Instability of Sunspot Equilibria in Real Business Cycle Models Under Adaptive Learning*

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Abstract

We provide conditions under which a general, reduced-form class of real business cycle (RBC) models has rational expectations equilibria that are both indeterminate and stable under adaptive learning. Indeterminacy of equilibrium allows for the possibility that non-fundamental "sunspot" variable realizations can be used to drive the model, and several researchers have offered calibrated structural models where sunspot shocks play such a role. However, we show that the structural restrictions researchers have adopted lead to reduced-form systems that are always *unstable* under adaptive learning dynamics, thus calling into question the plausibility of these sunspot-driven RBC models.

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1 Introduction

It is now well known that dynamic general equilibrium real business cycle (RBC) models with production externalities and other types of nonconvexities may admit equilibria that are locally non-unique or indeterminate. Some researchers, following the lead of Farmer and Guo (1994), have exploited this possibility to derive models where realizations of non-fundamental "sunspot" variables play a prominent role in driving business cycle fluctuations.[1] One critique of this approach has been that the calibrations of the structural models necessary to obtain indeterminacy are empirically implausible.[2] However, a more recent generation of RBC models with a variety of different nonconvexities has been successful at delivering indeterminate equilibria using empirically plausible calibrations of the structural model. Furthermore, RBC models with indeterminate equilibria in combination with sunspot and demand shocks can explain a variety of features of the macroeconomic data at business cycle frequencies that more traditional RBC models with determinate equilibria and technology (supply) shocks have a difficult time explaining (see, for example, Wen (1998), Benhabib and Wen, (2004)). Consequently, many have come to view these models of sunspot–driven business cycles as quite promising.

A second critique of sunspot equilibria in RBC models with nonconvexities is that these equilibria are unstable under adaptive learning dynamics. Specifically, suppose that agents possess the correct reduced form specification of the model but must learn the true, i.e., rational expectations equilibrium (REE) parameterization of the system using some kind of adaptive inference technique such as recursive least squares. While agents are attempting to learn this parameterization, their forecasts of future endogenous variables will necessarily differ from rational expectations forecasts. The question, then, is whether their adaptive learning process leads them toward or away from the REE. If the learning process leads agents to the REE, that equilibrium is said to be stable under adaptive learning, or "expectationally stable" (E-stability).[3] Otherwise the REE is expectationally unstable. Clearly, stability under adaptive learning provides an important robustness check on the plausibility of REE.

[1]See, e.g. Farmer (1999) or Benhabib and Farmer (1999) for surveys of this literature.

[2]See, e.g. Aiyagari (1995).

[3]See, e.g. Evans and Honkapohja (2001) for an introduction to the stability of REE under adaptive learning. The stability of indeterminate REE under adaptive learning dynamics has been demonstrated, e.g., by Woodford (1990), Duffy (1994) and Evans and Honkapohja (1994), in the context of simple, dynamic nonlinear models, e.g. overlapping generations models. Demonstrating the stability of sunspot equilibria in multivariate, RBC–type models has proved to be more elusive.

Evans and Honkapohja (2001) report numerical calculations showing that the REE of the Farmer-Guo (1994) model is *unstable* under adaptive learning when the model is calibrated according to Farmer and Guo's own parametric specification. Packalén (1999) goes further and shows, numerically, that there exist calibrations of the Farmer-Guo model for which the REE is both indeterminate and E-stable, but these calibrations are far away from those thought to be empirically relevant.[4] Rudanko (2002) explores the stability, under adaptive learning, of the indeterminate REE of the Schmitt-Grohé-Uribe (1997) model. She shows, numerically, that the empirically plausible sunspot equilibria of this model are unstable under adaptive learning. Evans and McGough (2002) also examine the stability under adaptive learning of indeterminate REE in the Schmitt-Grohé-Uribe (1997) model as well as in the Farmer–Guo (1994) and Benhabib and Farmer (1996) models. They identify what they call a "stability puzzle." For a general reduced–form system of equations that includes all three of models as special cases, they can find parameter regions for which the rational expectations equilibrium is both indeterminate and stable under adaptive learning. However, when they restrict attention to versions of the reduced form model consistent with calibrations of the three structural models, they find that the sunspot equilibria are always *unstable* under adaptive learning.

In this paper we offer a resolution to the stability puzzle identified by Evans and McGough (2002). Specifically, we study a general, reduced-form system of equations that encompasses all of the one–sector RBC models that have appeared in the literature. We provide conditions under which the AR(1) REE solution to this system is 1) E–stable, 2) indeterminate, and 3) jointly E-stable and indeterminate. We next show how three RBC models with nonconvexities, which have appeared in the literature on sunspot-driven business cycles – models due to Farmer and Guo (1994), Schmitt-Grohé and Uribe (1997) and Wen (1998) – have reduced forms that map into our general reduced form system. Using the conditions we derived for indeterminate and E–stable REE in the general system, we show analytically how the structural parameter restrictions imposed by these three models prevent REE from being simultaneously indeterminate and stable under adaptive learning behavior. In the appendix, we show how our analysis extends to several alternative AR(1) REE solutions as well as to the class of "common factor" solutions studied by Evans and McGough (2002, 2003). Finally, we discuss the generality of our findings and offer a resolution to the puzzle identified

[4] Packalén (1999) does provide analytic results showing that RBC models with determinate REE are E–stable, but he is unable to provide analytic results in the case of indeterminate REE of RBC models – the case we consider in this paper.

by Evans and McGough. Specifically, we show that it is possible to find parameterizations of the general, reduced-form model that yield indeterminate and E-stable REE. However, as we explain, such parameterizations should be inconsistent with structural model restrictions as confirmed by our analytic findings for the three RBC models with nonconvexities. Thus it appears that there really is no puzzle: the REE of one-sector, RBC models that are appropriately calibrated according to structural model restrictions cannot be simultaneously indeterminate and E-stable.

The results reported in this paper comprise several important contributions to the literature. First, as our literature review reveals, previous research on the stability under adaptive learning, of sunspot equilibria in RBC models have relied on *numerical analyses* to establish findings. For example, Evans and McGough (2002)'s findings are based on a numerical analysis of the equilibria of all three models using the same calibrations adopted by the researchers who developed those models. By contrast, in this paper we provide exact, *analytic conditions* under which the REE of a general, reduced–form RBC model can be both indeterminate and E–stable. To our knowledge this is the first paper to provide such conditions for this "irregular" class of RBC models.[5] Second, using these analytic conditions, we show how structural model restrictions rule out the possibility that the REE of three, one–sector RBC models that have appeared in the literature, can be simultaneously indeterminate and E–stable. Finally, and perhaps most importantly, our finding that the REE of three sunspot-driven RBC models are unstable under adaptive learning dynamics represents a critique of these models that is distinct from a more often heard critique of these models – that indeterminacy of the REE requires empirically implausible calibrations. Our, "instability–under adaptive–learning" critique should cast further doubt on the plausibility of this class of RBC models as descriptors of business cycle phenomena.

2 General conditions for E-stability and Indeterminacy

We begin by presenting a general reduced form system of equations that characterizes equilibria in a variety of different one-sector RBC models. We derive our main findings using this general reduced form. In particular, we provide conditions under which the rational expectations equilibrium is 1) "learnable" or *expectationally stable* (E–stable) under adaptive learning behavior and 2) *indeterminate*, thereby allowing non–fundamental sunspot variable realizations, or "sunspot shocks"

[5] Indeed, Evans and Honkapohja (2001, p. 386) suggested that the absence of analytic results for this class of models was an open area of research.

to drive the business cycle, either in concert with or without fundamental technology shocks.[6] We then combine the two sets of conditions to yield necessary conditions for both E–stability and indeterminacy of equilibrium.

