices. Suppose there is such a cycle A_i, A_{i+1}, A_{i+2}, $\cdots$, A_k $(k \geq 3)$, where A_i prefers to join A_{i+1}, etc., and A_k prefers to join A_i. Let a_i be the weight of A_i and $\overline{A_iA_j}$ be the distance between A_i and A_j. Then, for a cycle to exist, we must have

(1) $a_{i+1}/(a_i + a_{i+1}) \cdot \overline{A_iA_{i+1}} < a_k/(a_k + a_i) \cdot \overline{A_kA_i}$

(2) $a_{i+2}/(a_{i+1} + a_{i+2}) \cdot \overline{A_{i+1}A_{i+2}} < a_i/(a_i + a_{i+1}) \cdot \overline{A_iA_{i+1}}$

(3) $a_k/(a_{k-1} + a_k) \cdot \overline{A_{k-1}A_k} < a_{k-2}/(a_{k-2} + a_{k-1}) \cdot \overline{A_{k-2}A_{k-1}}$

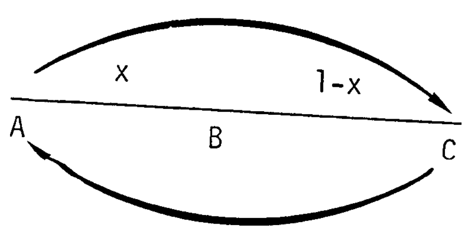
(4) $a_i/(a_k + a_i) \cdot \overline{A_kA_i} < a_{k-1}/(a_{k-1} + a_k) \cdot \overline{A_{k-1}A_k}$.

If you multiply those inequalities together, the left product is equal to the right-hand product, and hence cannot be less than the right-hand product. This contradiction shows that no cycle can have length 3 or greater. (Alternatively, one could note that the product of the $k - i$ initial inequalities contradicts the last inequality.)

Our process of protocoalition formation has another "nice" property. When we are dealing with a unidimensional (issue) space, our model invariably leads to connected winning coalitions. Such coalitions have been observed in cabinet formation (Axelrod, 1970; DeSwaan, 1973), opinion coalitions in the U.S. Supreme Court (Rohde & Spaeth, 1976) and in legislative decision making (MacRae, 1970).

Result 4. In unidimensional space, the process of protocoalition formation described above generates connected protocoalitions.

Proof: Again, we show a proof by contradiction. For a protocoalition to form which was not connected would require a situation such as that pictured below:



For such a situation to occur requires $\overleftarrow{AC} < \overleftarrow{BC}$ and $\overleftarrow{CA} < \overleftarrow{BA}$, i.e., it requires

(5) $a/(a + c) < b(1 - x)/(b + c)$,

and

(6) $c/(a + c) < bx/(a + b)$.

But, if Eq. (5) holds we must have

(7) $ac < b(c - ax - cx)$.

Similarly, if Eq. (6) holds we must have

(8) $ac < b(-c + ax + cx)$.

Since b is positive, the right-hand side of either Eq. (7) and/or Eq. (8) must be positive. Let us assume that the right-hand side of Eq. (8) is positive. Dividing Eq. (7) by Eq. (8), we obtain $1 < -1$, which is a contradiction. Identical results obtain if it is Eq. (7) which is assumed positive.

In order to see if Result 4 can be generalized to the N-dimensional case, we require a notion of "connectedness" applicable to N-space. In unidimensional space, a (proto)coalition can be said to be connected when it includes all actors on any line segment connecting any two members. A natural generalization of this to N-space is as follows:

Definition 1. A (proto)coalition shall be said to be *connected in N-space* (or, equivalently, *N-connected*) when it includes all actors in or on any convex hull defined by any $N + 1$ members of the (proto)coalition. As far as we are aware, this generalization of the connectedness notion has never previously been proposed. We believe the only author to look at connectedness in N-space to be Rosenthal (1969), who uses a graph theory model in which connected subgraphs indicate relative closeness in an otherwise ordinal and unidimensional space. Our definition of N-connectedness can be made clear by a simple two-dimensional example with six actors (see Fig. 1).

