Multiple Equilibria in Heterogeneous Expectations Models

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Abstract

This paper fills an important gap in the literature on determinacy and existence of sunspot equilibria in stochastic linear self-referential models. The results in this paper demonstrate that heterogeneity in expectations may alter a model's regions of determinacy. We show how to associate with a heterogeneous expectations model (HE-model) a rational expectations model (ARE-model), the solutions to which are the equilibria of the HE-model. This association recasts the analysis of determinacy in the HE-model to analysis of determinacy in the ARE-model. We proceed to study a forward looking model and find that the stability properties of rational expectations models may not be robust. In particular, we present new results showing that in some models even a very small fraction of non-rational agents may preclude rational agents from coordinating on sunspots. Also, we present parameterizations of the model in which sunspots exist in the heterogeneous model when they do not exist in the rational expectations version of the model.

KEYWORDS: Heterogeneous Beliefs, Adaptive Expectations, Rational Expectations, Sunspot Equilibria

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1 Introduction

It is well-known that forward-looking linear rational expectations models may exhibit sunspot equilibria, and moreover, these sunspot equilibria may be stable under learning (Evans and Honkapohja 2002). These issues are not just theoretical curiosa. A large literature has developed studying indeterminacy in RBC-type models (Farmer 1999), and recently there has been considerable interest in whether monetary policy can rule out the existence of sunspot equilibria.[1] An important open issue, however, is whether sunspot equilibria which exist under rational expectations also exist under heterogeneous expectations. This paper addresses whether the existence of sunspot equilibria in rational expectations models is robust to heterogeneity in expectations.

Most research involving dynamic macroeconomic models imposes rational expectations and thus the assumption that agents' expectations are homogeneous. However, recent empirical evidence indicates that expectations are heterogeneous. (Branch 2004) and (Carroll 2003) find that survey data suggest agents have a variety of methods which they use to form beliefs about future inflation. Heterogeneity across demographic groups was noted by (Bryan and Venkatu 2001a,b). In addition, recent theoretical work demonstrates that heterogeneity may arise in models where agents weigh the benefits and costs of various predictors. (Brock and Hommes 1997) show that when agents choose between costly rational and costless naive expectations, agents will be distributed across both predictors across all time periods. Previous work, though, has not studied the connection between the number and nature of equilibria under rational and heterogeneous expectations.

Because the evidence for heterogeneous expectations is so compelling, we address its dynamic effects on an economy. We are primarily interested in the impact of heterogeneity on the existence of sunspot equilibria. We define a *Heterogeneous Expectations Equilibrium* (HEE) as a non-explosive solution to the reduced form expectational difference equation given that expectations are formed heterogeneously. We then show how to write a heterogeneous expectations model (HE-model) as an associated rational expectations model (ARE-model), the solutions to which are also solutions to the HE-model. We show that re-casting heterogenous expectations models into a rational expectations framework is possible so long as heterogenous expectations can be written as a linear combination of the various forecasting approaches agents use. Thus we are able to test for the existence of sunspot equilibria in the usual way: by analyzing whether the ARE-model is determinate or indeterminate. Our

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[1]See (Woodford 2003) for an extensive overview.

methodology, adapted from (Blanchard and Khan 1980), is sufficiently general that it can be applied to most linear stochastic self-referential models.

We find that imposing heterogeneous expectations may alter the regions of the parameter space corresponding to determinacy and indeterminacy. To illustrate this point we develop the analysis in general and then focus on the simplest heterogeneous expectations model; we assume a fixed proportion of agents are rational and the remaining are simple adaptive. We then compare the regions of determinacy, indeterminacy and explosiveness in the HE-model with those of same model under the assumption that all agents are rational (the RE-model).[2] The determinacy of the HE-model depends on the weight assigned to each predictor and the weight assigned to past information in the adaptive predictor. When the adaptive predictor is naive, and when there is a unique equilibrium in the RE-model, then there will be a unique HEE. When there are multiple equilibria in the RE-model there will also be multiple HEE only if a majority of agents are rational, otherwise the model is explosive. We show if adaptive agents are mean-reverting then it may be possible for there to be a unique equilibrium in the HE-model even though under rational expectations sunspot equilibria exist; thus heterogeneity may have a stabilizing influence.[3] Finally, we find that if adaptive agents are trend-setting then the HE-model may exhibit sunspot equilibria even though under rational expectations the equilibrium will be unique; thus heterogeneity may be destabilizing.[4]

This paper proceeds as follows. Section 2 presents the theory of heterogenous expectations and shows how a Heterogenous Expectations Equilibrium can be represented in terms of an Associated Rational Expectations Equilibrium. Section 3 presents results on determinacy and indeterminacy in the heterogeneous expectations model. Finally, Section 4 concludes.

2   Heterogeneous Expectations Models

Informally, a heterogeneous expectations model is a reduced form model with fewer restrictions placed on the associated expectations operator. In this section we more formally define the notion of an HE-model, we show how to analyze its equilibria via an associated rational expectations model, and we

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[2]Note that the RE-model is distinct from the ARE-model, which incorporates the heterogeneity of the HE-model.