2.1 Reduced form model

The general reduced form of a sunspot-driven RBC model can be characterized by the following system of 2 equations:

$$k_{t+1} = d_k k_t + d_c c_t, \tag{1}$$

$$c_t = b_k E_t k_{t+1} + b_c E_t c_{t+1}, \tag{2}$$

where, for simplicity, we assume there are no fundamental shocks. Here, k_t denotes the capital stock at time t, c_t denotes consumption at time t, and d_k, d_c, b_k and b_c are coefficients. The REE is found by assuming that agents use equations (1–2) to form expectations for future values of capital and consumption. Since we are interested in the stability of this REE under adaptive learning, we instead assume that while agents know the functional form of these equations, they are initially uninformed as to the correct, REE coefficient values for these equations. Specifically, let $y_t' = (k_t, c_t)$, be the vector of endogenous variables and imagine that agents have a perceived law of motion (PLM) of the AR(1) form:

$$y_t = a_0 + a_y y_{t-1} + a_s s_t + \epsilon_t$$

where s_t represents a vector of non-fundamental expectation errors or sunspot variables and ϵ_t is a vector of random variables with 0 mean. We focus on these "general form" AR(1) representations as they are the ones that have been used in the RBC literature. In the appendix we show that our findings in this section also extend to "common-factor" AR(1) representations, studied by Evans and McGough (2002, 2003).

As equation (1) is already consistent with the AR(1) representation –it does not involve expectations of future endogenous variables, so there can be no expectation errors or sunspot shocks – we can assume that agents know the coefficients of equation (1), d_k, d_c.[7] Alternatively, the coefficients

[6] If the REE of the class of RBC models we examine is *determinate* or locally unique, then expectational errors must be a unique function of fundamental technology shocks alone; non–fundamental sunspot shocks cannot matter.

[7] Packalén (1999) and Evans and McGough (2002) make the same simplifying assumption.

of this equation could be learned as well, though this will not change any of our results. Hence the relevant perceived law of motion consists of the single equation for c_t which we write as

$$c_t = a_1 + a_k k_{t-1} + a_c c_{t-1} + a_f f_t + \varepsilon_t, \tag{3}$$

where f_t is the sunspot variable, and ε is a noise variable with 0 mean. This perceived law of motion corresponds to a particular, general form AR(1) solution class where it is assumed that agents cannot observe current consumption and capital, even though capital is predetermined, and so is known at time t. It is possible to relax this timing assumption for capital, as we show in the appendix, without changing any of our results. We also show in the appendix that we can dispense with the constant term, a_1, without changing our results.

2.2 Sunspot REE

Given the PLM (3), agents form expectations (in lieu of rational expectations) as follows:

$$\begin{aligned} E_t c_t &= c_t = a_1 + a_k k_{t-1} + a_c c_{t-1} + a_f f_t && (4)\\ E_t k_t &= d_k k_{t-1} + d_c c_{t-1} && (5)\\ E_t k_{t+1} &= d_k E_t k_t + d_c E_t c_t && (6)\\ E_t c_{t+1} &= a_1 + a_k E_t k_t + a_c E_t c_t && (7) \end{aligned}$$

Substituting (4)–(5) into (6) and (7) and collecting terms, we get

$$\begin{aligned} E_t c_{t+1} &= a_1(1+a_c) + a_k(d_k + a_c)k_{t-1} + (a_c^2 + a_k d_c)c_{t-1} + a_c a_f f_t && (8)\\ E_t k_{t+1} &= a_1 d_c + (d_k^2 + d_c a_k)k_{t-1} + d_c(d_k + a_c)c_{t-1} + d_c a_f f_t && (9) \end{aligned}$$

Finally, substituting (8) and (9) into (2), we get a mapping, T, between the perceived law of motion and the actual law of motion for c_t:

$$c_t = T(a_1) + T(a_k)k_{t-1} + T(a_c)c_{t-1} + T(a_f)f_t + \varepsilon_t, \tag{10}$$

where

$$\begin{aligned} T(a_1) &= a_1[d_c b_k + b_c(1+a_c)] && (11)\\ T(a_k) &= b_k(d_k^2 + d_c a_k) + b_c(a_k d_k + a_c a_k) && (12)\\ T(a_c) &= b_k d_c(d_k + a_c) + b_c(a_c^2 + a_k d_c) && (13)\\ T(a_f) &= a_f(b_k d_c + b_c a_c) && (14) \end{aligned}$$

The actual law of motion (10) together with equation (1) comprise the data generating process for the economy under adaptive learning.

The rational expectations solution where agents condition on sunspots is just a fixed point of this T-mapping under the restriction that $a_f \neq 0$; if $a_f = 0$, sunspots would not enter into agents' expectations. RE solutions can be found by application of the method of undetermined coefficients, i.e., by setting the coefficients in the perceived law of motion (3) equal to their corresponding T-map coefficients in the actual law of motion, (10). Notice that the coefficient a_f appears only in the map $T(a_f)$, so the restriction that $a_f \neq 0$ immediately implies that $b_k d_c + b_c a_c = 1$. Using the latter restriction, application of the method of undetermined coefficients yields the following set of sunspot REE:

$$a_1 = 0, \; a_k = -\frac{b_k d_k}{b_c}, \; a_c = \frac{1 - b_k d_c}{b_c}, \quad \text{with } a_f \text{ indeterminate.} \tag{15}$$

Notice that there is a continuum of such sunspot REE, indexed by different values for a_f.

2.3 Conditions for E-stability

We next examine the stability of these sunspot REE under adaptive learning, using the concept of expectational (E)–stability. Specifically, let a be the vector of coefficients in the perceived law of motion and $T(a)$ be the vector of coefficients in the actual law of motion. A rational expectations solution is said to be expectationally stable, or E-stable if it is locally asymptotically stable under the equation

$$\frac{da}{d\tau} = T(a) - a.$$

That is, if this differential equation, evaluated at the REE values for a, is locally stable. The time variable τ in this equation refers to notional time.[8] Intuitively, we are checking whether the adjustment of the PLM coefficients toward the ALM coefficients is leading agents toward the REE and not away form it, within a small neighborhood of the REE. RE solutions are said to be E-stable if all eigenvalues of $\frac{d(T(a)-a)}{da}$, when evaluated at the solution, have negative real parts.

Turning specifically to the set of sunspot REE, let $a = (a_1, a_k, a_c, a_f)'$ and $T(a) = (T(a_1), T(a_k), T(a_c), T(a_f))'$. The expression for $\frac{d(T(a)-a)}{da}$, evaluated at the REE values, is

[8] It turns out that there is a deep connection between the stability of the RE solution under this differential equation, and the stability of the RE solution under a real-time adaptive learning algorithm such as recursive least squares learning. See Evans and Honkapohja (2001) for details.

given by:

$$\begin{bmatrix} b_c & 0 & 0 & 0 \\ 0 & b_c d_k & -b_k d_k & 0 \\ 0 & b_c d_c & 1 - b_k d_c & 0 \\ 0 & 0 & b_c \overline{a}_f & 0 \end{bmatrix}, \tag{16}$$

where $\overline{a}_f$ is a REE value for a_f. The eigenvalues of this matrix are determined by the equation

$$(b_c - \lambda)(-\lambda)[\lambda^2 + (b_k d_c - b_c d_k - 1)\lambda + b_c d_k] = 0.$$

The first two eigenvalues are given by:

$$\lambda_1 = b_c, \tag{17}$$

$$\lambda_2 = 0. \tag{18}$$

The other two eigenvalues are determined by the quadratic formula

$$\frac{1 - b_k d_c + b_c d_k \pm \sqrt{(1 - b_k d_c + b_c d_k)^2 - 4 b_c d_k}}{2}$$

