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Stochastic asymmetric Blotto games: An experimental study

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ABSTRACT

We consider a model where two players compete for n items having different common values in a Blotto game. Players must decide how to allocate their common budgets across all n items. The winner of each item is determined stochastically using a lottery mechanism which yields a unique equilibrium in pure strategies. We analyze behavior under two competing payoff objectives found in the Blotto games literature that have not been previously compared: (i) players aim to maximize their total expected payoff and (ii) players maximize the probability of winning a majority value of all n items. We report results from an experiment where subjects face both payoff objectives and we find support for the differing theoretical predictions.

1. Introduction

Resource allocation plays a central role in both economics and politics. A well-studied game-theoretic representation of the resource allocation problem is the Colonel Blotto game (Borel, 1921) which models the problem as a two-player non-cooperative game. In the canonical " Blotto" game, there are $(2k+1)$ battlefields of equal value. The two players have fixed endowments of resources, e.g., troops or money, which they must simultaneously allocate to each of the $(2k+1)$ battlefields. The winner of each battlefield is determined according to which player allocated the greater amount of resources to that battlefield. The standard objective function for each player is to win $k+1$ or more (a majority) of the battlefields although an alternative objective function is to win as many battlefields as possible.

In this paper we experimentally study behavior in two-player, n-item (or battlefield) *stochastic asymmetric* Blotto games where the n items have commonly known but asymmetric values, and the winner of each item is determined stochastically using a simple lottery mechanism. The lottery mechanism for awarding each item makes the payoff function continuous,

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by contrast with the canonical deterministic (or auction) version of the Blotto game, resulting in a unique pure strategy equilibrium allocation of players' endowments across the n items. Thus, the stochastic Blotto game environment provides crisp predictions for an experimental evaluation by contrast with the deterministic version of the Blotto game which yields a multiplicity of possible equilibria typically in mixed strategies. Within the stochastic asymmetric Blotto game environment, we compare and contrast behavior under two different objective functions. In the first case, as originally studied by Friedman (1958), the players' objective is to maximize their *total* expected earnings from allocating their budget across all n battlefields. In the second case, as originally studied by Lake (1978), the players' objective is to win a *majority* of the values of the n battlefields. To our knowledge, there is no prior experimental work comparing these two different, but commonly used objective functions for Blotto games.

A motivating example for the "total rule" version of the stochastic asymmetric Blotto game comes from advertising decisions by two firms in a duopoly setting. The two firms produce and sell an identical good and must decide how to allocate their advertising budgets over n markets having various different sales potentials that are common to both firms. As advertising is not perfectly effective in stimulating sales of the good, increased expenditures in any given market may have only a stochastically larger impact on each firms' sales of the good in that market. However, both duopolists' goal is to maximize the *total market share* for their product and not just a majority share as in the electoral college example.

A motivating example for the "majority rule" version of the stochastic asymmetric Blotto game is the U.S. electoral college mechanism for determining the President of the United States. The electoral college system is a two-player, stochastic asymmetric majority-rule Blotto game in the sense that: (1) there are typically just two presidential candidates; (2) the 51 states (including the District of Columbia) have different numbers of electoral votes (values) ranging from a minimum of 3 to a maximum (as of 2012) of 55 votes (for the state of California); (3) the presidential candidate who spends the most resources (campaign expenditures) on any given state has a greater chance of winning that state's electoral votes, but does not necessarily win, i.e., the winner is stochastically determined.[1] (4) the overall winner of the Presidency is the candidate who earns a *majority* of the total electoral votes (currently 538), that is, the objective is to achieve a majority of the asymmetric and stochastically awarded prize values (electoral votes).

We note that while these two objective functions (total and majority) might seem to be quite similar, equilibrium resource allocation by the two players under each objective function can be dramatically different. Under the total rule, equilibrium bids are proportional to the values of each of the n battlefields, while under the majority rule, equilibrium bids reflect differences in the pivotality of each battlefield in achieving a majority share of the value of all n battlefields. Thus, the motivation for our study is to explore whether the differences in equilibrium predictions for the two objective functions find empirical support in a laboratory experiment. We present results from a within-subjects experimental design involving two different versions of a 4-item stochastic asymmetric Blotto game where subjects make resource allocation decisions under both the total and majority rules. Our experiment yields support for the different equilibrium allocation predictions under the two different payoff objective functions.

The rest of this paper is organized as follows. The next section reviews the related literature. Section 3 provides a theoretical framework for comparing equilibria under the two different objective functions for the stochastic asymmetric Blotto game. Section 4 describes our experimental design and Section 5 reports our main experimental findings. Section 6 concludes with a summary and some suggestions for future research.

2. Related literature

The Colonel Blotto game is one of the oldest games in game theory. It was originally proposed by Borel (1921), who considered the $n = 3$ battlefield case. Gross and Wagner (1950) considered the more general $n \geq 2$ case and introduced the name Colonel Blotto. Other theoretical contributions include: Borel and Ville (1938), Blackett (1954, 1958), Bellman (1969), Shubik and Weber (1981) and Weinstein (2012). Recent work on the Colonel Blotto game has considered asymmetries in the players' resources (Hart, 2008; Roberson, 2006; Macdonell and Mastronardi, 2012); non-constant-sum versions of the game (Kvasov, 2007; Powell, 2009; Hortala-Vallve and Llorente-Saguer, 2010, 2012; Roberson and Kvasov, 2012); more general payoffs (Golman and Page, 2009); incomplete information (Adamo and Matros, 2009; Kovenock and Roberson, 2011); alliances among players (Kovenock and Roberson, 2012a); and various applications in political economy (Brams and Davis, 1973, 1974; Colantoni et al., 1975; Young, 1978; Bartels, 1985; Shaw, 1999; Laslier, 2002; Laslier and Picard, 2002; Sahuguet and Persico, 2006; Powell, 2007; Roberson, 2008; Bell and Wilson, 2012). Much of the literature is surveyed in Kovenock and Roberson (2012b).

More relevant to this paper, Friedman (1958) was the first to provide analytic results for the stochastic asymmetric "total rule"Blotto game where the winner of each item i is determined by a lottery in which the chance of winning is proportional to each player's allocation of resources to item i, i.e., a standard Tullock (1980) contest success function.[2] Friedman describes

[1] For example in the 2012 U.S. Presidential election, candidate Barack Obama and affiliated political action committees spent $69.3 million USD in the state of Florida while candidate Mitt Romney and affiliated political action committees spent $81.3 million USD in Florida (Bell and Wilson, 2012). Despite being out spent in Florida, candidate Obama nevertheless won Florida's 29 electoral votes in the 2012 presidential election.

[2] Monahan (1987) and Robson (2005) generalize Friedman's model. Monahan allows costly effort and Robson considers more general versions of Tullock's contest success function.

a unique pure-strategy Nash equilibrium which we test in our experiment. Lake (1978) was the first to study a stochastic asymmetric "majority rule" Blotto game. Lake studied the case of equal budget constraints, which we also employ in our experiment. The equilibrium prediction in the majority rule case with equal budget constraints is that allocation of resources to item i should equal the Banzhaf Power Index for item i. In a companion paper, Duffy and Matros (2015), we generalize Lake's (1978) theoretical findings to asymmetric stochastic Blotto games involving n players and asymmetric budget constraints, i.e., $X \neq Y$ and we theoretically compare and contrast the predictions for the stochastic asymmetric Blotto game under the total and majority rule objectives.

Snyder (1989) analyzes an election model where players have cost functions instead of given budgets (or endowments). He was the first to compare equilibrium behavior under two different assumptions about the candidates' objectives (1) the total rule, where candidates maximize the expected total number of votes, and (2) the majority rule, where candidates maximize their probabilities to win a majority of the total votes. Klumpp and Polborn (2006) use Snyder's framework to develop a costly advertising model of political competition (both simultaneous and sequential) in which candidates have to win the majority of a number of electoral districts in order to obtain a certain prize. In both of these papers, the setting differs from the environment we study due to the presence of cost functions as opposed to endowments and budget constraints. More importantly in both Snyder (1989) and Klumpp and Polborn (2006), the value of each item (or of each electoral district) is the same. By contrast we study the more general case where the value of the individual items may be different from one another, as in the number of electoral votes per state in the U.S. electoral college, or the sales potential of different markets for a good.

Even more relevant to this paper are several experimental studies of Blotto games. Avrahami and Kareev (2009) consider a version of the Blotto game where players have identical valuations for the n items but different budgets. In their lottery version of the game, only one box is opened for each player and the winner is determined by comparing allocations in those two boxes. Avrahami et al. (2014) consider a similar lottery version of Blotto where the battlefields also differ in their likelihood of being chosen which is similar to the asymmetric values we assigned to battlefields in our setting.

Modzelewski et al. (2009) analyze the classic (auction) Blotto game with 6 identically valued battlefields and equal resources. Arad and Rubinstein (2012) consider a large web-based experiment of a tournament version of the Blotto game with 6 battles and equal resources. Chowdhury et al. (2013) investigate two types of Blotto games: stochastic (lottery) and deterministic (auction) versions where all battlefields have identical values but where the two players may have asymmetric resources. Several recent experimental studies consider Blotto games with asymmetric item values as in this paper, but these studies all use deterministic (auction) rules for determining the winner of each item. Hortala-Vallve and Llorente-Saguer (2010) examine an incomplete information Blotto game with both asymmetric and heterogeneous battlefield valuations so that the game is no longer zero-sum. Hortala-Vallve et al. (2013) and Hortala-Vallve and Llorente-Saguer (2015) report on further experiments in similar settings but where the players can either communicate with one another or there is complete information about the player's heterogeneous valuations. Montero et al. (2016) conduct an experimental test of Young's (1978) model where two lobbyists allocate resources to politicians having different voting powers and seek to obtain a majority of votes cast for their opposed positions. Mago and Sheremeta (2016) examine behavior in simultaneous and sequential Blotto games with symmetric battlefield values, a deterministic (auction) rule and a majoritarian objective.