[3]Economies exhibiting sunspot equilibria are often described as unstable because agents may coordinate on one of many possible equilibrium paths.

[4]Though the results section focuses on a univariate model, it seems very likely that similar results will hold in a multivariate setting.

briefly discuss some of the subtleties that arise with our definition.

2.1 Heterogenous Expectations Equilibria

We consider univariate reduced form models with the following recursive expectational structure:

$$z_t = \hat{A}\hat{E}_t z_{t+1} + v_t. \tag{1}$$

Here v_t is a zero mean exogenous i.i.d. process. Underlying this "reduced form model" is assumed to be a finite number of agent types with α_i representing the proportion of agents of type i. We think of $\hat{E}$ as an aggregate expectations operator, that is, a weighted average of individual expectations operators: $\hat{E}_t = \sum_i \alpha_i \hat{E}_t^i$, where $\hat{E}_t^i$ captures the expectations of agents of type i. The expectations operator $\hat{E}_t^i$ should be thought of as a function of I_t, the information available to agents at time t (we assume informational symmetry).[5,6] We restrict attention to models of the form (1) for simplicity and to obtain straightforward and precise results in Section 3; however, as will be clear below, our method of analysis extends to multivariate settings.

Two brief remarks are warranted. Here we assume that one component of $\hat{E}_t$ captures rational expectations, that is, one component is the conditional expectations operator. We also assume that the remaining components are adaptive or 'backwards looking' expectations. These restrictions insure that equation (2) below takes the form of a standard RE model.

Notice that, in case $\hat{E}_t$ is the conditional expectations operator, the HE-model above reduces to a familiar rational expectations model. In particular, the Cagan model of inflation has this form. Furthermore, the homogeneous expectations version of this reduced form model has been studied extensively in case of bounded rationality by (Evans and Honkapohja, 2001) among others. In writing our model (1), we have replaced the usual conditional expectations operator with a more general aggregate expectations operator. This replacement, while arguably reasonable, relies on the assumption that heterogeneity in expectations at the behavioral level aggregates in a straightforward manner

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[5]More carefully, $\hat{E}_t^i$ should be a function which is measurable with respect to the σ-algebra generated by the information available at time t. This is required so that its conditional expectation can be evaluated by rational agents.

[6]Having defined a general expectations operator as a function of available information, an interesting question arises: what restrictions should be placed on these functions so that agents have expectations which are, in some sense, reasonable? Addressing this question requires an axiomatic approach to defining general expectations operators, and would lead us astray from the central point of the current paper; therefore we address this issue much more carefully in a companion paper.

to heterogeneity in expectations at the reduced form level. For a discussion of this issue, see (Preston 2003).

A (linear, one-step-ahead, forward looking) *Heterogeneous Expectations Model* (HE-model) is a 5-tuple $(v_t, \hat{\Omega}, I_t, \hat{E}_t, (1))$ where v_t is an exogenous sequence of random vectors, $\hat{\Omega} = (\Omega, \mathcal{B}, P)$ is a probability space representing the stochastic process v_t, I_t is an increasing sequence of information sets, $\hat{E}_t$ is an aggregate expectations operator, and (1) is the expectational difference equation. Abusing notation slightly, we will typically identify an HE-model with its expectational difference equation.

A *heterogenous expectations equilibrium* (HEE) is any non-explosive solution to the expectational difference equation, given a particular aggregate expectations operator.[7] In this paper we are primarily interested in comparing the number of equilibrium sequences under HE to the number of equilibrium sequences under rational expectations (RE). We, thus, extend the notion of uniqueness under RE to the HE-model. An HE-model is said to be determinate if there exists a unique non-explosive solution to (1). We now show that it is possible to write the HE-model (1) in terms of an associated rational expectations model. Solutions to the ARE-model will be solutions to the HE-model and thus it is possible to apply (Blanchard and Kahn 1980) to determine the conditions for determinacy.

Assume agents of type 1 are rational. Then

$$\hat{E}_t = \alpha_1 E_t + \sum_{i \geq 2} \alpha_i \hat{E}_t^i.$$

Recall I_t is the collection of variables known at time t. Because of our assumption on the expectations operators of agents, we may then define

$$\hat{F}(I_t) = \hat{E}_t z_{t+1} - \alpha_1 E_t z_{t+1}$$

as the 'wedge' between the aggregate expectations and rational expectations. Inserting this equation into the reduced form (1) yields

$$z_t = A E_t z_{t+1} + F(I_t) + v_t \tag{2}$$

for appropriately defined A and function F. Equation (2), together with the information sets I_t, defines a possibly non-linear rational expectations model,

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[7]The notion of "non-explosive" can be made formal: for example one may require the process be weakly asymptotically stationary or uniformly bounded in conditional mean. See (Evans and McGough 2003) for details. The specific notion of non-explosiveness chosen will not impact our results.

possibly with lags, which we call the rational model associated to the HE-model or the ARE-model. Notice that a rational expectations equilibrium of the ARE-model (2) is a heterogeneous expectations equilibrium of the HE-model (1).