The necessary conditions for both of these roots to be negative are

$$\lambda_3 \lambda_4 = b_c d_k > 0 \tag{19}$$

$$\lambda_3 + \lambda_4 = 1 - b_k d_c + b_c d_k < 0 \tag{20}$$

The presence of a zero eigenvalue (18) can be problematic in assessing the stability of a system under adaptive learning. The zero eigenvalue is clearly due to the presence of the sunspot variable f_t in the perceived law of motion. As it turns out, the differential equation for a_f is given by

$$\frac{da_f}{d\tau} = a_f(b_k d_c + b_c a_c - 1),$$

which is a separable equation that can be directly integrated as:

$$a_f(\tau) = a_f(0) \exp\left\{ \int_0^\tau (b_k d_c + b_c a_c(u) - 1) du \right\}.$$

So long as $a_c \rightarrow \frac{1 - b_k d_c}{b_c}$ exponentially as $\tau \rightarrow +\infty$, a_f will also converge to a finite value, so the zero eigenvalue will not hinder our analysis of the stability of the system under adaptive learning. We therefore choose to ignore it and state the necessary conditions for E–stability as follows.

Proposition 1 *The necessary conditions for the system (1) and (2) to be E-stable are (19), (20) and*

$$b_c < 0. \tag{21}$$

Condition (21) in particular will play a critical role in our subsequent analysis.

2.4 Conditions for indeterminacy

Indeterminacy refers to local nonuniqueness of the solution paths leading to a RE solution. This local nonuniqueness is what allows nonfundamental sunspot shocks to enter into expectations and thus play a role in driving the business cycle; if the REE was instead determinate, expectations would have to be a unique function of fundamental variables only, and sunspot shocks would play no role. Hence, the importance of establishing conditions under which the REE is indeterminate. Note that indeterminacy of the RE solution path is quite distinct from stability of the REE under adaptive learning dynamics.

To assess whether the REE is indeterminate, we begin by imposing the rational expectations assumption that $E_t y_{t+1} = y_{t+1}$ and rewriting the general reduced form system (1-2) as:

$$\begin{bmatrix} d_k & d_c \\ 0 & 1 \end{bmatrix} \begin{bmatrix} k_t \\ c_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b_k & b_c \end{bmatrix} \left(\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_{t+1} \end{bmatrix} \right)$$

where $\varepsilon_{t+1} = c_{t+1} - E_t c_{t+1}$, is the forecast error (or sunspot variable). This RE system can be rewritten as:

$$\begin{bmatrix} k_{t+1} \\ c_{t+1} \end{bmatrix} = J \begin{bmatrix} k_t \\ c_t \end{bmatrix} + R\varepsilon_{t+1}, \tag{22}$$

where

$$J = \begin{bmatrix} d_k & d_c \\ -\frac{b_k d_k}{b_c} & \frac{1-b_k d_c}{b_c} \end{bmatrix} \text{and } R = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

The determinant and trace of the Jacobian, J can be obtained as

$$\begin{aligned} \det(J) &= \frac{d_k}{b_c}, \\ tr(J) &= \frac{1 - b_k d_c + d_k b_c}{b_c}. \end{aligned}$$

Indeterminacy of equilibrium in this model requires that both eigenvalues of J lie inside the unit circle.[9] Since the trace of the Jacobian measures the sum of the roots and the determinant measures the product, the necessary conditions for indeterminacy are

$$-1 < \det(J) < 1, \tag{23}$$

$$-1 - \det(J) < tr(J) < 1 + \det(J). \tag{24}$$

[9] See, e.g. Farmer (1999) for the general conditions necessary for equilibrium to be indeterminate in systems such as (1–2). For the class of one–sector RBC models examined here with one predetermined variable (capital) and one non-predetermined variable (consumption), indeterminacy of REE obtains if the dynamical system (22) is a sink. By contrast, determinacy of REE requires one eigenvalue to lie outside and one to lie inside the unit circle so that the dynamical system is a saddle.

Proposition 2 *The necessary conditions for the system (1–2) to have stationary sunspot equilibria are:*

$$-1 < \det(J) = \frac{d_k}{b_c} < 1, \tag{25}$$

$$-1 - \det(J) < tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c} < 1 + \det(J). \tag{26}$$

2.5 Necessary conditions for both E-stability and indeterminacy

If the reduced form model has indeterminate equilibria that are also E-stable, conditions (19) - (21), (25) and (26) must be satisfied simultaneously. Consider the condition (19). This condition implies that $\det(J) = d_k/b_c > 0$. Combining this condition with (25) we find that the determinant of the Jacobian must satisfy

$$0 < \det(J) = \frac{d_k}{b_c} < 1. \tag{27}$$

Furthermore, (21) and (27) together imply that

$$d_k < 0.$$

Similarly, the combination of (20), (21) and (26) requires that

$$0 < tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c} < 1 + \det(J). \tag{28}$$

Proposition 3 *The E-stability requirement imposes further restrictions on the coefficients of the sunspot model (RBC model with indeterminate equilibria). It requires that conditions (21), (27), and (28) hold simultaneously. In particular, (21) states that*

$$b_c < 0.$$

Note that a positive determinant implies that both roots of J have the same sign, and a positive trace implies that the sign of the roots is positive. Hence we have the following corollary.

Corollary 1 *The necessary conditions for the stationary sunspot equilibria in (1–2) to be E-stable are that both roots of the Jacobian matrix (22) have positive real parts* and $b_c < 0$*.*

Let us consider first, the requirement in Corollary 1 that both roots of the Jacobian matrix (22) have positive real parts. This condition can be viewed as a restriction to systems that display empirically plausible adjustment dynamics. Specifically, we will find it useful to work with the following definition.

Definition 1 *The adjustment dynamics of the RE system (22) are empirically plausible if the eigenvalues of the Jacobian matrix J are positive, or in the case of complex eigenvalues, if the real part of the eigenvalues is positive.*

By empirically plausible, we mean that there is no *period-by-period* oscillatory convergence to the steady state as would occur if one or both of the two eigenvalues of J were negative. Instead, the dynamics should involve monotonic or slow cyclic adjustment to the steady state as would occur if both eigenvalues were positive or if the real part of a complex pair of eigenvalues were positive. The former adjustment dynamics are typical in standard RBC models (with determinate REE) while the latter appear in RBC models with indeterminate REE and are in line with observed impulse responses in aggregate data for output, investment and consumption, (see, e.g., Farmer 1999, section 7.6.2). Hence, we label such systems as having "empirically plausible adjustment dynamics." If both eigenvalues are positive or have positive real parts when they are complex, their sum, the $tr(J)$, must be positive, and they must have the same sign so that their product, $\det(J)$, is positive. Interestingly, these two conditions are already contained in (27) and (28). As we show in the appendix, a class of "common factor" REE solutions to the RBC model, examined by Evans and McGough (2002), does not satisfy Definition 1.

Consider next the requirement in Corollary 1 that $b_c < 0$. This restriction *is* consistent with condition (27), i.e. $\det(J) > 0$, so long as $d_k < 0$, and satisfaction of condition (28), i.e. $tr(J) > 0$, further requires that $1 - b_k d_c + d_k b_c < 0$. Therefore, it *is possible* for REE to be both indeterminate and E–stable (satisfying the necessary conditions in Corollary 1) and having adjustment dynamics that satisfy Definition 1. Thus, Evans and McGough (2002) are correct in their claim, based on a numerical analysis, that there exist parameterizations of the general, reduced form model (1–2) that give rise to E-stable, indeterminate, "general–form" REE.