By contrast, the contribution of this paper is to experimentally compare and contrast Blotto games where: (1) the items (battlefields) have asymmetric but commonly known values, (2) the items (battlefields) are awarded according to a (stochastic) lottery mechanism so that the unique equilibrium prediction is always in pure strategies and (3) the objective function is either to maximize total expected payoff or to win a majority of the value of all items. Our main focus is on whether subjects appreciate the subtleties of these two different objective functions in allocating resources across the differently-valued battlefields. We are not aware of any prior experimental comparison of these two different, but commonly studied objective functions for Blotto games. As noted in the introduction, these two different versions of the asymmetric, stochastic Blotto game are of real-world interest in understanding advertising decisions by competing duopolists and resource allocation by U.S. presidential candidates competing to win the Electoral College.

3. Stochastic asymmetric Blotto games

The game we study involves two players x and y, and n items. Player x has a given budget of size X and player y has a given budget of size Y. Let $N = \{1, \ldots, n\}$ denote the set of the n items (or battlefields). Each item i has a known value, $W_i > 0$, that is the same for both players. The two players compete for these items by simultaneously allocating their budgets across all n items. A pure strategy for player x is a nonnegative n-dimensional vector $(x_1, \ldots, x_n)$, such that $\sum_{j=1}^{n} x_j = X$ and x_i is player x's spending on item i. A pure strategy for player y, $(y_1, \ldots, y_n)$, is determined analogously with $\sum_{j=1}^{n} y_j = Y$. Each item is allocated by means of a lottery in which player x obtains item i with probability $x_i/(x_i + y_i)$ and player y obtains item i with probability $y_i/(x_i + y_i)$.[3,4] Without loss of generality, for the rest of the paper we shall assume that

$$W_1 \geq \cdots \geq W_n > 0. \tag{1}$$

[3] We assume that if $x_i = y_i = 0$, then each player has 0.5 probability to win item i.

[4] We assume that all lotteries are statistically independent.

Denote the total value of all n items by:

$$W = \sum_{i=1}^{n} W_i.$$

3.1. Plurality: maximizing the expected value

Friedman (1958) describes a Nash equilibrium where both players seek to maximize their expected item values:

$$\max_{(x_1,\ldots,x_n)} \sum_{j=1}^{n} \frac{x_j}{x_j + y_j} W_j,$$

$$s.t. \sum_{j=1}^{n} x_j = X \text{ and } x_j \geq 0 \ \forall j,$$

for player x and analogously for player y.

Theorem 1. Friedman, 1958 *The stochastic Blotto game where players seek to maximize their expected item values has a unique Nash equilibrium. In this Nash equilibrium,*

$$(x_1, \ldots, x_n) = \left(\frac{W_1}{W}, \ldots, \frac{W_n}{W}\right) X, \tag{2}$$

$$(y_1, \ldots, y_n) = \left(\frac{W_1}{W}, \ldots, \frac{W_n}{W}\right) Y. \tag{3}$$

The expected equilibrium payoffs are $\frac{X}{X+Y}W$ *for player x and* $\frac{Y}{X+Y}W$ *for player y.*

We highlight several important features of Theorem 1. First, the Nash equilibrium is *unique*. Second, both players compete for *all* items in the Nash equilibrium of this version of the Blotto game. Third, the Nash equilibrium has a monotonic property: the player with the *greater budget* has a greater chance to win *each item*.[5]

3.2. Weighted majority: maximizing the probability of winning a majority

We now assume that each player wants to maximize her probability to win a majority of all items' values as in the U.S. electoral college example. Note that each possible coalition of items $\{W_1, \ldots, W_n\}$ can be represented by a binary, n-dimensional characteristic vector $(t_1, \ldots, t_n)$, where $t_i \in \{0, 1\}$ for any $i = 1, \ldots, n$. If $t_i = 1$, then item i belongs to the coalition, and if $t_i = 0$, then item i does not belong to the coalition. We will use the corresponding characteristic vector to represent a coalition in the rest of the paper. There are 2^n such coalitions.

Denote by **V** the set of winning coalitions under the win-a-majority-of-item-values objective. Then coalition $(t_1, \ldots, t_n) \in \mathbf{V}$, if

$$\sum_{j=1}^{n} t_j W_j > \frac{W}{2}.$$

We next make a technical assumption that guarantees a unique majority winner in all realizations of individual lotteries.

Assumption 1. $(4) \sum_{j=1}^{n} t_j W_j \neq \frac{W}{2}$ forany coalition $(t_1, \ldots, t_n)$.

A stronger version of Assumption 1 is typical in the literature. Usually in this literature, all values are the same, $W_1 = \cdots = W_n$, in which case Assumption 1 becomes $n = 2k + 1$, $k = 1, 2, \ldots$.

[5] We note further that if one wanted to relax the assumption of risk neutral preferences, it is a simple matter to transform the maximization problem in this section to one of expected utility maximization, e.g., by replacing W_j in the maximization problem with $U(W_j)$ where $U()$ is some utility function. The equilibrium bid vector in the expected utility case would generally differ from the risk neutral case, but it would continue to be unique. For the majority rule case discussed in the next section, such utility transformations will not affect equilibrium bids as payoffs are winner-take-all.

A player wins the stochastic majority Blotto game if she gets a winning coalition. Without loss of generality, a player receives a payoff of 1 from winning the game and a payoff of 0 from losing the game. Player x maximizes her chance to get a winning coalition by solving the following maximization problem:

$$\max_{(x_1,\ldots,x_n)} \sum_{(t_1,\ldots,t_n)\in \mathbf{V}} \prod_{j:t_j=1} \frac{x_j}{x_j+y_j} \prod_{k:t_k=0} \frac{y_k}{x_k+y_k}, \tag{5}$$

$$s.t. \sum_{j=1}^{n} x_j = X, \tag{6}$$

where $\prod_{j:t_j=1} \frac{x_j}{x_j+y_j}$ is the probability to win all the items that belong to the coalition $(t_1, \ldots, t_n)$ and $\prod_{k:t_k=0} \frac{y_k}{x_k+y_k}$ is the probability to lose all the items that do not belong to the coalition $(t_1, \ldots, t_n)$.

Similarly, player y solves the following maximization problem:

$$\max_{(y_1,\ldots,y_n)} \sum_{(t_1,\ldots,t_n)\in \mathbf{V}} \prod_{j:t_j=1} \frac{y_j}{x_j+y_j} \prod_{k:t_k=0} \frac{x_k}{x_k+y_k}, \tag{7}$$

$$s.t. \sum_{j=1}^{n} y_j = Y. \tag{8}$$

We will need the following definition.

Definition 1. An item i is *pivotal* in coalitions $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$ and $(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$ if $(t_1, \ldots, t_{i-1}, 1, t_{i+1}, \ldots, t_n)$ is a winning coalition and $(t_1, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n)$ is a losing coalition.

Denote by $\mathbf{V}_i$ a set of winning coalitions where item i is pivotal. Denote by $Piv(i)$ the number of winning coalitions in which item i is pivotal, or

$$Piv(i) = \|\mathbf{V}_i\| = \sum_{(t_1,\ldots,t_n)\in \mathbf{V}_i} 1. \tag{9}$$

We now introduce the Banzhaf Power Index[6] for item i in the following way:

$$BPI(i) = \frac{Piv(i)}{Piv(1)+\cdots+Piv(N)} = \frac{\sum_{(t_1,\ldots,t_n)\in \mathbf{V}_i} 1}{\sum_{(t_1,\ldots,t_n)\in \mathbf{V}_1} 1+\cdots+\sum_{(t_1,\ldots,t_n)\in \mathbf{V}_n} 1}. \tag{10}$$

Intuitively, the Banzhaf Power Index for item i measures the probability that item i will be pivotal as part of a winning coalition. Using this definition, and following Lake (1978), we can now characterize the Nash equilibrium for the asymmetric stochastic Blotto game where both players seek to maximize their probability of winning a *majority* of all items' values.

Theorem 2. Lake, 1978 *Suppose that conditions* (4) *and* $X=Y$ *hold. Then, the stochastic Blotto game where players seek to maximize their chance of winning a majority of the item values has a unique pure-strategy Nash equilibrium in which* $x_i = BPI(i)X$ *and* $y_i = BPI(i)Y$, *for* $i = 1, 2, \ldots, n$.

A comparison of equilibrium allocations under the two different, but seemingly similar objective functions – the total and the majority rules – as given in Theorems 1 and 2 reveals that the unique equilibrium allocations of resources to the n items can be quite different from one another. A natural question is whether these differences are simply a theoretical curiosity or whether they are indeed empirically relevant to subjects incentivized to play according to the two different objective functions. To address this important question we designed an experiment to test equilibrium predictions in stochastic asymmetric Blotto games under the two different objective functions.

4. Experimental design

The main objective of our experiment is to test the predictions found in Theorems 1 and 2 regarding the equilibrium allocation of players' endowments toward winning the n prizes under the two different payoff rules. Toward that goal we chose to consider a two player, $n=4$ item asymmetric value, stochastic Blotto game. We focus on the $n=4$ case because it represents the smallest value of n for which the differences in allocations under the two payoff rules can be sufficiently distinct from one another. For simplicity, we also study the case where the two players have equal budgets, leaving the case

[6] See Banzhaf (1965) for discussion.