In this example the coalition $\{A, B, C, F\}$ is 2-connected, since there are no actors who are not themselves members of the coalition in any of the 3 convex hulls defined by any 3 of the 4 actors in the coalition (see Fig. 1). On the other hand, the coalition $\{A, B, C\}$ would not be 2-connected, since actor F is not a member, though he is within the convex hull defined by A, B, and C.

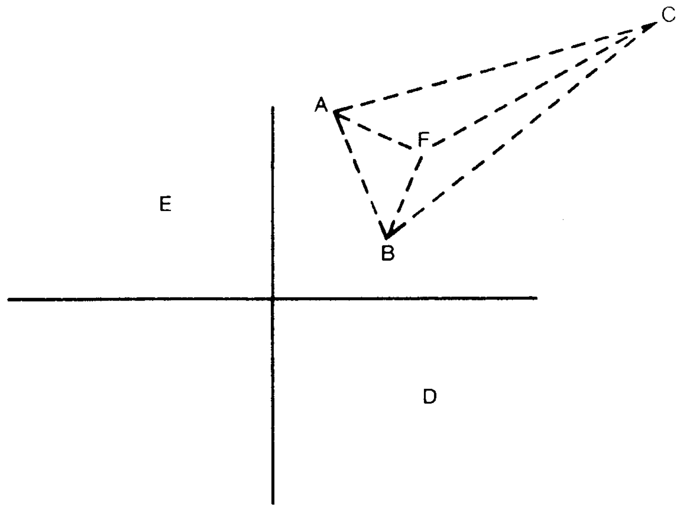


FIG. 1. MODEL OF N-CONNECTEDNESS.

It is important to see that N-connectedness need not imply $(N-1)$-connectedness. If we look at the actors in the two-dimensional example in Fig. 1 above and consider their projections onto the one-dimensional space defined by the x-axis, we obtain a unidimensional alignment as follows:

$$\underline{E \quad A \quad B \quad F \quad D \quad C}$$

Although the coalition $\{A, B, F, C\}$ is connected in two-dimensional space, it is not connected in the one-dimensional space defined by the projections onto the x-axis, or in the one-dimensional space defined by projections onto the y-axis:

$$\begin{array}{|l} C \\ A \\ F \\ E \\ B \\ D \end{array}$$

This example leads us to consider another extension of the connectedness notion. One may define full-connectedness as follows.

Definition 2. A (proto)coalition shall be said to be *fully connected in N-space* (or equivalently, *fully connected*) when it is j-connected for all integers: $0 < j \leq N$. In the example given in Fig. 1, $\{A, B, E, F\}$ would be a fully connected (proto)coalition, while $\{A, B, C, F\}$ would not be.

N-connectedness and full-connectedness are, we believe, concepts of considerable potential empirical importance, since if a coalition is not N-connected, there are, given ideological location, "natural" members of the coalition who are not part of it. Such actors might be expected to vociferously seek their "natural" rewards or to force some kind of coalitional realignment. Similarly, if a (proto)coalition is not fully connected, there necessarily exists a dimension (or dimensions) of choice which has the possibility of splitting the coalition, since for choices constrained to such a dimension(s), some coalition members will be closer to actors outside the coalition than to actors within it.

For N-dimensional space $(N > 2)$, unlike that in unidimensional space, we may show that the process of protocoalition described above need not produce N-connected coalitions. One specific counterexample is due to Philip Straffin (personal communication, April 10, 1979). The Straffin counterexample $(N = 2)$ offers a situation which seems empirically quite feasible, but a preliminary investigation of the model's behavior on seven cases of cabinet formation (three from Norway, two from Denmark, and two from the Federal Republic of Germany) for which two-dimensional party spaces have been obtained augurs for N-connectedness as the pratical norm, since in none of these seven cases did the model give rise to any protocoalitions which were not 2-connected. We conjecture, however, that the greater the dimensionality of the space, the higher the likelihood of the formation of a coalition which is not N-connected.

THREE ILLUSTRATIVE TESTS OF THE MODEL

To test our model, we shall first look at data on seven elections in three multiparty Western European democracies for which data is available on the spatial location of the parties in issues/ideological space. We shall use our model to predict which cabinet coalitions can be expected to form after those elections. The countries we shall look at are Norway, Denmark, and the Federal Republic of Germany.