On the other hand, Corollary 1 rules out another parameterization of the reduced form system that would satisfy Definition 1, specifically, $b_c > 0$ and $d_k > 0$, along with the further (positive trace) restriction that $1 - b_k d_c + d_k b_c > 0$. As we shall see, it is the latter parameterization of the reduced form system that emerges from structural model restrictions imposed by researchers working with sunspot-driven RBC models. Hence, the parameterizations of the reduced-form system that Evans and McGough study numerically are not ones that are consistent with structural model restrictions, and therein lies our resolution to their instability puzzle.

Indeed, in the next three sections, we show, analytically, that in the three leading sunspot–driven RBC models, model restrictions always imply that $b_c > 0$. Specifically, we first show how linearized versions of these models map into the reduced form system (1-2) that we examined in this section. We then demonstrate *analytically* that under the parameter restrictions placed on the structural models, the REE cannot be simultaneously indeterminate and stable under adaptive learning. We then discuss our resolution of the puzzle identified by Evans and McGough more generally in section 6.

3 The Wen (1998) model

The economy in Wen's (1998) model consists of a large number of identical consumer-producer households who solve:

$$\max_{\{c_t, n_t, k_t, u_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \left(\log c_t - \frac{n_t^{1+\gamma}}{1+\gamma} \right) \tag{29}$$

subject to:

$$c_t + x_t = \overline{e}_t \left(u_t k_t^{\alpha} \right) n_t^{1-\alpha} \tag{30}$$

$$k_{t+1} = x_t + (1 - \delta_t) k_t \tag{31}$$

$$\overline{e}_t = (\overline{u}_t \overline{k}_t)^{\alpha\eta} \overline{n}_t^{(1-\alpha)\eta} \tag{32}$$

$$\delta_t = \frac{1}{\theta} u_t^{\theta} \tag{33}$$

for a given initial stock of capital, $k_0 > 0$. We adopt Wen's (1998) notation. The choice variables are consumption, c_t, the number of hours worked, n_t, the capital stock, k_t, and the rate of capacity utilization, $u_t \in (0, 1)$. The restrictions on the parameters of the structural model are: $0 < \alpha < 1$, $0 < \beta < 1$, $\gamma \geq 0$, $\eta > 0$, and $\theta > 1$. The production externality, $\overline{e}_t$, is a function of the mean productive capacity, $\overline{u}_t \overline{k}_t$, and mean labor hours, $\overline{n}_t$. The rate of depreciation of the capital stock, $\delta_t \in (0, 1)$, is an increasing function of the capacity utilization rate, u_t. The restriction that $\theta > 1$ ensures that the optimal capacity utilization rate, u_t, lies in the interval $(0, 1)$. The restriction that $\eta > 1$ ensures increasing returns to scale in production which in important both for generating indeterminacy and for allowing capacity utilization to affect the extent of aggregate returns to scale. Indeed, Wen (1998) showed that the addition of variable capital utilization could significantly reduce the degree of increasing returns to scale needed to deliver indeterminate equilibria (e.g. by comparison with Farmer and Guo (1994)), from empirically implausible to empirically plausible levels. This feature of the Wen model has made it an attractive choice for other researchers interested in empirical applications of sunspot-driven RBC models e.g. Harrison and Weder (2002) use the Wen model with sunspot shocks to explain a number of features of the data found over the Great Depression era. Benhabib and Wen (2004) use the Wen model to show how shocks to aggregate demand can explain a number of business cycle anomalies that have eluded standard RBC models (without indeterminate equilibria).

Following Wen (1998), we can first solve for the optimal capacity utilization rate, u_t, and using this expression, derive a reduced–form aggregate production function of the form:

$$y_t = k_t^{a^*} n_t^{b^*},$$

where $a^* = \alpha(1+\eta)\tau_k$, $b^* = (1-\alpha)(1+\eta)\tau_n$, and $\tau_k = \frac{\theta-1}{\theta-\alpha(1+\eta)}, \tau_n = \frac{\theta}{\theta-\alpha(1+\eta)}$.

3.1 The reduced form

The Wen (1998) model can be linearized and written as a reduced form system of 2 equations:

$$\begin{bmatrix} k_{t+1} \\ n_{t+1} \end{bmatrix} = J \begin{bmatrix} k_t \\ n_t \end{bmatrix} + R\varepsilon_t, \tag{34}$$

where

$$J = \begin{bmatrix} 1 & (1+\gamma)c/k \\ \frac{(1-\beta)(1-a^*)}{1+\gamma-\beta b^*} & \frac{1+\gamma-b^*+[1+\beta(a^*-1)](1+\gamma)c/k}{1+\gamma-\beta b^*} \end{bmatrix}.$$

3.2 Requirements for indeterminacy

For this model to have multiple stationary sunspot equilibria, the conditions (23) and (24) must be satisfied. After lengthy algebra, one can show that

$$\begin{aligned} \det(J) &= \frac{1+\gamma-b^*+a^*(1+\gamma)c/k}{1+\gamma-\beta b^*} && (35)\\ &= \frac{1}{\beta}\left[1+\frac{\eta(1+\gamma)(1-\beta)\tau_n}{1+\gamma-\beta b^*}\right], && (36)\\ tr(J) &= 1+\det(J)+\frac{(1+\gamma)(1-\beta)(1-a^*)c/k}{1+\gamma-\beta b^*}. && (37) \end{aligned}$$

The crucial insight of (36) is that when there is no externality ($\eta = 0$), $\det(J) = 1/\beta > 1$, which violates the condition (23). For there to be indeterminacy, therefore, the second term of (36) must become negative as the externality becomes positive. Since $\eta(1+\gamma)(1-\beta)\tau_n > 0$, this requires that the denominator be negative:

$$1+\gamma-\beta b^* < 0. \tag{38}$$

Proposition 4 *A necessary condition for indeterminacy of REE in the Wen (1998) model is (38).*

3.3 E-instability

To check the conditions for E-stability, we need to convert the system (34) into the form of (1–2). After this is done, the mapping from the parameters of the model to those of (1–2) is given by:

$$b_c = \frac{\beta b^* - (1+\gamma)}{b^* - (1+\gamma)}, \tag{39}$$

$$b_k = \frac{(\beta - 1)[b^* - (1+\gamma)(1-a^*)]}{b^* - (1+\gamma)}, \tag{40}$$

$$d_k = 1 - \frac{a^*(1+\gamma)c/k}{b^* - (1+\gamma)}, \tag{41}$$

$$d_c = \frac{(1+\gamma)c/k}{b^* - (1+\gamma)}. \tag{42}$$

We can now use the conditions derived in the previous section to examine if the REE of this model is E-stable. Consider first equation (39). In Proposition 4 we have shown that $\beta b^* - (1+\gamma) > 0$. Since $0 < \beta < 1$, this implies that $b^* - (1+\gamma) > 0$. It is immediately evident that

$$b_c = \frac{\beta b^* - (1+\gamma)}{b^* - (1+\gamma)} > 0,$$

which exactly violates the required condition for E-stability (21).

Proposition 5 *The REE of the Wen (1998) model cannot be both indeterminate and E-stable since condition (21) is always violated when the REE is indeterminate.*

4 The Farmer and Guo (1994) model

In Farmer and Guo's (1994) model a large number of identical consumer-producer households solve:

$$\max_{C_t, L_t} E_0 \sum_{t=0}^{\infty} \rho^t \left(\log C_t - A \frac{L_t^{1-\gamma}}{1-\gamma} \right)$$

subject to:

$$\begin{aligned} K_{t+1} &\leq Y_t + (1-\delta)K_t - C_t \\ Y_t &= Z_t K_t^{\alpha} L_t^{\beta} \\ Z_t &= Z_{t-1}^{\theta} \eta_t \end{aligned}$$

Here we are using the same notation as in Farmer and Guo (1994): C_t denotes consumption, L_t denotes labor supply, K_t is the capital stock, Y_t is output, Z_t is a productivity shock and η_t is an i.i.d. random variable with unit mean. The structural model parameter restrictions are: $\gamma < 0$, $0 < \rho < 1$, $0 < \delta < 1$, $0 < \theta < 1$ and, most importantly, $\alpha + \beta > 1$, so that the technology exhibits increasing returns.[10] However, from the perspective of individual producers, the production technology is Cobb-Douglas with constant returns, where a and b represent capital and labor's shares of output, respectively, and $a + b = 1$. Farmer and Guo assume that

$$\alpha = a/\lambda, \quad \beta = b/\lambda \tag{43}$$

with $0 < \lambda < 1$ to insure increasing returns to scale.