Table 1
Equilibrium predictions for prize vector Version 1.

Prize vector	Eq. token allocation under	
Version 1	Majority rule	Total rule
$W_1 = 35$	40	$120 \times \frac{35}{100} = 42$
$W_2 = 30$	40	$120 \times \frac{30}{100} = 36$
$W_3 = 25$	40	$120 \times \frac{25}{100} = 30$
$W_4 = 10$	0	$120 \times \frac{10}{100} = 12$

of asymmetric budgets to future research. Under these conditions, the equilibrium allocations are unique under both rules (Theorems 1 and 2).

Our experiment involves a 2 × 2 ×2 experimental design involving three main treatment variables: (1) the payoff rule: players either seek to maximize the total expected value of prize items ("the total rule") or to maximize their chance of winning a majority of the value of all items ("the majority rule"); (2) the vector of values of the four prize items (labeled Version 1 and Version 2): we chose to consider two different prize value vectors to test some of the comparative statics implications of the theory as detailed below; (3) the order in which subjects allocated resources to the four prizes: we consider the case where the prizes were presented in descending order (highest to lowest value prize) or in ascending order (lowest to highest value prize). This last treatment was intended to check whether there might be any behavioral biases arising from the order in which prizes of different values were presented. In addition, as we use a within-subjects experimental design, we also vary the order in which subjects make decisions under a given rule, total or majority as detailed below.

Under the total rule, a player receives in points the total value of any and all items won. Under both prize Versions 1 and 2, the total value of all four items sums to 100 points. Thus under the total rule, a player can get a positive number of points if she wins one or more of the four items. The game is constant sum: one player's earnings are 100 minus the earnings of the other player.

The majority rule game is also constant sum but the payoff earnings are different. Under majority rule, a player wins the game if the value of all her won items is greater than the value of the items won by her opponent. The winner of the majority rule game thus receives 100 points and the loser receives *zero* points, that is, the majority rule game is "winner take all." The prize valuations respect our Assumption 1 (condition (4)) so the majority winner is always unambiguous in our setup.

In all four treatments, the play of a round of the asymmetric stochastic Blotto game proceeds as follows. First, subjects are randomly and anonymously paired with one another with all possible pairings being equally likely in their matching group of size six. Next, both players in each pair are given a budget of 120 tokens and instructed that each must allocate all of their 120 tokens toward winning the four items. Both players submit their token allocations simultaneously and without communication using a computer interface developed for this study. The program checks that the four allocations made by each player to the four items exactly sum up to 120; if not, then a player is prompted to resubmit his or her allocation until the budget of 120 tokens has been fully exhausted. Thus, the pure strategy of player x is a 4-dimensional vector $(x_1, \ldots, x_4)$, such that $\sum_{i=1}^{4} x_i = 120$ and the pure strategy of his opponent, player y, is a 4-dimensional vector $(y_1, \ldots, y_4)$, such that $\sum_{i=1}^{4} y_i = 120$.

The four items have different but commonly known valuations, W_i, $i = 1, \ldots, 4$. These valuations remained constant across all rounds of a session. Item i is allocated by means of a lottery in which player x wins item i with probability $x_i/(x_i + y_i)$ and player y wins item i with probability $y_i/(x_i + y_i)$. In the event that $x_i = y_i = 0$, then each of the two players has a 0.5 probability of winning item i. The lottery mechanism for awarding prizes was carefully explained to subjects in the written instructions. Example scenarios were presented and subjects had to successfully complete a quiz demonstrating their knowledge of the lottery mechanism used to determine the winners of each of the four items prior to the start of the experiment.

As mentioned, we considered two versions for the distribution of values across the four items.

4.1. Version 1

In this treatment, the four items had the known values:

$$W_1 = 35,\ W_2 = 30,\ W_3 = 25,\ W_4 = 10.$$

The value of these four items sum to 100 points. Given that both players have a fixed budget of 120 tokens, Table 1 gives the unique equilibrium predictions regarding the allocation of tokens across the four items under the total and majority rules for prize vector Version 1. These predictions follow from Theorems 1 and 2.

Notice that our parameterization of the experiment induces distinct differences in equilibrium bids between the two rules. A complete characterization of the points and dollar earnings possible in this version of the experiment (and the next) is given in the payoff table of the experimental instructions found in Appendix A of the online supplementary materials.

Table 2
Equilibrium predictions for prize vector Version 2.

Prize vector	Eq. token allocation under	
Version 2	Majority rule	Total rule
$W_1 = 45$	60	$120 \times \frac{45}{100} = 54$
$W_2 = 25$	20	$120 \times \frac{25}{100} = 30$
$W_3 = 20$	20	$120 \times \frac{20}{100} = 24$
$W_4 = 10$	20	$120 \times \frac{10}{100} = 12$

4.2. Version 2

In this treatment, the four items had the known values:

$$W_1 = 45,\ W_2 = 25,\ W_3 = 20,\ W_4 = 10.$$

The values of the four items again sum to 100 points. Notice that Version 2 differs from Version 1 only in the values assigned to the first three items; the value of the fourth prize is 10 in both treatments, (prize versions), a feature we will later exploit in our analysis of the experimental data. Under the maintained assumption that both players have a fixed budget of 120 tokens, Table 2 gives the equilibrium predictions regarding the allocation of tokens across the four items under the majority and total rules for prize vector Version 2.

Notice that the two different versions for the vector of prize values (Versions 1 and 2) yield strikingly contrasting predictions as to how subjects should allocate their tokens, and indeed, that is why we chose these two different sets of prize values. In particular, observe that for Version 1, under the Majority rule, there should be zero allocation to an item that can never be pivotal (item 4). Further, when payoffs are determined by the Majority rule, allocations should be the same across the first three items under Version 1 while under Version 2, allocations should be the same across the last three items. By contrast, for both prize vector versions, allocations under the Total rule should be proportional to the prize values.

4.3. Interface

In each period of the experiment, subjects saw two screens. The first screen was the decision screen, where subjects chose how to allocate their tokens across the four items. Subjects were reminded on this decision screen of the prize values of the four items which were presented to subjects either in descending order from highest to lowest value or in ascending order from lowest to highest value (a further treatment condition). Next to each prize value, subjects entered their bids and the program checked that bids for all items were non-negative and summed to 120, the common endowment of tokens for each subject.

After all pairs had submitted their bids the winners of each of the four items in each pair were determined by the computer program and the results were shown to subjects on a results screen, which utilized pie charts to convey the chances that both subjects in a pair had of winning the four different items. An illustration of this results screen for Version 1 under the total rule is shown in Fig. 1. This results screen further revealed the number of tokens that each player in the pair had allocated toward winning the four items and reported on whether a player had won or lost each prize, their total points earned and their dollar earnings for that round. The results screen for the majority rule treatment was the same as shown in Fig. 1 for the total rule treatment, except that “round earnings” would either be \$20 or \$0, depending on whether or not a player’s “total points earned” exceeded 50 of the 100 possible points (a majority). It was public knowledge that the point earnings of the other player in each match were always 100 minus the player’s own point earnings. Subjects only saw results for their own pair.

4.4. Session characteristics

Each session involved 18 subjects divided up into 3 matching groups of size 6. Subjects were randomly matched only with the five other members of their matching group in all rounds of each session. Thus, each session yields three independent observations or what we label ‘groups’. We conducted two sessions of each of our four treatments. Thus we have data from 8 sessions involving $8 \times 18 = 144$ subjects divided up into 24 matching groups.

In each session, the vector of prize values $\{W_1, W_2, W_3, W_4\}$ was set either according to Version 1 or Version 2 and the prize vector remained constant for the duration of the session. As noted earlier, we used a within-subjects design where subjects in each matching group/session played the first 20 rounds under either the majority or the total rule treatment. Following the 20th round, the experiment was paused and subjects were given new, continuation instructions informing them that in the remaining 20 rounds they would be playing under the opposite rule, either the total or the majority rule treatment. This treatment change was *not* announced in advance. Thus each session involved play of 20 rounds of both payoff rules (40 rounds total) all under the *same* vector of item values. Sessions 1–4 used Version 1 for the prize values, {35, 30, 25, 10}, while sessions 5–8 used Version 2 for the prize values, {45, 25, 20, 10}. We chose a within-subjects design as it is statistically more powerful than a between-subject design since in a within-subject design, each subject serves as their