Test I: Norway

For Norway we show in Fig. 2 the results of a two-dimensional spatial representation of the perceived location of Norwegian political parties in 1965 taken from Converse and Valen (1971). Seats won in the 1961, 1965, and 1969 elections are also shown in this figure.

There were five parties with seats in the Norwegian legislature in 1965 and 1969; six in 1961. Seventy-six votes are needed for a majority. Numbered circles in Fig. 2 represent the stages of protocoalition formation. In both 1965 and 1969, our model results in a prediction of a winning coalition of the Liberals, Christian, Center, and Conservative Parties by the third stage of protocoalition formation. This coalition prediction is confirmed by the data. For 1961 our model gives rise to a prediction that the Labor Party and the Socialist Peoples Party will emerge as the winning coalition on the first round of protocoalition formation. This outcome prediction is also confirmed by the data.

For all three elections, the winning coalitions we have correctly predicted are 2-connected, minimal winning, and connected in 1-space with respect to projections onto the x-axis but not with respect to the y-axis. However, our model gives rise to rather more focused predictions than others in the literature. For example, there are four other minimal winning coalitions that could have formed in 1965 and 1969 and five in 1961. The coalition we predict is the unique minimal resources coalition in 1965 and 1969 and one of the two such in 1961. It is one of the minimal winning coalitions with the fewest actors in 1961, but is not a member of that set in 1965 and 1969. For the three Norwegian elections we compare the predictions of our model and those of five other coalition models in Table 1.

We have replicated this analysis of these three Norwegian elections on other spatial representations of the Norwegian party system and obtained virtually identical results. We used the model to predict the 1969 cabinet coalition from a two-dimensional representation for 1969 survey data on voters given by Converse and Valen (1971), and to predict the 1961, 1963, and 1969 coalitions from a two-dimensional representation based on a nonmetric scaling analysis performed by Groennings (1970)

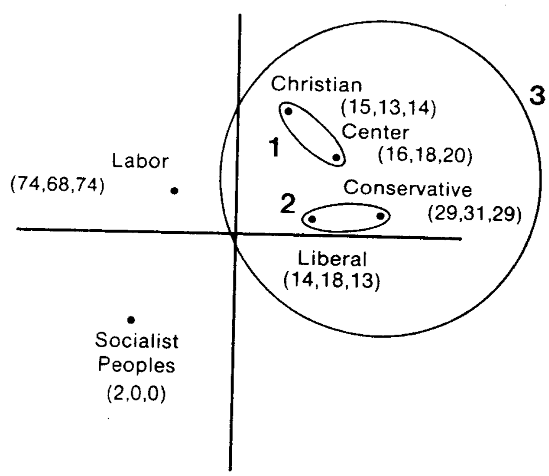


FIG. 2. PERCEIVED PARTY LOCATIONS IN NORWAY, 1965—TWO-DIMENSIONAL SOLUTION. (Adopted from Converse and Valen (1971, Fig. 4, p. 134). First entry in each vector represents that party's seats in the 1961 Norwegian legislature; second entry is for the 1965 legislature; third entry is for the 1969 legislature. Numbered circles represent stages of protocoalition formation in 1965 and 1969. In 1961 the process requires only one stage, union between the Socialist Peoples Party and the Labor Party. The Communist Party has been omitted because it was not seen as a viable coalition partner.)

TABLE 1
Comparisons of the Predictions of Six Models of Coalition Formation for Norwegian Cabinet Coalitions in 1961, 1965, AND 1969.*

Year	Observed support coalition	Predicted Coalitions: Protocoalition model	Aumann-Maschler (1964), Peleg (1972) bargaining set	Gamson (1961) minimal winning	Leiserson (1970b) fewest actor	Riker (1962) least resource	McKelvey-Ordeshook-Winer (1978) competitive solution
1961	1,6	1,6	1,2 1,3 1,4 1,5 1,6	1,2 1,3 1,4 1,5 1,6 2,3,4,5,6	1,2 1,3 1,4 1,5 1,6	1,6 2,3,4,5,6	1,5 1,6 2,3,4,5,6
1965	2,3,4,5	2,3,4,5	1,2 1,3 1,4 1,5	1,2 1,3 1,4 1,5 2,3,4,5	1,2 1,3 1,4 1,5	2,3,4,5	1,4 1,5 2,3,4,5
1969	2,3,4,5	2,3,4,5	1,2 1,3 1,4 1,5	1,2 1,3 1,4 1,5 2,3,4,5	1,2 1,3 1,4 1,5	2,3,4,5	1,4 1,5 2,3,4,5