4.1 The reduced form

Omitting fundamental shocks, Z_t, the model can be reduced to:

$$\begin{aligned}
Y_t &= K_t^{\alpha} L_t^{\beta}, && (44)\\
AC_t/L_t^{\gamma} &= bY_t/L_t, && (45)\\
K_{t+1} &= Y_t + (1-\delta)K_t - C_t, && (46)\\
\frac{1}{C_t} &= \rho E_t[\frac{1}{C_{t+1}}(a\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta)], && (47)
\end{aligned}$$

where equations (45) and (47) are the first order conditions from the representative agent's problem.

The two dynamic equations can be linearized as

$$\begin{aligned}
c_t &= E_t c_{t+1} + \rho\frac{y}{k}(E_t k_{t+1} - E_t y_{t+1}), && (48)\\
k_{t+1} &= \frac{y}{k} y_t + (1-\delta)k_t - \frac{c}{k} c_t. && (49)
\end{aligned}$$

Combining the linearized versions of (44) and (45),

$$\begin{aligned}
y_t &= \alpha k_t + \beta l_t,\\
c_t + (1-\gamma) l_t &= y_t,
\end{aligned}$$

we obtain:

$$y_t = \frac{\beta}{\beta - 1 + \gamma} c_t - \frac{\alpha(1-\gamma)}{\beta - 1 + \gamma} k_t.$$

[10] An alternative interpretation of the latter restriction involving monopolistically competitive firms is also possible – see Farmer and Guo (1994) for the details.

Substituting this equation into the two dynamic equations (48-49), we get

$$\begin{aligned} c_t &= (1+\rho a\frac{y}{k}\frac{\beta}{1-\beta-\gamma})E_t c_{t+1} + \rho a\frac{y}{k}[1-\frac{\alpha(1-\gamma)}{1-\beta-\gamma}]E_t k_{t+1}, \\ k_{t+1} &= [\frac{y}{k}\frac{\alpha(1-\gamma)}{1-\gamma-\beta}+1-\delta]k_t - (\frac{c}{k}+\frac{y}{k}\frac{\beta}{1-\gamma-\beta})c_t. \end{aligned}$$

Mapping this system into our general reduced form representation, (1–2), the first critical coefficient is

$$\begin{aligned} b_c &= 1+\rho a\frac{y}{k}\frac{\beta}{1-\beta-\gamma}, \\ &= \frac{1-\gamma-\beta\rho(1-\delta)}{1-\beta-\gamma}, \end{aligned}$$

where the second equality comes from the steady state version of (47): $\rho a\frac{y}{k} = 1-\rho(1-\delta)$. The second critical coefficient is

$$\begin{aligned} d_k &= \frac{y}{k}\frac{\alpha(1-\gamma)}{1-\gamma-\beta}+1-\delta, \\ &= \frac{\frac{1-\rho(1-\delta)}{\rho a}\alpha(1-\gamma)+(1-\delta)(1-\gamma-\beta)}{1-\gamma-\beta}, \end{aligned}$$

where the second equality again comes from using (47).

4.2 Requirements for indeterminacy

We only need a subset of the necessary conditions for indeterminacy to make our point. As we proved in section 2.4, one necessary condition for indeterminacy is that

$$-1 < \frac{d_k}{b_c} < 1.$$

In the Farmer–Guo model we have

$$\frac{d_k}{b_c} = \frac{\frac{1-\rho(1-\delta)}{\rho a}\alpha(1-\gamma)+(1-\delta)(1-\gamma-\beta)}{1-\gamma-\beta\rho(1-\delta)}.$$

Substituting equation (43) into (4.2) and simplifying terms, we get

$$\begin{aligned} \frac{d_k}{b_c} &= \frac{\frac{1}{\rho}\{\frac{1}{\lambda}(1-\gamma)[1-\rho(1-\delta)]+\rho(1-\delta)(1-\gamma-\frac{b}{\lambda})\}}{1-\gamma-\frac{b}{\lambda}(1-\delta)\rho}, \\ &= \frac{1}{\rho}\left\{1+\frac{(1-\frac{1}{\lambda})(1-\gamma)[\rho(1-\delta)-1]}{1-\gamma-\frac{b}{\lambda}(1-\delta)\rho}\right\}. \end{aligned} \tag{50}$$

Suppose the economy has constant returns to scale ($\lambda = 1$). In that case equation (50) indicates that $\frac{d_k}{b_c} = 1/\rho > 1$, since the discount factor ρ must be less than 1. Of course in the case of constant returns, the condition for indeterminacy is violated. For the equilibrium to be indeterminate, we need increasing returns ($\lambda < 1$), and we further require that the second term in the bracket of expression (50) must be negative. Setting $\lambda < 1$, it is easy to verify that the numerator of this second term is positive, given that $1 - 1/\lambda < 0$, $\gamma < 0$, and $\rho(1-\delta) - 1 = -\rho ay/k < 0$. Therefore, for indeterminacy to obtain, the denominator of this second term must be negative, that is, we must have that

$$\beta\rho(1-\delta) > 1 - \gamma, \tag{51}$$

where we have made use of the definition of β given in (scale). Since $0 < \rho < 0$ and $0 < 1 - \delta < 1$, it further follows that:

$$\beta > 1 - \gamma. \tag{52}$$

Proposition 6 *A necessary condition for indeterminacy of REE in the Farmer and Guo (1994) model is (52).*

4.3 E-instability

We can now show that if the Farmer and Guo (1994) model satisfies the above necessary condition for indeterminacy, then the REE of that model must be E-unstable under adaptive learning. We only have to check the necessary condition

$$b_c < 0.$$

In this model,

$$b_c = \frac{\beta\rho(1-\delta) - (1-\gamma)}{\beta - (1-\gamma)}.$$

If condition (52) holds, the numerator must be positive. Combining condition (51), we have

$$\begin{aligned} b_c &= \frac{\beta\rho(1-\delta) - (1-\gamma)}{\beta - (1-\gamma)} \\ &> \frac{1-\gamma-(1-\gamma)}{\beta-(1-\gamma)} = 0 \end{aligned}$$

It follows that the equilibrium of this model is E-unstable under adaptive learning.

Proposition 7 *The REE of the Farmer–Guo (1994) model cannot be both indeterminate and E-stable since condition (21) is always violated when the REE is indeterminate.*

5 The Schmitt-Grohé and Uribe (1997) model

We focus on the simpler version of Schmitt-Grohé and Uribe's (1997) model where there is no capital income tax. We consider a discrete-time version of the model with labor income taxes only and adopt Schmitt-Grohé and Uribe's (1997) notation. A large number of identical consumer-producer households solve:

$$\max_{C_t, H_t} E_0 \sum_{t=0}^{\infty} \beta^t (\log C_t - AH_t)$$

subject to:

$$\begin{aligned} K_{t+1} &\leq Y_t + (1-\delta)K_t - C_t - G \\ Y_t &= K_t^a L_t^b \\ G &= \tau_t b Y_t \end{aligned}$$

Here, C_t denotes consumption, H_t is hours worked, K_t is the capital stock, and Y_t is output produced according to a Cobb-Douglas technology with constant returns, $a + b = 1$. Government revenue, G, is obtained through taxes on labor income at rate $\tau_t \in (0, 1)$. The discount factor satisfies $0 < \beta < 1$, as does the rate of depreciation of the capital stock, $0 < \delta < 1$, and we assume $A > 0$.

