Model Solutions for Odd-Numbered Problems in Section 2.4

Problem 2.4.1 Prove that for all vectors \( u \) and \( v \) in \( V \),

\[ \|u + v\|^2 + \|u - v\|^2 = 2(\|u\|^2 + \|v\|^2). \]

Proof Let \( u \) and \( v \) be vectors in \( V \). Then

\[ \|u + v\|^2 + \|u - v\|^2 = \langle u + v, u + v \rangle + \langle u - v, u - v \rangle = 2\langle u, u \rangle + 2\langle v, v \rangle = 2(\|u\|^2 + \|v\|^2). \]

Problem 2.4.3 (The measure of a straight angle is \( \pi \).) Let \( p, q, r \) be (distinct) collinear points, and suppose that \( q \) is between \( p \) and \( r \) (i.e., \( \overrightarrow{pq} = \lambda \overrightarrow{pr} \) with \( 0 < \lambda < 1 \)). Show that \( \angle(p, q, r) = \pi \).

Proof Since \( \overrightarrow{pq} = \lambda \overrightarrow{pr} \), we have \( \overrightarrow{qp} = -\lambda \overrightarrow{pr} \) and \( \overrightarrow{qr} = (1 - \lambda) \overrightarrow{pr} \). Hence,

\[ \langle \overrightarrow{qp}, \overrightarrow{qr} \rangle = \langle -\lambda \overrightarrow{pr}, (1 - \lambda) \overrightarrow{pr} \rangle = -\lambda(1 - \lambda)\|\overrightarrow{pr}\|^2 \]

and, since \( 0 < \lambda < 1 \),

\[ \|\overrightarrow{qp}\|\|\overrightarrow{qr}\| = \| -\lambda \overrightarrow{pr}\|\|(1 - \lambda) \overrightarrow{pr}\| = \lambda(1 - \lambda)\|\overrightarrow{pr}\|^2. \]

It follows that

\[ \cos(\angle(p, q, r)) = \frac{\langle \overrightarrow{qp}, \overrightarrow{qr} \rangle}{\|\overrightarrow{qp}\| \|\overrightarrow{qr}\|} = -1. \]

The only number between 0 and \( \pi \) whose cosine is \(-1\) is \( \pi \). So, \( \angle(p, q, r) = \pi \).

Problem 2.4.5 (Right Angle in a Semicircle Theorem) Let \( p, q, r \) be (distinct) points such that (i) \( p, o, r \) are collinear, and (ii) \( \|\overrightarrow{op}\| = \|\overrightarrow{or}\| = \|\overrightarrow{dr}\| \). (So \( q \) lies on a semicircle with diameter \( LS(p, r) \) and center \( o \).) Show that \( \overrightarrow{qp} \perp \overrightarrow{qr} \), and so \( \angle(p, q, r) = \frac{\pi}{2} \).

Proof By (i), we have \( \overrightarrow{op} = a \overrightarrow{or} \) for some \( a \). Hence, \( \|\overrightarrow{op}\| = |a| \|\overrightarrow{or}\| \) and, therefore, by (ii), \( |a| = 1 \). Now \( a \) cannot be \( 1 \). For if \( \overrightarrow{op} = \overrightarrow{or} \), then

\[ \overrightarrow{op} = \overrightarrow{or} + \overrightarrow{rp} = \overrightarrow{op} + \overrightarrow{rp}. \]
And so it would follow that $\vec{r}_p = 0$, which is impossible since $p$ and $r$ are distinct. So $a = -1$ and $\vec{a}_o = -\vec{o}_o$. This implies that

$$\vec{q}_r = \vec{q}_o + \vec{o}_r = -\vec{o}_q - \vec{o}_p.$$ 

We also clearly have

$$\vec{q}_p = \vec{q}_o + \vec{o}_p = -\vec{o}_q + \vec{o}_p.$$ 

Hence, by (ii) again,

$$\langle \vec{q}_p, \vec{q}_r \rangle = \langle -\vec{o}_q + \vec{o}_p, -\vec{o}_q - \vec{o}_p \rangle = \langle -\vec{o}_q, -\vec{o}_q \rangle - \langle \vec{o}_p, \vec{o}_p \rangle$$

Thus, $\vec{q}_p \perp \vec{q}_r$ and

$$\cos(\angle(p,q,r)) = \frac{\langle \vec{q}_p, \vec{q}_r \rangle}{\|\vec{q}_p\| \|\vec{q}_r\|} = 0.$$ 

The only number between 0 and $\pi$ whose cosine is 0 is $\frac{\pi}{2}$. So, $\angle(p,q,r) = \frac{\pi}{2}$. □