* Source of spatial map: Converse and Valen (1971, Fig. 4, p. 134). See also Ordeshook and Winer (1980).
1 = Labor Party (Social Democratic Party), 2 = Conservative Party, 3 = Center Party, 4 = Christian People's Party, 5 = Liberal Party, 6 = Socialist People's Party.

on 1963 survey data. Our outcome predictions remained unchanged, and only with the Groennings (1970) spatial representation do we get even a different protocoalitional dynamic. Thus, at least in the Norwegian case our model does not appear unduly sensitive to alternative methods in specifying party space. This is particularly important because in the Groennings (1970) array the Labor Party is actually shown as (slightly) closer to the Liberal Party than it is to the Socialist Peoples Party; yet our model (based on *weighted* distance) does not predict a Liberal-Labor coalition.

Test II: Denmark

For Denmark, Rusk and Borre (1974) use a nonmetric multidimensional scaling technique to obtain a two-dimensional configuration of ideal points for the ten major Danish political parties based on 1973 voter responses to a party thermometer scale. We show this configuration in Fig. 3. Next to each party is a two-place vector which indicates the number of seats held by the party in the 1971 and 1973 elections, respectively. Eighty-eight seats were needed to form a majority government. In 1971 only five parties obtained seats in the legislature, and the cabinet "coalition" consisted of a one-party minority government. However, the minority party, the Social Democrats, had the parliamentary support of the Socialist People's Party, and we have treated those two parties as in a coalition together.

In 1973, all ten parties held legislative seats with the Agrarian Liberal Party forming a one-party minority government. However, by 1974, five other parties were providing parliamentary support for key elements of the Agrarian Liberal Party program, especially its economic policy. We have treated these parties as in a coalition together. We might note that the Progress Party was the last to join the support coalition, and only joined on certain issues.

In the space defined by voter perception of the Danish political parties, our model would predict that on the first round of protocoalition formation in 1971, the Social Peoples Party and the Center Democratic Party would form a protocoalition and also the Conservative Party and the Agrarian Democrats would coalesce into a protocoalition (see Fig. 3). Since the protocoalition consisting of the Social Democratic Party and the Socialist Peoples Party is a winning coalition, our model predicts that to be the coalition which would form in 1971. Treating the parliamentary support coalition as the winning coalition, our prediction is confirmed by the data. The coalition we predict is among five possible minimal winning coalitions. It is also the unique minimal resource coalition and one of the minimal winning coalitions with the least possible number of parties.

Let us now turn to the 1973 election. We show in Fig. 3, in the form of concentric elipses, the protocoalition formation stages our model leads us to expect for the 1973 election. The innermost elipses are those protocoalitions which would occur on the first round of protocoalition formation. Our model requires four stages of protocoalition formation to form a winning coalition. At this fourth stage, there will be two protocoalitions, one of which—the one consisting of the Progressive Party, the Center Democratic Party, the Christian Peoples Party, the Conservative Party, the Agrarian Liberal Party, and the Radical Liberal Party—will be winning. This is the coalition our model predicts will form. Treating the parliamentary support coalition as the winning coalition, despite the fact that we have, quite counterintuitively, predicted a supraminimal coalition, our *Fourth Stage* prediction is confirmed by the data. (Once the Progressive Party joins the coalition consisting of the other five parties, either the Center Democratic Party or the Christian People's Party is superfluous). For Denmark in 1973 our model yields a confirmed prediction which is distinct from that of any other coalition models and far more specific than that of most other coalition theory models (see Table 2 below: cf. Winer, 1979). We should also note that if the Progressive Party is not to be counted as a coalition partner, our model still appears accurate, since the nonmajority coalition arising at Stage 3 of our process consists of the Conservative Party, Agrarian Liberal Party, Radical Liberal Party, Center Democratic Party, and Christian People's Party (see Fig. 3). Once the Progressive Party joins, both the Center Democratic Party and the Christian People's Party are superfluous.