5.1 The reduced form

This model can be reduced to

$$Y_t = K_t^a H_t^b \tag{53}$$

$$AC_t = b(1-\tau_t)Y_t/H_t \tag{54}$$

$$K_{t+1} = Y_t + (1-\delta)K_t - C_t - G \tag{55}$$

$$\frac{1}{C_t} = \beta E_t[\frac{1}{C_{t+1}}(a\frac{Y_{t+1}}{K_{t+1}} + 1 - \delta)] \tag{56}$$

$$G = \tau_t b Y_t, \tag{57}$$

Equations (54) and (56) are the first order conditions from the representative agent's problem. Let lower case letters denote deviations from steady state values. We can eliminate y_t and τ_t by using the linearized version of (53), (54) and (57):

$$y_t = ak_t + bh_t$$

$$
\begin{aligned}
c_t &= y_t - h_t - \frac{\tau}{1-\tau}\tau_t \\
0 &= \tau_t + y_t,
\end{aligned}
$$

Combining these equations we obtain

$$
y_t = \frac{(1-\tau)b}{b-1+\tau}c_t - \frac{(1-\tau)a}{b-1+\tau}k_t.
$$

Substituting the equation above into the two linearized dynamic equations, we get

$$
\begin{aligned}
c_t &= [1 - \beta a \frac{y}{k}\frac{(1-\tau)b}{b-1+\tau}]E_t c_{t+1} + a\beta\frac{y}{k}[1 + \frac{(1-\tau)a}{b-1+\tau}]E_t k_{t+1}, \\
k_{t+1} &= (\frac{y}{k}\frac{b(1-\tau)}{b-1+\tau} - \frac{c}{k})c_t + [\frac{y}{k}\frac{(1-\tau)a}{1-b-\tau} + 1 - \delta]k_t.
\end{aligned}
$$

Again, the two critical coefficients are:

$$
\begin{aligned}
b_c &= 1 - \beta a \frac{y}{k}\frac{(1-\tau)b}{b-1+\tau} \\
&= \frac{b-1+\tau-[1-\beta(1-\delta)]b(1-\tau)}{b-1+\tau}, \\
d_k &= \frac{y}{k}\frac{(1-\tau)a}{1-b-\tau} + 1 - \delta \\
&= \frac{[1/\beta - (1-\delta)](1-\tau) + (1-\delta)(1-b-\tau)}{1-b-\tau},
\end{aligned}
$$

where the second equality in these expressions follows from the steady state value of equation (57): $\beta a \frac{y}{k} = 1 - \beta(1-\delta)$.

5.2 Requirements for indeterminacy

It is again sufficient for our purposes to check the indeterminacy condition:

$$
-1 < \frac{d_k}{b_c} < 1.
$$

In this model we have:

$$
\begin{aligned}
\frac{d_k}{b_c} &= \frac{1/\beta\{[1-\beta(1-\delta)](\tau-1) + \beta(1-\delta)(\tau+b-1)\}}{b-1+\tau-[1-\beta(1-\delta)]b(1-\tau)}, \\
&= \frac{1}{\beta}\left\{1 + \frac{-\tau b[1-\beta(1-\delta)]}{b-1+\tau-[1-\beta(1-\delta)]b(1-\tau)}\right\}. \qquad (58)
\end{aligned}
$$

When there is no labor income tax ($\tau = 0$), expression (58) is equal to $1/\beta > 1$, and the equilibrium is always determinate. To have indeterminacy, we require both that $\tau > 0$, and that the second term in the bracketed part of expression (58) is negative. It is obvious that the numerator of this second term is negative, so the denominator must be positive to have indeterminacy. That is we must have that

$$b - 1 + \tau > b(1-\tau)[1-\beta(1-\delta)]. \tag{59}$$

Since the right-hand-side of (59) is positive, it further follows that we must have

$$b - 1 + \tau > 0. \tag{60}$$

Proposition 8 *A necessary condition for indeterminacy of REE in the Schmitt-Grohé and Uribe (1997) model is (60).*

5.3 E-instability

Next we show that when indeterminacy holds, the condition for E-stability $b_c < 0$ will be violated. Combining the expression for b_c with (59) and (60), we have

$$\begin{aligned} b_c &= 1 - \frac{b(1-\tau)[1-\beta(1-\delta)]}{b-1+\tau} \\ &> 1 - \frac{b-1+\tau}{b-1+\tau} = 0. \end{aligned}$$

This equilibrium of this model is therefore E-unstable under adaptive learning.

Proposition 9 *The REE of the Schmitt-Grohé and Uribe (1997) model cannot be both indeterminate and E-stable as condition (21) is always violated when the REE is indeterminate.*

6 Discussion

While we have applied our conditions for indeterminacy and stability under learning to just three RBC models, we believe that our instability conclusion is even more general. In all three RBC models that we consider, the necessary condition for E-stability, $b_c < 0$, is always violated. We believe this is not a coincidence. All structural models that are calibrated to match empirical facts should violate this condition. To see why, recall that $b_c < 0$ together with the necessary condition (27). implies that $d_k < 0$.

However, in this case, two of the four coefficients in the general, reduced–form system (1)–(2), which we reproduce below, are negative:

$$k_{t+1} = d_k k_t + d_c c_t$$

$$c_t = b_k E_t k_{t+1} + b_c E_t c_{t+1}$$

A negative coefficient for d_k yields the counterfactual implication that capital stocks are *negatively autocorrelated*. Similarly, a negative coefficient for b_c yields the counterfactual implication that the representative household *decreases* current consumption whenever future consumption is expected to rise. Such behavior is clearly inconsistent with the notion of consumption "smoothing" in dynamic economies. Neither type of behavior is likely to emerge from structural model restrictions, indeed such restrictions should lead to the opposite case where $b_c > 0$ and $d_k > 0$. The latter parameterization would, *in addition*, yield adjustment dynamics (impulse responses) that are consistent with our Definition 1. However, under this parameterization of the reduced form system, the REE would always be E–unstable! Hence, in any appropriately calibrated sunspot-driven RBC model, we expect that our findings will continue to apply, that is, the equilibrium of such models cannot be jointly indeterminate and stable under adaptive learning.

7 Conclusions

In this paper we have examined the conditions for indeterminacy and stability under adaptive learning for a general reduced form model that characterizes a number of linearized, one–sector real business cycle models. We have provided simple, analytic conditions under which the equilibrium of this system is both stable under adaptive learning behavior and indeterminate, so that sunspot shocks can play a role in driving the business cycle. To our knowledge, such conditions have not previously appeared in the literature. These conditions imply that, in principle, it is possible for agents to learn the REE of sunspot-driven RBC models. However, we also find that for three sunspot-driven RBC models that have appeared in the literature, structural model parameter restrictions imply that the REE of these models cannot be simultaneously indeterminate and E-stable under adaptive learning. This finding comprises a critique of these models that is distinct from a more often-heard critique of these models – that the calibrations of the structural models necessary to obtain *indeterminate* REE are empirically implausible. Our critique is instead that agents would never learn to implement such REE.