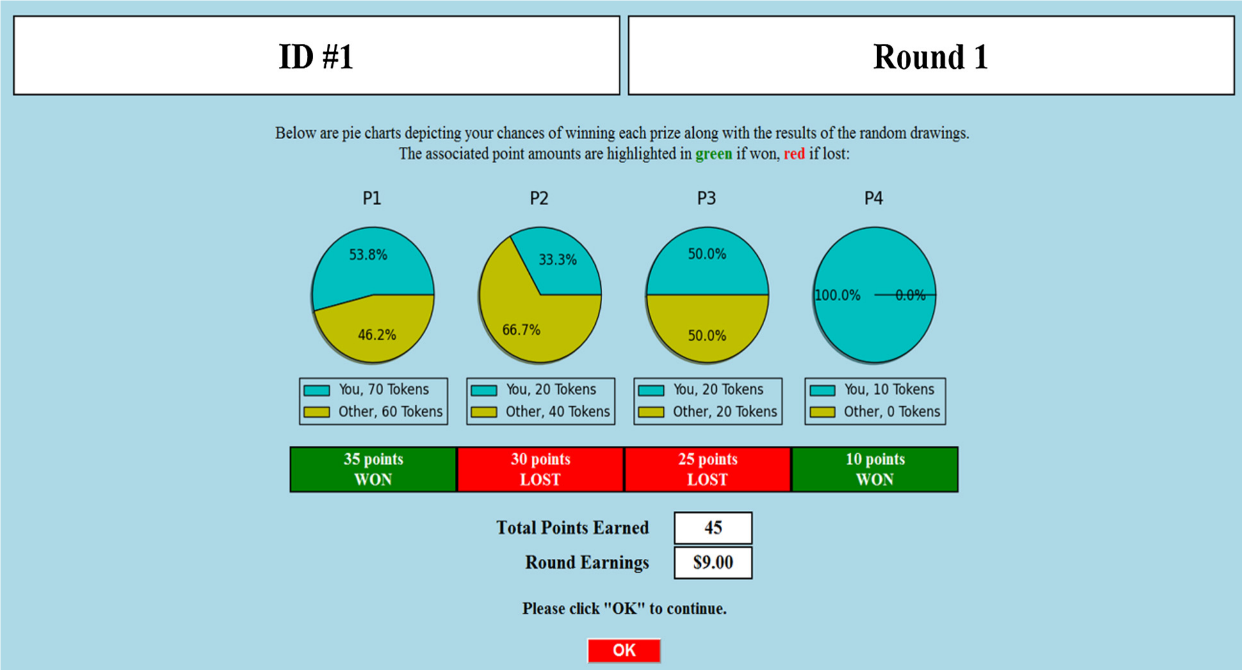

Fig. 1. Illustration of the depiction of the results from play of treatment Version 1 under the total rule.

Table 3
Characteristics of experimental sessions.

Session number	Item value list order	Payoff rule treatment order	No. of subjects	No. of indep. obs.	Group nos.
1	{35, 30, 25, 10}	Majority then total	18	3	1–3
2	{35, 30, 25, 10}	Total then majority	18	3	4–6
3	{10, 25, 30, 35}	Majority then total	18	3	7–9
4	{10, 25, 30, 35}	Total then majority	18	3	10–12
5	{45, 25, 20, 10}	Majority then total	18	3	1–3
6	{45, 25, 20, 10}	Total then majority	18	3	4–6
7	{10, 20, 25, 45}	Majority then total	18	3	7–9
8	{10, 20, 25, 45}	Total then majority	18	3	10–12

own control (e.g., being in both objective function treatments) thereby minimizing the effects of individual differences.[7] At the same time, within-subject designs mean that treatment order can matter and to minimize this possibility, in one-half of the sessions of each payoff vector treatment (Version 1 or Version 2) subjects played under the majority rule for the first 20 rounds followed by play under the total rule for the last 20 rounds, while in the other half of the sessions of each payoff vector treatment this order was reversed. As noted earlier, we also varied the order in which the list of prize values was presented to subjects and their resource allocations were elicited. In half of the sessions of each payoff vector treatment, prize values were presented and resource allocations were elicited in descending order (e.g., {35, 30, 25, 10}) while in the other half they were presented and elicited in ascending order (e.g., {10, 25, 30, 35}). The precise details of our eight experimental sessions are summarized in Table 3.

A copy of the written instructions used in session 1 where the majority rule was played in the first 20 rounds and the total rule was played in the last 20 rounds all under prize Version 1 with descending prize values is given in Appendix A of the online supplementary materials. Instructions for other sessions/treatments are similar.

The experiment was programmed using Willow, a Python-based toolkit for conducting economics experiments.[8] Subjects were recruited from the undergraduate population of the University of Pittsburgh and the experiments were all conducted in the Pittsburgh Experimental Economics Laboratory. No subject participated in more than one session of this study.

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[7] See, e.g., Camerer (2003), pp. 41–42.
[8] Willow was developed by Jaap Weel at George Mason University and is available at: http://econwillow.sourceforge.net/.

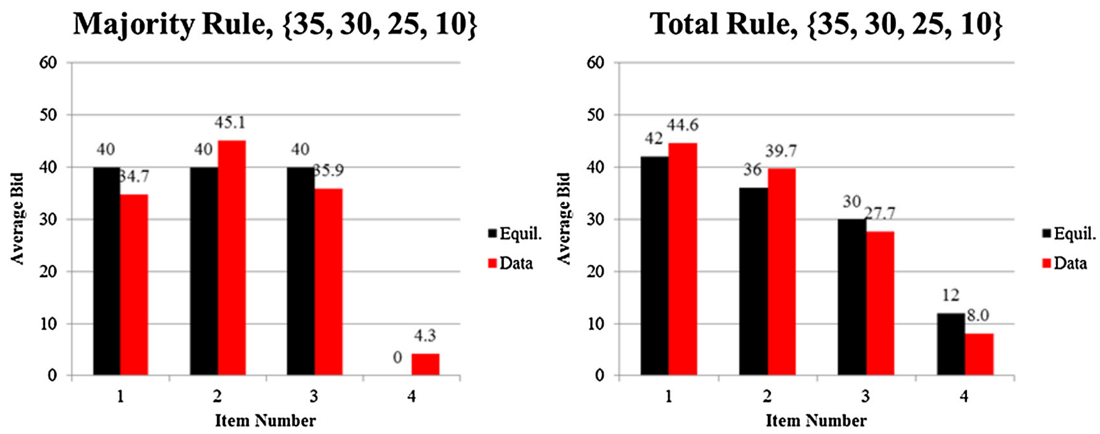

Fig. 2. Equilibrium and average actual bids under the majority and total rules, Version 1 {35, 30, 25, 10}. Pooled data from all rounds played by all 12 groups. For each item, the left bar indicates the equilibrium prediction while the right bar is the average amount bid over all rounds by all 12 groups.

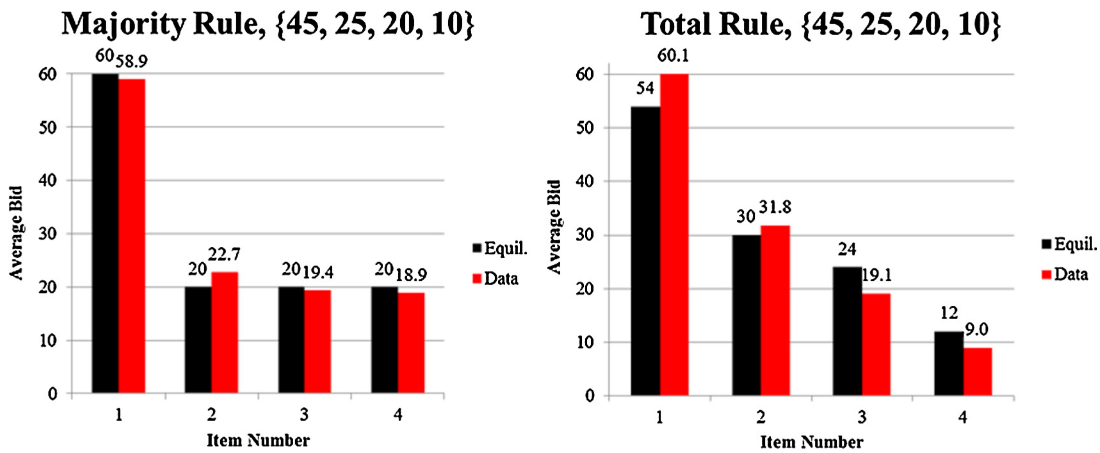

Fig. 3. Equilibrium and average actual bids under the majority and total rules, Version 2 {45, 25, 20, 10}. For each item, the left bar indicates the equilibrium prediction while the right bar is the average amount bid over all rounds by all 12 groups.

4.5. Payments

Subjects were paid their point earnings from two randomly chosen rounds, one from the first 20 rounds played and one from the second 20 rounds played. Points from these two randomly chosen rounds were converted into dollars at the fixed and known rate of 1 point = 20 cents (so 100 points = $20). In addition, subjects were given a $5 show-up payment. Thus the maximum total payoff that subjects could earn (including the $5 show-up payment) was $45, the minimum payment was $5 and the average payoff was $25 for participation in a 90-minute experimental session.

5. Experimental results

The main comparative statics implications of our treatments concern the allocation of subjects' endowment of 120 tokens across the four items having the prize values of Version 1 or Version 2 and under the different payoff rules, the majority rule or the total rule. Our experimental results consist of a number of different findings.

Finding 1. Mean bids are close to Nash equilibrium bids but are significantly different in three of the four main treatments. Nevertheless, NE bids remain a best response to the data in all four treatments.

Support for Finding 1 can be found in Figs. 2 and 3 which show the mean bids for each of the four items by all subjects participating in each of the four treatments using pooled data from all 12 groups of the four main treatments: Version 1, Majority and Total Rule and Version 2, Majority and Total Rule.[9] The Nash equilibrium bid for each item is given by the

[9] For now we ignore other treatment variables such as the order of the payoff rules or whether prizes were presented in ascending or descending order. We will address the impact of these treatment variables below.

Table 4
Mean bids compared with Nash bids.