Because the spatial array for Denmark,

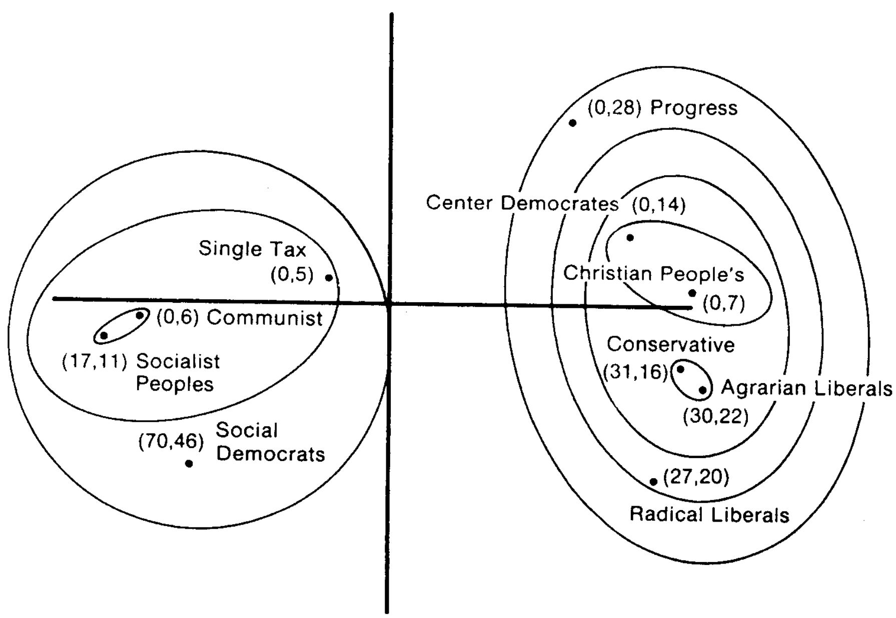


FIG. 3. PERCEIVED PARTY LOCATIONS IN DENMARK, 1973. (Spacial configuration from Rusk and Borre (1974, Fig. 3, p. 341). First entry in each vector represents that party's seats in the 1971 Danish legislature; second entry is for the 1973 legislature. Ellipses represent protocoalition formation stages after the 1973 election.)

TABLE 2
Comparisons of the Predictions of Six Models of Coalition Formation for Danish Cabinet Coalitions in 1971 AND 1973.*

Year	Observed support coalition	Predicted Coalitions: Protocoalition model	Aumann-Maschler (1964), Peleg (1973) bargaining set	Gamson (1961) minimal winning	Leiserson (1970b) fewest actor	Riker (1962) least resource	McKelvey-Ordeshook-Winer (1978) competitive solution
1971	1,5	1,5	1,5 1,2 1,3 1,4	1,5 1,2 1,3 1,4 2,3,4,5	1,5 1,2 1,3 1,4	1,5	1,5 1,6 2,3,4,5
1973	2,3,4,6,7,8	2,3,4,6,7,8	not calculated	2,3,4,6,7 2,3,4,6,8 and sixty other predicted solutions	1,2,6 1,3,4 1,3,6 1,4,6 1,6,7	1,3,4 1,6,7 and nine other predicted solutions	2,3,4,6,7 2,3,4,6,8 and eight other predicted solutions

* Source for spatial configurations: Rusk and Borre (1974, p. 341, Fig. 3).
1= Social Democratic Party, 2 = Conservative Party, 3 = Agrarian Liberal Party, 4 = Radical Liberal Party, 5 = Socialist People's Party, 6 = Progressive Party, 7 = Center Demoncratic Party, 8 = Christian People's Party, 9 = Communist Party, 10 = Single Tax Party.

like that of Norway, is two-dimensional, a straightforward application of Axelrod's (1970) connected coalition model is, of course, impossible, but we can look at N-connectedness as before. The coalitions we predicted were 2-connected. If we look at the projections onto the x-axis, we see that the coalitions we predict were connected in 1-space as well as in 2-space; but this is not true if we look at projections onto the y-axis (see Fig. 3).