Our analysis provides a resolution of the puzzle identified by Evans and McGough (2002). REE of one-sector models with nonconvexities *can* be both indeterminate and E–stable, but only if the laws of motion for consumption and the capital stock are parameterized in a way that is *inconsistent* with structural model restrictions, or leads to empirically implausible adjustment dynamics. In the appendix, we show that a similar finding extends to a class of common-factor REE solutions studied by Evans and McGough (2002). In this sense, our results are consistent with and reminiscent of McCallum's (2002) finding that linear models with unique minimal state variable RE solutions (solutions that exclude extraneous state variables, e.g., sunspot variables) are always E–stable in any "well–formulated" model; by well–formulated, McCallum means restrictions on the parameter values of the system that avoid economically implausible dynamics.

Our findings would appear to cast serious doubt on the research agenda that seeks to use sunspot-driven RBC models to explain business cycle phenomena. However, we caution that our findings are limited to RBC models. Other researchers have shown that sunspot REE may be E-stable in monetary models with infinitely-lived agents (e.g., Evans et al. (2002)) or with overlapping generations of finitely lived agents (Woodford (1990), Duffy (1994) and Evans and Honkapohja (1994)). Further, our E-instability findings only apply to one-sector RBC models, and we have precise analytical results for only three such models that have appeared in the literature. Recently, researchers have shown that indeterminacy of equilibria is readily obtained in multi-sector RBC models under empirically relevant calibrations. Hence, it may not be the case that our findings extend to such sunspot-driven RBC models. We leave an analysis of the stability under learning of indeterminate equilibria in these multi-sector models to future research.

Appendix

In this appendix we consider the robustness of our findings in section 2 to two different assumptions regarding the perceived law of motion (3) and to a "common factor" representation of the REE solution.

Alternative timing assumption

As noted in section 2.1, the perceived law of motion (3) assumes that k_t is not known at time t when in fact it is predetermined by decisions made in period $t-1$. Assume, therefore, that agents use the alternative perceived law of motion,

$$c_t = a_1 + a_k k_t + a_c c_{t-1} + a_f f_t + \varepsilon_t, \tag{61}$$

in place of (3). Following the same steps as outlined above, one can show that if agents use (61) as their perceived law of motion, the actual law of motion will be given by:

$$c_t = T(a_1) + T(a_k)k_t + T(a_c)c_{t-1} + T(a_f)f_t + \varepsilon_t,$$

where

$$\begin{aligned} T(a_1) &= a_1[d_c b_k + b_c(1 + a_k d_c + a_c)], \\ T(a_k) &= b_k d_k + b_k d_c a_k + b_c a_k(d_k + a_k d_c + a_c), \\ T(a_c) &= b_k d_c a_c + b_c a_c(a_c + a_k d_c), \\ T(a_f) &= a_f[b_k d_c + b_c(a_k d_c + a_c)]. \end{aligned}$$

The matrix $\frac{d(T(a)-a)}{da}$, evaluated at this REE solution, can be written as:

$$\begin{bmatrix} b_c & 0 & 0 & 0 \\ 0 & b_c d_k - b_k d_c & -b_k & 0 \\ 0 & d_c & 1 & 0 \\ 0 & -b_c d_c \overline{a}_f & b_c \overline{a}_f & 0 \end{bmatrix}, \tag{62}$$

where $\overline{a}_f$ is again the REE value of a_f (c.f. (16)). While (62) differs from (16), the eigenvalues of (62) are found by solving the characteristic equation

$$\lambda(b_c - \lambda)[\lambda^2 + (b_k d_c - b_c d_k - 1)\lambda + b_c d_k] = 0,$$

which is the same characteristic equation obtained for matrix (16). It follows that Proposition 1 also holds in the case where agents use the alternative perceived law of motion (61) in place of (3).

Eliminating the constant term

A closer look at the matrix (16) (or 62) reveals that the eigenvalue b_c comes solely from the T-mapping for the coefficient a_1, the constant term in the agent's perceived law of motion. Since the variables in the reduced form model are all expressed as deviations from steady state values, it is not necessary to incorporate a constant term in the PLM. Indeed, for this reason, the reduced form equations of RBC models typically do not involve constant terms. However, if the variables of the model were not expressed in terms of deviations from steady state values, learning agents would need to include constant terms in their perceived laws of motion; the value of these intercept terms would depend on the steady state values. Indeed, one could argue that learning agents might not initially know steady state values of model variables, so it is more reasonable to assume that they use perceived laws of motion with constant terms included. Alternatively, the presence of the constant term could be regarded as a slight model misspecification; agents could, after all, learn that the coefficient on this constant term is zero in the rational expectations equilibrium.

Nevertheless, suppose that we eliminate the constant term. Under our original timing assumption for k, the PLM (3) now becomes

$$c_t = a_k k_{t-1} + a_c c_{t-1} + a_f f_t + \varepsilon_t.$$

With this modification, the necessary conditions for indeterminacy and E-stability will be reduced to (27) and (28) only. In this case b_c can be either positive or negative. If $b_c < 0$, the necessary conditions will be exactly the same as stated in Proposition 3. We therefore focus on the case where $b_c > 0$.

Suppose that $b_c > 0$. It follows immediately, from the necessary condition (20) and the definition of $tr(J)$ that we must have

$$tr(J) = \frac{1 - b_k d_c + d_k b_c}{b_c} < 0. \tag{63}$$

From the other necessary condition, (19) and the definition of $\det(J)$ we must also have

$$\det(J) = \frac{d_k}{b_c} > 0. \tag{64}$$

A negative trace and a positive determinant imply that both eigenvalues of J have *negative real parts*. This would violate our Definition 1 of empirically plausible adjustment dynamics, and by extension it would violate Corollary 1. Hence, it follows that if attention is restricted to REE with empirically plausible adjustment dynamics, our conditions for REE to be both indeterminate and E-stable are unchanged.

Proposition 10 *When the PLM has no constant term, the necessary conditions for E-stability of indeterminate REE with empirically plausible adjustment dynamics remains unchanged: (21), (27), and (28).*

Common-factor representation of the REE

Rational expectations equilibria may result from more than one *representation*, or linear recursion; the limiting solution of a particular recursion yields the REE. In this paper we have considered the standard, general-form AR(1) representation of the REE as reflected in the perceived law of motion (3) as that is the one that has been used in the RBC literature. However, there are many alternative representations that are possible. [11] Evans and McGough (2002, 2003) argue that the stability of a REE under learning may well depend on the representation used. In particular, they report that general form ARMA representations that include sunspot variables (such as the AR(1) representation we consider) are unstable under learning. However, there are alternative, "common factor" representations of these same REE sunspot solutions that are sometimes stable under learning. We therefore reexamine our findings using an "common factor" AR(1) representation in place of the general form AR(1) representation following Evans and McGough (2002).[12]

To obtain the "common factor" representation, we first diagonalize the Jacobian matrix J, in (22) writing:

$$J = S\Theta S^{-1} \tag{65}$$

where

$$\Theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}$$

with θ_1 and θ_2 representing the two eigenvalues of J in (22) and S representing the 2×2 matrix of the associated eigenvectors, with

$$S^{-1} = \begin{bmatrix} s^{11} & s^{12} \\ s^{21} & s^{22} \end{bmatrix}.$$

In what follows, we will use the normalization $s^{i1} = 1$, $i = 1, 2$.

[11] See, e.g., Evans and Honkapohja (1986) for an extensive discussion of this issue.

[12] Evans and McGough (2002) consider an encompassing reduced-form system similar to our own (1-2), though they allow a fundamental, autoregressive shock process as well. We will continue to ignore the fundamental shock as it is unimportant to our results.

We will focus on the case where the REE is indeterminate, i.e. where both roots θ_1, θ_2 lie inside the unit circle, though common factor representations are possible for any system so long as all the eigenvalues are real and have norm less than one. We note that, in practice, the eigenvalues of indeterminate systems used in the sunspot-driven RBC literature are typically *complex*. So, strictly speaking, it would seem that common-factor representations are not so relevant to understanding the stability properties of the equilibria that have appeared in this literature.