Treatment	Majority Rule Version 1				Total Rule Version 1			
Prize value	35.00	30.00	25.00	10.00	35.00	30.00	25.00	10.00
Nash bid	40.00	40.00	40.00	0.00	42.00	36.00	30.00	12.00
Mean bid	34.73	45.08	35.92	4.26	44.58	39.69	27.70	8.04
(StDev)	(8.29)	(5.66)	(6.39)	(1.95)	(4.48)	(3.78)	(2.76)	(2.79)
WSRT p-value	0.06	0.01	0.12	[a]	0.07	0.01	0.02	0.00
Mean bid L5	33.33	45.87	36.20	2.92	44.71	38.95	28.18	8.16
(StDev) L5	(10.41)	(6.93)	(8.94)	(1.94)	(4.39)	(4.56)	(2.81)	(3.15)
WSRT L5 p-value	0.07	0.02	0.21	[a]	0.05	0.03	0.05	0.01
Treatment	Majority Rule Version 2				Total Rule Version 2			
Prize value	45.00	25.00	20.00	10.00	45.00	25.00	20.00	10.00
Nash bid	60.00	20.00	20.00	20.00	54.00	30.00	24.00	12.00
Mean bid	58.94	22.72	19.42	18.92	60.06	31.84	19.07	9.03
(StDev)	(6.90)	(3.48)	(4.60)	(4.66)	(5.73)	(1.64)	(4.40)	(3.10)
WSRT p-value	0.35	0.01	0.72	0.58	0.00	0.00	0.00	0.01
Mean bid L5	56.89	22.78	20.56	19.77	61.43	31.21	18.67	8.70
(StDev) L5	(9.32)	(5.17)	(6.27)	(6.58)	(9.46)	(2.23)	(5.16)	(4.45)
WSRT L5 p-value	0.39	0.12	0.81	0.88	0.03	0.10	0.01	0.03

[a] Since the Nash bid prediction in this case equals the lower bound of 0, a Wilcoxon signed rank test is not appropriate. Still, mean bids for this item are the closest to zero. Bids for this item were greater than zero 57.3 percent of the time over all rounds but only 54.2 percent over the last 5 rounds (L5). According to a two-tailed binomial test, the frequency of positive bids over all rounds is significantly greater than 50 percent ($p < 0.01$) but not over the last 5 rounds ($p = 0.13$). Note also that mean bids for this item are 2 standard deviations greater than 0 over all rounds, but only 1.5 standard deviations greater than 0 over the last 5 rounds.

left-most bar (the one colored black), while the second (red (or lighter)–colored) bars represent the mean amounts bid on each item by subjects. Figs. 2 and 3 give the impression as conveyed in Finding 1 that Nash equilibrium provides a good but imperfect way of characterizing bidding behavior in our experiment. Table 4 reports the same mean bids shown in Figs. 2 and 3 along with their associated standard deviations for each treatment/prize value over all 20 rounds and over the last 5 rounds only (L5).[10] In addition, Table 4 reports p-values from a two-sided, one-sample Wilcoxon signed-rank test (WSRT) of the null hypothesis that the 12 independent group-level mean bids for each prize/treatment condition (as reported in Tables 10–13 of Appendix B, online supplementary materials) equal the Nash equilibrium predicted bids for that same prize/treatment condition. The p-values from applying this test are reported both for mean bids over all rounds and for mean bids over the last 5 rounds (L5) only. Table 4 reveals that mean bids over all rounds or over the last 5 rounds are close to but often statistically significantly different from Nash equilibrium predictions at the 5 percent level. The sole exception occurs for mean bids under the Majority Rule, Version 2 treatment where by the last 5 rounds, we cannot reject the null hypothesis that the median of the mean bids differs from the Nash equilibrium bid for all four items ($p > 0.10$ for all four tests).

While mean bids are not a perfect match to the Nash equilibrium point predictions, we did check whether some other bidding strategy would yield a better outcome. Specifically, we took the vector of mean bids from each of the four treatments over all rounds as shown in Figs. 2 and 3 and we asked what a best response to those mean bids would be.[11] As reported in Finding 1, we found that for all four treatments, a best response to the vector of mean bids in the data was, in fact, the Nash equilibrium bid vector! Hence the Nash equilibrium remains the relevant benchmark for analysis.

We next consider the comparative statics predictions of the theory with regard to a change in the payoff rule on individual behavior. Specifically, we have:

Finding 2. Individual bids respond to the change in the payoff rule in a manner that is consistent with equilibrium predictions.

Support for Finding 2 comes from Table 5 which reports results from a generalized least squares (GLS) random effects estimation of the linear model:

$$b_{W,i,t} = \beta_0 + \beta_1 t + \beta_2 \delta^T + \beta_3 \delta^{RO} + \beta_4 \delta^A + \epsilon_{i,t}. \quad (11)$$

Here, $b_{W,i,t}$ denotes the amount bid on the item with value W by player i in round t, δ^T is a dummy variable equal to 1 when the total rule was in effect (so the baseline is the majority rule), δ^{RO} is a dummy variable equal to 1 when the Rule Order was 20 rounds of the total rule followed by 20 rounds of the majority rule (so the baseline is majority then total) and δ^A is a dummy variable equal to 1 when the four items were presented to subjects for bidding in ascending order of valuation,

[10] Tables 10–13 in Appendix B, online supplementary materials provide mean bids for each of the 12 groups per treatment over various subintervals of time.
[11] Results are available upon request.

Table 5
Regression analysis of bids for the four prizes under Version 1 (left columns) or Version 2 (right columns) all data from all sessions.

	Version 1: {35, 30, 25, 10}				Version 2: {45, 25, 20, 10}			
	b_{35}	b_{30}	b_{25}	b_{10}	b_{45}	b_{25}	b_{20}	b_{10}
Constant	29.98***	45.41***	40.01***	4.60***	55.03***	23.19***	21.66***	20.10***
	(1.76)	(1.19)	(1.32)	(0.96)	(2.58)	(1.52)	(1.57)	(1.24)
δ^T	9.85***	−5.39***	−8.23***	3.78***	1.19	8.92***	−0.54	−9.56***
	(3.22)	(2.17)	(2.02)	(0.86)	(1.62)	(1.21)	(1.10)	(1.43)
δ^{RO}	1.98	2.11	−4.62***	0.55	3.29	−1.04	−0.37	−1.88
	(1.45)	(1.42)	(1.13)	(1.05)	(2.85)	(0.72)	(2.20)	(1.22)
δ^A	4.63***	−2.50*	−2.97***	0.82	3.92	1.81**	−2.51	−3.23**
	(1.45)	(1.42)	(1.13)	(1.05)	(3.00)	(0.76)	(2.20)	(1.23)
Round (t)	0.14	−0.02	−0.03	−0.10**	0.02	−0.06	−0.06	0.10
	(0.12)	(0.07)	(0.08)	(0.04)	(0.10)	(0.06)	(0.08)	(0.08)
R^2	0.09	0.05	0.11	0.07	0.01	0.11	0.01	0.18
No. obs.	2880	2880	2880	2880	2880	2880	2880	2880

***,**,*, indicate significance, respectively, at the 0.01, 0.05, and 0.10 levels.

e.g., {10, 25, 30, 35} (so the baseline is descending order, e.g., {35, 30, 25, 10}). The regression (11) was run separately for each prize value under either Version 1 or Version 2 (8 regressions). The regressions make use of bid data from all four sessions of either Version 1 or Version 2, consisting of 2880 individual-level observations on amounts bid over all 40 rounds for each prize value, W. The regression estimates and robust standard errors obtained from clustering on matching groups are reported in Table 5.

Table 5 reveals that for nearly every prize, the coefficient on the dummy variable, δ^T, is significantly different from zero, and in all cases where this coefficient is significantly different from zero, the coefficient has the theoretically predicted sign (negative or positive). For example, consider bids for the prize with value 35, b_{35}, under Version 1 – the first column of Table 5. In the baseline, majority rule case, the coefficient on the constant term indicates an average bid of 29.98 for this prize which is somewhat less than the predicted bid of 40 for this treatment condition. However, under the total rule there is a statistically significant increase in the average bid by the amount 9.85 (the coefficient on δ^T) taking the average bid up to 39.83 (ignoring other explanatory factors); the latter is close to the Nash equilibrium bid prediction of 42 for the total rule treatment. Bids for the other three prizes under Version 1 and bids b_{25} and b_{10} under Version 2 display similarly theoretically consistent reactions when the payoff rule is made the total rule (from the baseline majority rule). The two exceptions are for b_{45} and b_{20} under Version 2; in those two cases the coefficient on δ^T is not significantly different from zero, so that the payoff rule change does not affect bidding for those two prizes. Notice, however, that the estimated mean bids under the baseline majority rule treatment (the coefficients on the constant terms as reported in Table 5) are already close to the Nash equilibrium bids under the Total Rule for Version 2 for these two prize amounts, i.e., $b^{ne}_{45} = 54$ and $b^{ne}_{20} = 24$, and consequently the change in the payoff rule does not have much effect on bidding for these two prize items.

Table 5 further reveals that there is not much change in bidding over time, as the estimated coefficients on the round t variable are, with a single exception, not significantly different from zero. The sole exception is for bids on the item with value 10 (b_{10}) under Version 1 where, in the baseline majority rule treatment, subjects do learn to bid less on this item, which is never pivotal. Similarly, the payoff rule treatment ordering does not appear to matter for bidding behavior; with a single exception (b_{25} for Version 1), the coefficient on the dummy variable δ^{RO} is never significantly different from zero indicating that whether the payoff rule treatment order was 20 rounds of majority rule followed by 20 rounds of the total rule (the baseline rule order) or the reverse rule order was not much of a factor in subjects' bidding behavior. On the other hand, Table 5 also reveals that the coefficient on the δ^A dummy variable is frequently significantly different from zero (in 5 out of the 8 bid regressions), indicating that the presentation of the four values in ascending order as opposed to descending order (the baseline) has some effect on bidding behavior. In particular, there is some evidence that the ascending order increases bids on the higher value items while decreasing bids on the lower value items, suggesting that players "save" in their initial bids for items and spend more on later items in the list. While the latter finding is surprising as it is not predicted by the theory, a mitigating factor is that the impact of the prize order on bids is, economically speaking, rather small amounting to a change of less than 5 tokens (or 4.2% of a player's budget) in all cases. Nevertheless, the latter finding suggests that we were correct to consider some variation in the prize order so that overall mean bids as presented in Figs. 2 and 3 and Table 4 take account of the changes in bids that can be induced by variations in the presentations of the prize values.