Test III: Federal Republic of Germany

Using an unfolding technique developed by Poole (1978), Winer (1979) used data from a 1972 survey of the German electorate to develop a two-dimensional configuration of ideal points for the four German political parties. We show this spatial array in Fig. 4 for data from one wave of the survey. The vector entries in this figure reflect seats in the 1969 and 1972 German elections.

Our model gives rise to identical predictions for 1969 and 1972. The elipses in Fig. 4 represent the protocoalitions which our model predicts will form on the first round of protocoalition. The formation of one of those protocoalitions—that consisting of the Free Democratic Party and the Social Democratic Party—is a winning coalition. Thus, our model predicts that to be the coalition which will form. For both elections, this prediction is confirmed by the data. The coalition we predict is 2-connected. If we look at Fig. 4 in terms of projections onto the axes, we see that the coalitions we predict are connected in 1-space with respect to the x-axis, but not necessarily with respect to the y-axis. This coalition is minimal winning in both 1969 and 1972. In both years there are three other possible minimal winning coalitions.

We show in Table 3 the comparative predictions for these two German cabinets of six different coalition models. Despite the limited number of actors and the rather

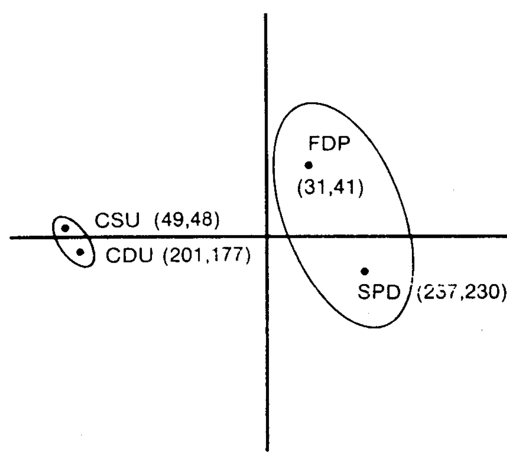


FIG. 4. PERCEIVED PARTY LOCATIONS IN THE FEDERAL REPUBLIC OF GERMANY, 1969. (Spacial configuration from Winer (1979). First entry in each vector represents the party's seats in the 1969 German legislature; second entry is for the 1972 legislature.)

TABLE 3
Comparisons of the Predictions of Six Models of Coalition Formation for the Federal Republic of Germany Cabinet Coalitions in 1969 AND 1972.*

Year	Observed support coalition	Predicted Coalitions: Protocoalition model	Aumann-Maschler (1964), Peleg (1973) bargaining set	Gamson (1961) minimal winning	Leiserson (1970b) fewest actor	Riker (1961) least resource	McKelvey-Ordeshook-Winer (1978) competitive solution
1969	3,4	3,4	3,4	3,4	3,4	3,4	3,4
			1,4	1,4	1,4		2,4
			2,4	2,4	2,4		1,2,3
				1,2,3			
1972	3,4	3,4	3,4	3,4	3,4	1,2,3	3,4
			1,4	1,4	1,4		2,4
			2,4	2,4	2,4		1,2,3
				1,2,3			

* Source for spatial map: Winer (1979); see also Ordeshook and Winer (1980).
1 = Christian Social Union, 2 ;= Christian Democratic Union, 3 = Free Democratic Party, 4 = Social Democratic Party.

simple spatial array, most other models come up with multiple predictions. We do not wish to make too much of our results since in 1972 the Free Democratic Party ran as the incumbent partner of the Social Democratic Party and we are using 1972 voter perceptions to locate the parties. Moreover, even in 1969 the Social Democratic Party–Free Democratic Party coalition might have been to many of the voters to be predetermined. Furthermore, it is misleading in this period to really treat the Christian Democratic Union and Christian Social Union as independent parties. However, this observation still does not invalidate the basic point that *perceived* proximity correlates perfectly with coalitional choices when we use our weighted distance model, i.e., ideological factors and not just coalitional resources or number of actors are important in determining coalitional alignments.