Using the notation $y_t' = (k_t, c_t)$, and the transformation $z_t = S^{-1}y_t$, the system (22) can be rewritten as the uncoupled system:

$$z_t = \Theta z_{t-1} + \eta_t$$

where $\eta_t = S^{-1}R\epsilon_t$, or equivalently, as

$$\begin{bmatrix} 1-\theta_1 L & 0 \\ 0 & 1-\theta_2 L \end{bmatrix} z_t = \eta_t$$

were L represents the lag operator. Next, premultiply both sides of the system above as follows:

$$\begin{bmatrix} 1 & 0 \\ 0 & (1-\theta_2 L)^{-1} \end{bmatrix} \begin{bmatrix} 1-\theta_1 L & 0 \\ 0 & 1-\theta_2 L \end{bmatrix} z_t = \begin{bmatrix} 1 & 0 \\ 0 & (1-\theta_2 L)^{-1} \end{bmatrix} \eta_t$$

The equation for the non-predetermined consumption process can be isolated as:

$$z_{2t} = (1-\theta_2 L)^{-1}\eta_{2t} = \xi_t,$$

where the second equality corresponds to a definition of the variable ξ_t. Using the transformation, $z_{2t} = s^{21}c_t + s^{22}k_t$, and the normalization $s^{21} = 1$, we may write:

$$c_t = -s^{22}k_t + \xi_t.$$

The law of motion for the capital stock, is unchanged by the transformation, since this equation was uncoupled to begin; note also that $\eta_{1t} = 0$. Using the law of motion for k_t in the equation for c_t given above, we can write the "common factor" representation of the system as:

$$\begin{aligned} c_t &= -s^{22}d_c c_{t-1} - s^{22}d_k k_{t-1} + \xi_t \\ k_t &= d_c c_{t-1} + d_k k_{t-1} \\ \xi_t &= \theta_2 \xi_{t-1} + \eta_{2t} \end{aligned}$$

where the last equation follows from the definition of ξ_t given above and $\eta_{2t} = \varepsilon_t$, the sunspot variable.

The perceived law of motion is now

$$c_t = a_1 + a_c c_{t-1} + a_k k_{t-1} + a_f \xi_t,$$

which is nearly identical to the PLM (3); the only difference is that f_t is replaced by the autoregressive variable ξ_t. We can derive the expected values just as before, which are given by:

$$E_t c_t = a_1 + a_k k_{t-1} + a_c c_{t-1} + a_f \xi_t, \tag{66}$$

$$E_t k_t = d_k k_{t-1} + d_c c_{t-1}, \tag{67}$$

$$E_t c_{t+1} = a_1(1 + a_c) + a_k(d_k + a_c)k_{t-1} + (a_c^2 + a_k d_c)c_{t-1} + (a_c a_f + af\theta_2)\xi_t, \tag{68}$$

$$E_t k_{t+1} = a_1 d_c + (d_k^2 + d_c a_k)k_{t-1} + d_c(d_k + a_c)c_{t-1} + d_c a_f \xi_t. \tag{69}$$

Using these expectations, it is easy to show that the T-maps $T(a_1)$, $T(a_k)$ and $T(a_c)$ will be the same as in the general case, as given by equations (11–13). The only T-mapping that will differ is the one for $T(a_f)$. In place of (14) we now have:

$$T(a_f) = a_f[b_k d_c + b_c(a_c + \theta_2)] \tag{70}$$

Because of this change, the RE solution – the fixed point of the T–mapping – will be different. Specifically, if $T(a_f)$ is now given by (70), setting $a_f = T(a_f)$ now yields the set of sunspot REE:

$$a_c = \frac{1 - b_k d_c}{b_c} - \theta_2, \tag{71}$$

with a_f indeterminate,

where θ_2 is an eigenvalue of the system. One possible solution is $a_f = 0$. However that solution does not interest us as it implies that sunspots have no effect on the system.

Solutions for the other two coefficients are obtained as

$$a_1 = 0 \tag{72}$$

$$a_k = \frac{b_k d_k^2}{b_c(\theta_2 - d_k)} \tag{73}$$

The key matrix $\frac{d(T(a)-a)}{da}$, evaluated at this new REE solution can be written as:

$$\begin{bmatrix} b_c(1-\theta_2) & 0 & 0 & 0 \\ 0 & b_c(d_k - \theta_2) & \frac{b_k d_k^2}{\theta_2 - d_k} & 0 \\ 0 & b_c d_c & 1 - b_k d_c & 0 \\ 0 & 0 & b_c \overline{a}_f & 0 \end{bmatrix} \tag{74}$$

(c.f. (74) with (16)). The eigenvalues of (74) are determined by the equation

$$[b_c(1-\theta_2)-\lambda](-\lambda)\left\{\lambda^2-[b_c(d_k-\theta_2)+(1-b_kd_c)]\lambda+b_c(d_k-\theta_2)(1-b_kd_c)+\frac{b_cd_cb_kd_k^2}{d_k-\theta_2}\right\}=0.$$

The first two eigenvalues are

$$\lambda_1 = b_c(1-\theta_2), \quad (75)$$

$$\lambda_2 = 0. \quad (76)$$

The remaining two eigenvalues are solutions to:

$$\lambda^2-[b_c(d_k-\theta_2)+(1-b_kd_c)]\lambda+b_c(d_k-\theta_2)(1-b_kd_c)+\frac{b_cd_cb_kd_k^2}{d_k-\theta_2}=0.$$

The full set of necessary conditions for E-stability are thus:

$$b_c(1-\theta_2) < 0, \quad (77)$$

$$b_c(d_k-\theta_2)+1-b_kd_c > 0, \quad (78)$$

$$b_c(d_k-\theta_2)(1-b_kd_c)+\frac{b_cd_cb_kd_k^2}{d_k-\theta_2} > 0. \quad (79)$$

Recall that for the system (22), the trace of the Jacobian matrix J was given by:

$$tr(J)=\theta_1+\theta_2=\frac{1-b_kd_c+d_kb_c}{b_c}$$

Substituting out θ_2 with this expression in (78), we can rewrite the necessary condition (78) as:

$$b_c\theta_1 > 0. \quad (80)$$

We now make use of Definition 1 concerning empirically plausible adjustment dynamics and impose our indeterminacy conditions. Since common factor representations require the eigenvalues to be real, we must have $\theta_1,\ \theta_2 \in (0,1)$. It follows immediately that (77) implies $b_c < 0$ and (80) implies $b_c > 0$ - a contradiction. Therefore, if the system is E-stable under adaptive learning, it cannot be both indeterminate and have empirically plausible adjustment dynamics.

Proposition 11 *The common factor AR(1) REE solution to an RBC model with empirically plausible adjustment dynamics cannot be both indeterminate and E–stable under adaptive learning.*

There would not be any contradictions, however, if the eigenvalues of J were negative, though such a case would violate our Definition 1 for empirically plausible adjustment dynamics. Thus we have the following corollary:

Corollary 2 *If an RBC model has negative roots, then it is possible for its common factor representation to be both indeterminate and E-stable. However, such models would exhibit period-by-period oscillatory adjustment dynamics that would not be empirically plausible.*

Corollary (2) provides us with another reconciliation between our findings and those of Evans and McGough (2002), who found parameter values for which the RE system was indeterminate and a common factor representation of the REE was stable under learning. Our analysis suggests that this can only occur if at least one eigenvalue of the RE system is negative, which would be ruled out by any reasonably calibrated, empirically plausible model of adjustment dynamics.

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