The next findings address in further detail some of the comparative statics predictions of the theory, according to whether the majority or total rule was in place.

Finding 3. Under the majority rule, consistent with the theoretical predictions there is near zero spending on an item that can never be pivotal. Spending on an item with the same absolute value is higher when that item can be pivotal.

Under the majority rule treatment, the item with the lowest prize value of 10 can never be pivotal when the prize values are as given in Version 1 and thus bids for this item should be zero. Table 4 reveals that bids for the item with prize value 10 under Majority Rule Version 1 are indeed low, averaging 4.26 over all rounds and just 2.92 over the last 5 rounds of this

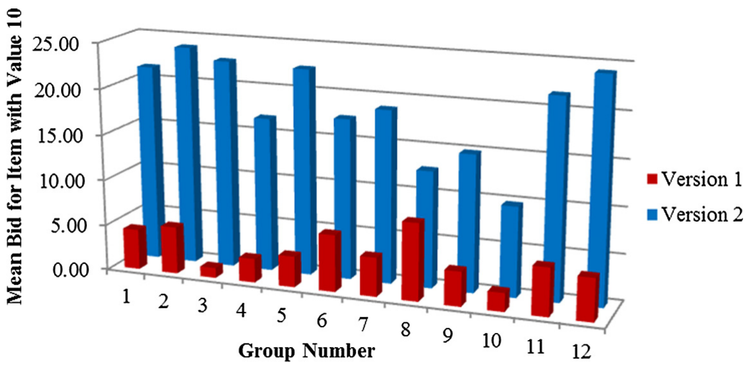

Fig. 4. Mean bids over all 20 rounds for the item with value 10: each of the 12 groups for the majority Version 1 treatment compared with each of the 12 groups for the majority Version 2 treatment.

Table 6
Pairwise bidding companions, majority treatments.

Null hypothesis	Majority Rule Version 1			Majority Rule Version 2		
	$\bar{b}_{35} = \bar{b}_{30}$	$\bar{b}_{30} = \bar{b}_{25}$	$\bar{b}_{35} = \bar{b}_{25}$	$\bar{b}_{25} = \bar{b}_{20}$	$\bar{b}_{20} = \bar{b}_{10}$	$\bar{b}_{25} = \bar{b}_{10}$
WSRT p-value	0.0386	0.0063	1.0000	0.3877	0.7744	0.1460
WSRT L5 p-value	0.3877	0.0386	0.7744	0.3877	0.3877	0.7744

treatment. While these bid amounts are different from zero, it is instructive to compare bids for the item with value 10 under both versions of the Majority rule treatment. In particular, under the Majority Rule Version 2, the lowest valued item also has a prize value of 10 and can be pivotal. Thus bids for this prize should be strictly positive (the Nash prediction is a bid of 20). Consider the 12 independent group-level mean bids over all 20 rounds for the prize with value 10 under Majority Rule, Version 1 as reported in Table 10 and under the Majority Rule, Version 2 as reported in Table 12 (see Appendix B, online supplementary materials). For convenience these bids are also presented in Fig. 4.

This figure makes it clear that bids for the item with value 10 under the Majority Rule Version 1 are stochastically dominated by bids for the item with value 10 under the Majority Rule Version 2. Equivalently, bids for the three items with values above 10 are stochastically greater under Majority Rule Version 1 as compared with Majority Rule Version 2. Indeed a Wilcoxon Mann-Whitney test using the 12 independent (group) observations for each treatment condition confirms that we can reject the null hypothesis of no difference in bids for the item with value 10 (or bids on all items with values greater than 10) in favor of the alternative that bids for the item with value 10 (or bids for all items greater than value 10) are higher (lower) under Version 2 than under Version 1 ($p < 0.01$, one-sided test). Note that this same finding holds for *any* of the 7 mean bid subsamples reported in Tables 10 and 12 of Appendix B, online supplementary materials.[12] This is strong evidence that pivotality concerns play an important role as identified in the theory.

Finding 4. Under the majority rule, consistent with the theoretical predictions, we observe significantly higher spending on items that can be pivotal more often, e.g., item 1 in Version 2.

Using a Wilcoxon Signed Ranks test for matched pairs on mean bids from the Majority Rule Version 2 treatment (see Table 12, online Appendix B) we test whether the 12 independent (group) mean bids (over all 20 rounds) for the item with value 45 are significantly greater than mean bids for each of other three items having values 25, 20 and 10, respectively. We find that we can easily reject the null hypothesis of no difference in mean bids on the item with value 45 versus each of the items having values 25, 20 or 10, respectively, in favor of the alternative that bids are higher for the item with value 45 ($p < 0.01$ for all three pair-wise comparisons, one-sided test).

Finding 5. Under the majority rule, we observe roughly similar levels of spending on all items that can be pivotal the same number of times, e.g., items 1–3 under Version 1 and items 2–4 under Version 2, albeit with some variance, especially in the case of Version 1.

Support for this finding comes from Table 6 which reports the results of several Wilcoxon Signed Ranks test for matched pairs on bid data from the two Majority Rule treatments. The test is performed on matched pairs of the group level bid averages for two items ($\bar{b}_i$) over all rounds or over the last 5 rounds only (L5). The table reports the p-value from various pairwise null hypotheses of no difference in bidding behavior (two-sided tests in all cases).

We observe that for Majority rule Version 2, the null hypothesis of no difference in average bids between items with values 25, 20, and 10 is never rejected ($p > 0.10$ in all pairwise comparisons). However, for Version 1, we observe that there is

[12] That is, this finding holds not only for all rounds played but also for the first round, for rounds 1–5, rounds 6–10, etc.

Table 7
p-Values from pairwise tests of the null hypothesis of no difference in bid amounts between treatments for the prize with the lowest value of 10. Tests are performed on session-level bid averages over all 20 rounds.

Treatment and prediction	Total V1, $\bar{b}_{10} = 12$	Total V2, $\bar{b}_{10} = 12$	Majority V2, $\bar{b}_{10} = 20$
Majority V1, $\bar{b}_{10} = 0$	0.0027[a]	0.0005	0.0000
Total V1, $\bar{b}_{10} = 12$		0.4356[b]	0.0001
Total V2, $\bar{b}_{10} = 12$			0.0013[a]

[a] Wilcoxon signed rank test; all other tests are Wilcoxon Mann Whitney.
[b] Two-sided test; all other tests are one-sided.

excess bidding on the item with prize value 30 relative to the items with prize values 35 and 25, respectively. The difference in mean bids for items with prize values 35 and 30 ceases to be significant by the final 5 rounds though the difference in mean bids for the two prizes with values 30 and 25 remains significant even in the final 5 rounds. The excessive bids for the item with value 30 relative to the items with values 35 and 25 which are equally pivotal may reflect a strategic calculation that one's opponent is more likely to bid for the item with the highest (35) or lowest (25) prize value among the three items that are equally pivotal toward winning a majority of item values in this treatment.

Finding 6. Under the total rule, consistent with theoretical predictions, we generally find that (1) mean bids $\bar{b}_i$, are positive for all items and (2) (over all 20 rounds) under Version 1 we generally have that $\bar{b}_{35} > \bar{b}_{30} > \bar{b}_{25} > \bar{b}_{10}$ while under Version 2 we have that $\bar{b}_{45} > \bar{b}_{25} > \bar{b}_{20} > \bar{b}_{10}$, that is, token allocations are, on average, increasing with item values.

Specifically, using the data for all rounds as reported in Tables 11 and 13 Appendix B, online supplementary materials, this ordering is observed to hold for 10 of the 12 groups under Version 1 and for all 12 groups under Version 2. Where it breaks down under the total rule for Version 1 is for groups 3 and 6 (see Table 11, online Appendix B) where overall average bids for the prize with value 35 are less than overall average bids on the prize with value 30, though average bids on the highest prize of 35 exceeded average bids on the two lowest prizes with values 25 and 10 in these two sessions. This finding is again suggestive of some type of strategic avoidance of bidding on the highest valued item in this treatment. However, we note that under Version 1, there is not as large a difference between the valuations of the first two prizes (a difference of just 5) and this small difference may have also played a role in bidding behavior. When the difference in valuations between the highest and second highest items is more substantial as in Version 2 (a difference of 20) evidence of strategic avoidance of bidding on the highest valued item under the total rule disappears completely.

The lowest prize, with a valuation of 10 is the same across all four of our treatments and is therefore a natural focus for comparisons across all treatments. Under the total rule, bids on this item are predicted to equal 12 under both treatments (Versions 1 and 2). Under the majority rule Version 1, as previously discussed in Finding 3 bids on this item should be 0 while under the majority rule Version 2, bids on this item should be 20.

Finding 7. Consistent with theoretical predictions, bids for the lowest prize having a common value of 10 in all four treatments are: (a) not significantly different from one another under both Versions 1 and 2 of the total rule, (b) significantly lower under majority rule Version 1 than for the other three treatments and (c) significantly higher under the majority rule Version 2 than for the other three treatments.