DISCUSSION

The model we have presented has, we believe, a number of nice properties, and some other properties which might be argued to be less desirable. On the negative side (1) our model, like the Cournot duopoly model, involves "rational" behavior only in a quite myopic sense. It is easy to generate situations where, say, in the unidimensional case, a right-centrist actor, by choosing to join in protocoalition with the actor to the right of him, is (it will turn out) in the longer run foreclosing participation in what will go on to become the winning center-left coalition and also rendering impossible the formation of a centrist coalition with itself as pivotal member. (2) Our model does not directly deal with the question of the stability of the (proto)coalitions whose formation it posits. At the heart of the model is the assumption that protocoalitions once formed remain nondissolvable. Thus, we beg the question of whether actors can be tempted out of existing protocoalitions. To this accusation, we have two responses. First, since our model requires that before a (proto)coalition can form both partners must have no other coalitions which they prefer to join, the assumption that protocoalitions act as a bloc in subsequent rounds of the coalition process does not seem all that unreasonable. This is particularly true in those settings involving repeated interactions of a set of actors (such as cabinet negotiations over the course of several elections) where a reputation for keeping one's commitments could be expected to be valuable (cf. Hinckley, 1972).

Second, our model appears to give rise, at least in the unidimensional equal weight case, to outcomes which are stable. Let us look again at the example we first considered.

$A \qquad\qquad BC \qquad D \quad E$

In that example the coalition which emerges is $((B + C) + (D + E))$. If we assume, as above, that the payoffs to each member of a winning coalition are inversely proportional to their distance from the coalition's center of gravity (with side payments not permitted), then no other winning coalition gives rise to a payoff imputation which dominates that for $\{B, C, D, E\}$. Given our previous assumption as to spatial location, it is clear that both C and D would rather be in the $\{B, C, D, E\}$ coalition than in the coalition $\{A, B, C\}$; while D prefers the $\{B, C, D, E\}$ coalition to the $\{B, C, D\}$ coalition; and C prefers the $\{B, C, D, E\}$ coalition to the $\{C, D, E\}$ coalition. Similarly C and D would rather be in that coalition than in an $\{A, B, C, D\}$ coalition, and C, D, and E prefer that coalition to the coalition of the whole. Thus, once the $\{B, C, D, E\}$ coalition forms, even though supraminimal, it is stable. We are not, however, sure how far this example may be generalized.

(3) Our model assumes that protocoalitions locate at their center of gravity. This seems a strong assumption, but one which still avoids the necessity of a median voter result. On the other hand, the bargaining powers of two actors being equal, such a location as the outcome of the *pairwise* bargaining process we have posited does not seem unreasonable, particularly given the myopic nature of the process as we have envisioned it. Moreover, the center of gravity is also within the sets of imputations which are Pareto optimal for that coalition (cf. Hoffman & Plott, 1980, 5).

On the positive side: (1) Our model is dynamic rather than static. (2) Our model is conceptually simple and has a clear axiomatic base. (3) Our model emphasizes the motivations of actors in seeking to join in coalition with those who are closest to them in attitude, and it does not treat actors as interchangeable. As Zais and Kessel (1973, 141) ask: "What motivation does an extreme conservative have to join a very liberal protocoalition and convert it into a winning coalition when he does not approve of what the coalition intends to do?" (4) Our model emphasizes the reciprocity required for a coalition to form—both partners must have no other coalitions which they prefer to join and which it is feasible for them to join. (5) Our model is compatible with a good deal of empirical data on cabinet formation in that it requires connectedness in unidimensional policy spaces. (Connected coalitions seem prevalent in countries such as Italy and Denmark; see Axelrod, 1970; Grofman, 1979.) (6) Our model is adaptable to a wide range of coalition processes. If we identify the center of gravity of the winnning coalition as a policy or choice, then we have a model for predicting the outcome of committee decision making. If we introduce political parties or factional groups as protocoalitions predating the first stage of our process, then our model is applicable to the case of cabinet formation. If we concern ourselves not just with the winnning coalition but also with the other protocoalitions that exist at the last stage of the coalition process, we have a model which can, in principle, account for which justices join the opinion majority and which write separate concurring opinions in the U.S. Supreme Court. Consider, for example, the following situation.

	AFFIRM			*REVERSE*	
A	*BC*	*DE*	*FG*	*H*	*I*

Justices A through G wish to affirm a lower court ruling. Justices H and I wish to reverse it. The justices disagree, however, as to the extent an "absolutist" view of the first amendment should govern the rationale for their ruling. The eventual opinion coalition that will form is $(B + C) + (D + E) + (F + G)$. Justice A might be expected to write a separate but concurring opinion. We neglect "costs" of dissent which may dissuade a justice from writing a separate opinion (see Rohde & Spaeth, 1976). (7) To the extent that operationalization of a distance metric and of actors' weights is possible, our model provides *unique* predictions as to which coalition will form. This is a sharp contrast with other game-theoretic models which usually only serve to narrow the range of feasible coalitions. (8) Our model does not require minimal winning coalitions. However, a look at cabinet coalitions in Denmark (Grofman, 1979) suggests that any supraminimal coalition predicted by this model may well be unstable. In three of the four cases in Denmark where this model predicted a supraminimal coalition, no winning coalition emerged and there was a minority government. (9) Our model does not require centrist coalitions. While the median voter is a member of all winning coalitions, the center of gravity of winning coalitions need not be closest to that of the overall median voter. Thus, we are not a priori limited to predicting a politics of moderation. (10) Our model is one of few in the literature which, like Axelrod's (1970) hypothesis of connected minimal winning coalitions, incorporates both actors' weights and their location in ideological space. On balance, despite some drawbacks to the model, we believe that its positive features outweigh in importance its negative ones, especially given its good fit to cabinet coalition data from the three countries on which we have tested it.

However, we should emphasize that unlike many other researchers in this area, we do not believe that there is one "best" model of coalition formation. Once we begin to look at coalitional *dynamics*, we believe that we shall find that the nature of the decision process will have important consequences for the nature of the expected outcome(s). In particular, in spatial voting games it should make a great deal of difference whether the process is one of (more or less indissoluble) sequential (pairwise) agreements on which *actors* shall join the coalition versus one in which the set of actors considered to comprise the winning coalition is incidental to a public and general decision as to which *point* in N-space will be picked—with those who voted for this point against its rivals then being regarded as the members of the winning coalition. The former process will encourage a coalition which will then be likely to pick a point in its set of Pareto efficient imputation set, probably a "prominent" point, like a centroid. The latter process will be more likely, in our opinion, to give rise to a global optimum, such as the core if one exists. (Some experimental evidence which supports this point of view may be found in Hoffman & Plott, 1980.) The protocoalition dynamic we have proposed seems most appropriate for situations, like cabinet negotiations, which are likely to go on in secret, often involving bilateral negotiations and, we believe, often proceeding sequentially, i.e., key actors (parties) may provisionally "agree" to join the cabinet, and then additional support is solicited to generate sufficient voting strength to form a winning coalition.

In any case, we hope to have shown how a relatively simple dynamic coalition model could, in principle, be used to predict outcomes in a wide range of coalitional situations; and we hope our work will stimulate additional research, both theoretical and empirical, on dynamic models of coalition processes. At minimum, we have shown how Axelrod's model of connected coalitions may be generalized to the N-dimensional case, and our data argue for the view that ideological proximity correlates strongly with coalitional choice, and that simple resource or size models cannot accurately predict coalitional alignmens. The latter is a view which has also strongly been advocated by Budge and Fairley (1977, pp. 158–160), Budge and Valentine (1978), Morgan (1976) and DeSwann and Mokken (1980). In a world where policies matter, cabinet coalition politics is not a zero-sum game.

The chief difficulties with empirical applications of our model are that (1) it is applicable only to coalition formation where ideological proximity is the driving force of coalition dynamics; and (2) it requires data on spatial configurations at an interval level, and, thus, can be applied only in areas where such data can be obtained—or approximated by knowledgeable observers. Nonetheless, in all seven of the cases of cabinet formation we considered, our model predicted perfectly. No other coalition model of those we considered comes close to that predictive power for these data. However, these results must be regarded as only suggestive because of important limitations in the data we used as a test of our model; there is a problem of circularity—parties which have been in coalitions together may be seen as closer together. Unfortunately, a similar problem besets any model which incorporates data on ideological or policy proximity, e.g., the McKelvey, Ordeshook, and Winer (1978) competitive solution model or Axelrod's connected coalition model.

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