Support for this finding comes from pairwise Wilcoxon signed-rank tests or Mann-Whitney tests (as appropriate) using the 12 independent (group) observations on average bids for the item with value 10 for each of the four main treatments over all 20 rounds as reported in Tables 10–13, Appendix B, online supplementary materials. The *p*-values from the pairwise tests are summarized in Table 7. In all but one case, we can reject the null hypothesis of no difference in favor of the alternative directional prediction of the theory. The one case where we cannot reject the null hypothesis is in the comparison between the Total Version 1 and Total Version 2 treatments, where consistent with the theory, bids on the item with prize value 10 are predicted to be exactly the same (12); the fact that we cannot reject the null hypothesis in this case is thus also consistent with the theoretical prediction.

Having examined the behavior of *mean* bids across treatments we next consider the distribution of *individual* bids across our four treatments so as to assess whether these distributions also conform to predictions of the theory.

Finding 8. The distribution of individual bids is not degenerate at equilibrium predictions. However, these bid distributions are ordered in such a way as to be consistent with the comparative statics predictions of the theory.

Support for Finding 8 can be found in Figs. 5 and 6 which show the cumulative distribution functions (CDFs) of bid amounts between 0 and 120 tokens for each of the four prizes in each of the four treatments.

The CDFs presented in Figs. 5 and 6 reveal that the distribution of bid amounts for the four prizes do not correspond precisely with theoretical predictions. However, the observed *differences* between the bid distributions for the four prizes are strikingly consistent with the comparative statics predictions of the theory. More precisely, consider the CDFs for bid amounts under prize vector 1 as shown in Fig. 5. Under the majority rule (left panel of Fig. 5) equilibrium bids for the 3 highest prize values should all be 100 percent at a bid of 40. While these three bid distributions are clearly not degenerate at 40, the bid distributions for the prizes with values 35, 30, and 25 are all centered around 40 and are similar to one another. By contrast, under the majority rule the equilibrium bid for the lowest prize with a value of 10 should be 0 and indeed there

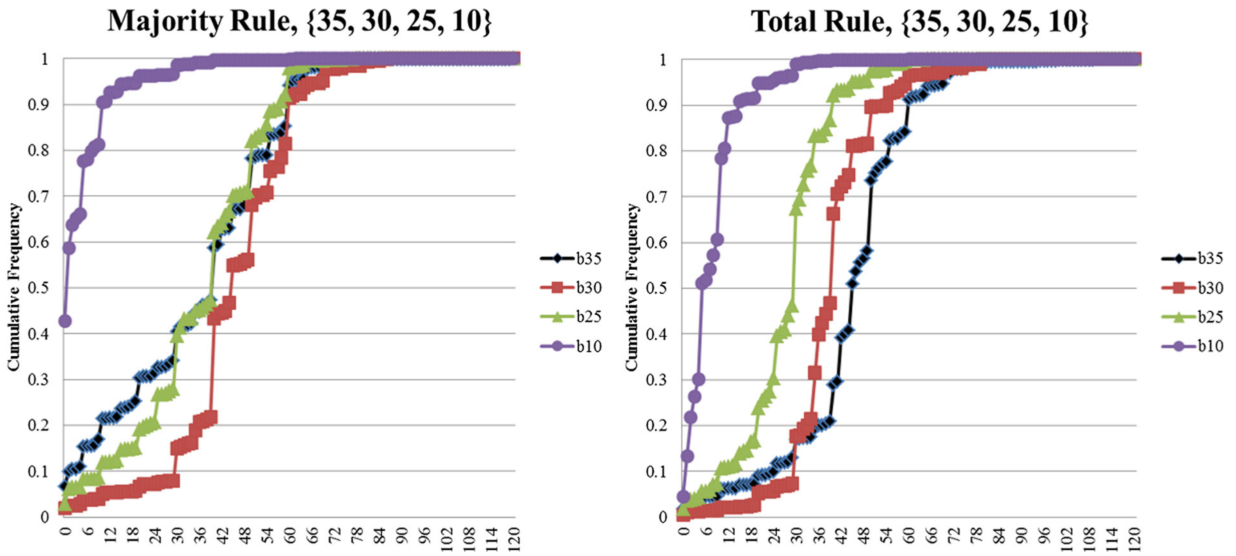

Fig. 5. Cumulative distribution of bid amounts on the four items under the majority and total rules, Version 1 {35, 30, 25, 10}.

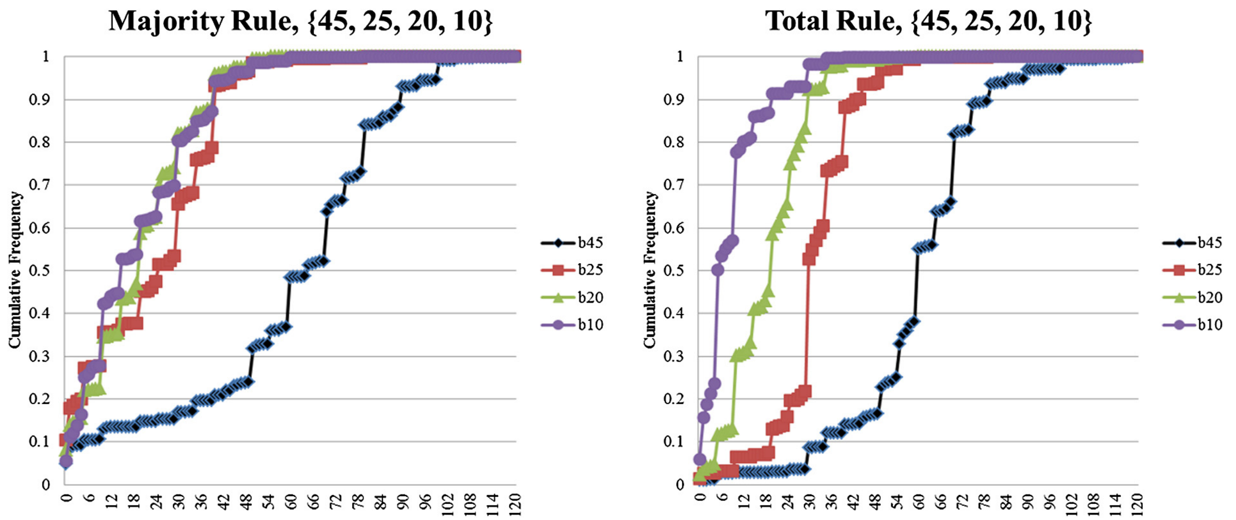

Fig. 6. Cumulative distribution of bid amounts on the four items under the majority and total rules, Version 2 {45, 25, 20, 10}.

is a large mass of bids (42.7%) at a bid of 0. Importantly, the bid distribution for this lowest value prize is clearly distinct from the bid distributions for the 3 highest value prizes. Consider next the distribution of bids under the total rule for prize vector 1 (right panel of Fig. 5). These distributions are all rather distinct from one another and reflect the monotonic prediction between bids and prize amounts for this treatment. Similar results are found in the CDFs of bids under prize vector 2 as shown in Fig. 6. Under the majority rule (left panel of Fig. 6) equilibrium bids for the 3 lowest value prizes should all be 100 percent at a bid of 20. While the distributions are clearly not degenerate at 20, they are closely clustered together and centered around 20. By contrast equilibrium bids for the highest prize with a value of 45 should all be at 60. While the distribution of bids for this highest value prize is not degenerate at 60, the distribution of bids for this prize is clearly distinct from the distribution of bids for the other three lower valued items. Under the total rule for Version 2 (right panel of Fig. 6) bid distributions are again distinct from one another and correspond precisely to the monotonic prediction that higher prize values are associated with higher bids.

Using the data presented in these CDFs, we also report on the frequencies with which we observed equilibrium bids in our data. Specifically, for each treatment and for each item, we report in Table 8 the frequencies with which players bid exactly the equilibrium predicted amounts, which are also shown in the table.

We generally find some small mass of bids at these predicted equilibrium frequencies, especially when the equilibrium prediction is a multiple of 5. This mass becomes even greater if one allows for bids that are close to but not precisely equal

Table 8
Frequencies of bids that equaled Nash equilibrium bid predictions.

Treatment conditions		Item number			
		1	2	3	4
Majority Version 1	Eq. bid	40	40	40	0
	Freq.	0.11	0.22	0.15	0.43
Total Version1	Eq. bid	42	36	30	12
	Freq.	0.10	0.08	0.21	0.07
Majority Version 2	Eq. bid	60	20	20	20
	Freq.	0.12	0.07	0.12	0.08
Total Version 2	Eq. bid	54	30	24	12
	Freq.	0.01	0.31	0.02	0.02

to the equilibrium bids as the CDFs make clear. For instance, for item 1 of the Total Rule, Version 2, if we consider bids of 54–55, the frequency of observed bids in this interval jumps to 0.09 from 0.01 for bids of exactly 54.

Finally, we discuss the adjustment of bids over time. At the aggregate level, there is some evidence of learning over time in comparisons of the mean bids made by groups over the first rounds 1–5 and the mean bids made by these same groups over the final rounds, 16–20, using the data of Tables 10–13, Appendix B, online supplementary materials. For example, consider mean group bids for the prize with value 35 under the Majority Rule, Version 1 as reported in top panel of Table 10, Appendix B online supplementary materials. The predicted Nash equilibrium bid for this item is 40. Notice that over the first five rounds (Rnds 1–5) 10 of the 12 groups have mean bids below 40 while only two groups have mean bids above 40. Of the 10 groups with mean initial bids below 40, 7 of these 10 groups had increased mean bids for this same item over the last 5 rounds (Rnds 16–20). Of the 2 groups with mean bids initially above 40, 1 had decreased its mean bid by the final 5 rounds. Thus, 8 of the 12 groups who bid for this item in this treatment – a majority – exhibit some evidence of aggregate equilibration toward Nash equilibrium bids over time. Carrying out a similar analysis for all other prize value/treatment conditions we can report that for at least 6 of the 12 groups, if the average group bid for an item over the first rounds 1–5 was below (above) the predicted Nash equilibrium bid for that item then the average bid by that same group for that same item over the last rounds 16–20 was higher (lower) in 14 of the 16 prize/treatment conditions reported on in Tables 10–13, online Appendix B.[13] This finding provides some evidence of aggregate equilibration toward equilibrium bids.

We look for further evidence of learning behavior by exploring how *individual* bids differed from Nash equilibrium bids over time. Recall from Finding 1 that NE bids remain a best response to the actual mean bids in all four treatments of our experiment; hence the mean squared deviation (MSD) of individual bids from NE bids is a good measure of each subject's deviation from best response behavior. We define the mean squared deviation of individual i's 4-element bid vector in period t from the vector of Nash equilibrium bids by $MSD_i(t) = \frac{1}{4}\sum_{j=1}^{4}(b_{i,j}(t) - b_j^{ne})^2$, where b_j^{ne} refers to the Nash equilibrium bid for item j, which depends on the prize vector, Version 1 or 2, and the rule, majority or total that was in place in period t. Using this MSD variable as the dependent variable, we ran GLS random effects regressions of the same form as equation (11), using the same explanatory variables described earlier in connection with Table 5, again with robust standard errors clustered on an individual's group membership. The results are reported in Table 9. Regressions using the data from both the Majority and Total Rule treatments are under the heading "All data"; we also report on separate regressions of the MSD under either the "Majority" rule alone or the "Total" rule alone for each prize vector (thus excluding the δ^T variable).

The regression results reported in Table 9 indicate that the mean squared deviations of individual's bids from NE predictions are significantly lower under the total rule treatment as compared with the majority rule treatment for both prize vectors, Versions 1 and 2. Indeed, the rule change seems to be the most significant factor in explaining the MSDs, judging from the large impact that the rule change has on the MSD. Evidence for this can be found in the significantly negative coefficient attached to the δ^T dummy variable in Table 9. We speculate that MSDs are higher under the Majority rule because the equilibrium requires that subjects consider the pivotality of each item, which can be difficult to calculate. By contrast, equilibrium bids under the Total rule are proportional to prize values and those calculations are easier for subjects to make, as well as being more intuitive. Other explanatory variables that sometimes play a significant role in these regressions are the dummy variable for whether the prize values were presented in ascending order and, in one instance only, the round number. An ascending prize order has a marginally negative impact on MSD, particularly under Version 2 where the ascending prize order is 10, 20, 25, 45. The coefficient on the round number is almost always negative suggesting that there is a slight reduction in MSD over time, but this coefficient is only significantly negative under the total rule for prize vector 1 and the coefficient is small in magnitude.

[13] The exceptions are for a prize value of 10 under the Majority rule, Version 2 (Table 12, online Appendix B) and for a prize value of 45 under the Total rule, Version 2 (Table 13, online Appendix B). In these two cases only 5 of the 12 groups exhibit evidence for equilibration in terms of the difference in their mean bids over the first and last 5 rounds.

Table 9
Regression analysis of MSDs of bids from NE predictions in Version 1 (left columns) or Version 2 (right columns) all data from all sessions.

	Version 1: {35, 30, 25, 10}			Version 2: {45, 25, 20, 10}		
	MSD of bids from NE pred.			MSD of bids from NE pred.		
	All data	Majority	Total	All data	Majority	Total
Constant	302.07***	274.38***	208.28***	404.97***	396.07***	223.03***
	28.16	40.87	27.95	53.28	52.37	53.09
δ^T	−121.08***			−190.84***		
	23.32			29.17		
δ^{RO}	40.54	70.07	11.27	20.76	43.37	45.70
	34.56	45.21	34.29	57.97	77.75	−0.04
δ^A	−63.77	−41.61	−86.19*	−107.54*	−142.45*	−72.63
	34.51	45.21	34.32	57.96	77.62	45.70
Round(t)	−1.82	−1.64	−1.97*	−0.10	1.33	−1.53
	1.13	1.98	1.08	1.67	2.74	1.78
R^2	0.09	0.03	0.05	0.13	0.05	0.03
Nobs	2880	1440	1440	2880	1440	1440

***,**,*, indicate significance, respectively, at the 0.01, 0.05, and 0.10 levels.

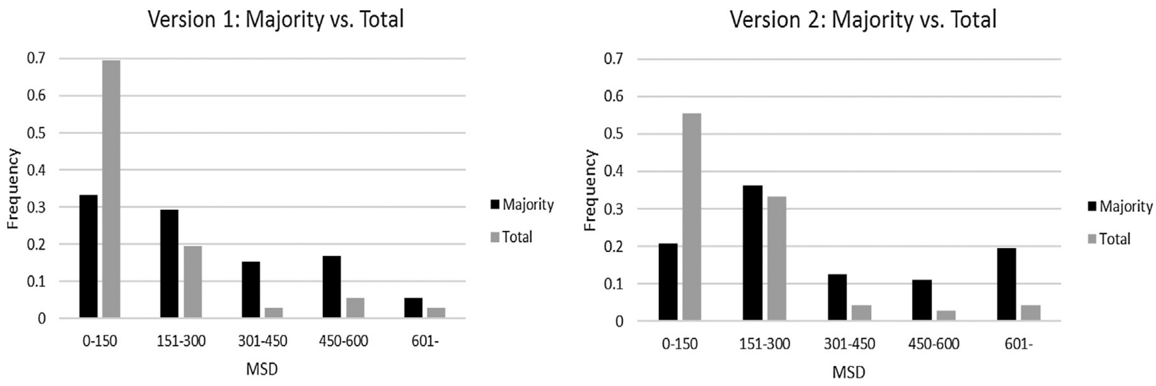

Fig. 7. Distribution of mean squared deviations (MSDs) from NE bids, majority vs. total, Version 1 (left), Version 2 (right).

In addition to a regression analysis involving MSDs, we also considered the distribution of MSDs between the two treatments, Majority and Total, and whether there is any correlation in subjects' MSD as we move from one treatment to the other all under the same vector of prize values (Version 1 or Version 2). Figure 7 shows the distribution of MSDs under the Majority and Total rules for Versions 1 and 2, respectively. Consistent with the results reported in Table 9, we observe that the MSDs under the Total rule are skewed more to left (closer to a MSD of 0) than are the MSDs under the Majority rule. This observation lends further support to the notion that subjects' had a more difficult time playing a best response under the Majority rule than under the Total rule. We also find that there is a correlation between each subject's MSD under the Majority rule and their MSD under the Total rule (again, for a given version of the prize vector). Specifically, for Prize Vector 1, the correlation coefficient, 0.52, is positive and statistically significant ($p < 0.01$, $N = 72$) while for Prize Vector 2, the correlation coefficient, 0.51, is also positive and statistically significant ($p < 0.01$, $N = 72$). Thus, subjects who were closer to (further from) playing best responses under one rule are also likely to be closer to (further from) from playing best responses under the other rule. Taken together, these results suggest that much of the adjustment in subjects' bids is a consequence of the rule change alone with very little modification to bids in response to experience or other factors under a given rule. We summarize these findings as follows:

Finding 9. Individual bids are significantly closer to equilibrium bids under the total rule than under the majority rule. Individual bids under both rules adjust only slowly toward NE bids with experience. Individual deviations from NE bids under the two different rules are positively correlated.

6. Summary and conclusions

The stochastic, asymmetric Blotto game has many applications, e.g., to warfare, advertising and political campaigns. In this paper we present results from an experimental study of this version of the Blotto game under two commonly used objective functions: a majority rule objective and a total expected payoff objective. The majority rule objective is particularly relevant to understanding electoral competitions in two party systems, e.g., the electoral college system for electing the U.S. president, while the total expected payoff version is relevant to understanding competition between duopoly firms for market share.

Despite the seeming similarity between the two objective functions, equilibrium bid allocations under the majority rule objective are quite different than under the total expected payoff objective. In particular, for the equal budget constraint case that we study, bids for each item under the majority rule objective are proportional to the Banzhaf index of an item's power. By contrast, bids under the total rule are proportional to the relative value of each item.

To test these theoretical predictions, we report the results of a laboratory experiment comparing bidding behavior in stochastic, asymmetric 4-item Blotto games under the majority rule objective with bidding for the same items under the total rule objective using a within-subjects design. We consider two different prize vectors so as to further test some of the comparative statics implications of the theory. Our experimental results are shown to be qualitatively (if not perfectly quantitatively) consistent with the theoretical predictions for how players should allocate their bids across the four items, confirming that the differing payoff function objectives matter for allocations.

Future research on this topic might proceed in several dimensions. First, one could attempt to incorporate some other potentially important features of the U.S. electoral college system (that we have left out) for instance, the fact that certain states (items) are ex-ante more likely to be won by one player or the other, or relaxing the assumption that the winner of an item gets all of that items' value.[14] Another possible extension would be to consider super-majority rules and examine how allocations are affected relative to the majority rule case. We leave these extensions to future research.

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Appendix A. Supplementary data

Supplementary data associated with this article including experimental instructions can be found, in the online version, at http://dx.doi.org/10.1016/j.jebo.2017.05.005.

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[14] Of the 51 states in the electoral college, all but two currently assign all of their electoral votes to the winner of the state. The two exceptions, Maine and Nebraska, assign electoral votes in a more proportional manner: 1 electoral vote is awarded to the winner of each Congressional district within the state and the remaining 2 electoral votes are awarded to the state-wide winner.

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