Writing the equilibrium path in terms of the 'forward-looking' ($E_t z_{t+1}$) and 'backward-looking' ($F(I_t)$) components highlights the main tension in heterogeneous expectations models. The equilibrium path can be influenced either by the anticipatory behavior of rational agents or the reflective beliefs of adaptive agents. It is well known that in homogeneous expectations models these belief mechanisms may yield very different dynamic equilibrium paths. The approach of writing a rational representation of the heterogeneous expectations equilibrium allows us to use standard tools to analyze the rational expectations equilibria satisfying (2) rather than analyzing non-explosive stochastic processes satisfying (1).

We summarize the discussion in this section as a proposition.

Proposition 1 *The sequence $\{z_t\}$ is a solution to the Heterogeneous Expectations model (1) if and only if it is a solution to the Associated Rational Model (2).*

3 A Simple Example of Heterogeneity and Determinacy

To explore the equilibrium implications of heterogeneity in expectations, we now apply Proposition 1 to a simple HE-model with two agent types. To this end, we first adapt to HE-models the notions of determinacy and predetermined variables.

3.1 Predetermined Variables and Determinacy

Having identified an HE-model with its ARE-model, we may simply generalize the definition of predetermined variables in such a way that the results of Blanchard and Kahn apply. Specifically, the predetermined variables of the HE-model are precisely the predetermined variables in the ARE-model. Notice, then, that the HE-model is determinate provided the number of free variables in the ARE-model is equal to the number of explosive roots in the ARE-model; and, the HE-model is indeterminate if the number of free variables is greater than the number of explosive roots.

Our interest in this paper is to illustrate how to use the insight of the previous subsection to characterize the HEE. To this end, we focus on the simple case in which a proportion α of agents are rational and the rest believe in mean reversion:

$$
\begin{aligned}
z_t &= \beta \hat{E}_t z_{t+1} + v_t \\
\hat{E}_t &= \alpha E_t + (1 - \alpha) E_t^*
\end{aligned}
\tag{3}
$$

where $E_t^*(z_{t+1}) = \theta z_{t-1}$ and is assumed to fix observables. This model has the same form as (1) except here we make heterogenous expectations explicit by focusing on a particular $\hat{E}$. The associated rational model is given by

$$
z_t = \beta \alpha E_t z_{t+1} + \beta (1 - \alpha) \theta z_{t-1} + v_t.
\tag{4}
$$

According to our definition, this HE-model has one predetermined variable: z_{t-1}. Multiplicity of equilibria can be analyzed by studying (4) in the usual way.

One further issue remains. If the HE-model is taken to have a solution z_t for $t \geq 0$ (as opposed to a doubly-infinite solution) the predetermined variables of the ARE-model must have initial conditions. We proceed to argue that once the ARE-model is determined, it is reasonable to simply assume that for each predetermined variable an initial condition exists, even though no such initial condition is explicitly given in the model. We will conclude that the assumed initial condition must be taken to have no special properties.

To illustrate this point, rewrite the ARE-model as

$$
z_t = \alpha \beta E_t z_{t+1} + (1 - \alpha) \beta E_t^* z_{t+1} + v_t.
$$

This equation holds for $t = 1$ only if z_0 is known; otherwise, $E_1^* z_2$ is not defined. Alternatively, by taking the initial forecasts $E_1^* z_2$ as given, we can, through the definition of E_1^*, back-out the implied initial condition z_0. In this way, taking z_0 as given is equivalent to taking the initial forecasts of non-rational agents as given. To remain agnostic about how non-rational agents arrive at their initial forecasts, we emphasize that no special attributes may be assigned to z_0.

We now turn to a characterization of heterogeneous expectations equilibria in this simple model.

3.2   Heterogeneity and Equilibrium Determinacy

To illustrate how the nature and number of equilibrium sequences in the RE and HE-models are related, we now turn to the careful analysis of the model

(3), rewritten here for convenience:

$$z_t = \beta \hat{E}_t z_{t+1}. \tag{5}$$

We have shut down the stochastic term, but the results presented here can be easily extended; we follow (Evans and Honkapohja 2002) in focusing on this simple case. We restrict attention in this section to $\beta > 0$.

We compare the regions of determinacy, indeterminacy and explosives of the HE-model (with aggregate expectations as defined in (6) below) to those of the RE-model, that is, model (5) under the assumption that $\hat{E} = E$. Notice that if $\beta < 1$ the unique non-explosive equilibrium of the RE-model is $z_t = 0$. When $\beta > 1$ then the RE-model also admits so-called sunspot solutions,

$$z_t = \beta^{-1} z_{t-1} + \varepsilon_t,$$

where ε_t is a martingale difference sequence. Thus, the RE-model is indeterminate if and only if $\beta > 1$. This result is not new. The question we now pose is, "Does heterogeneity alter the region of indeterminacy?"

We address this question using the same specification of the aggregate expectations operator as before: a proportion α of agents are rational and the rest believe in mean reversion:

$$\hat{E}_t z_{t+1} = \alpha E_t z_{t+1} + (1 - \alpha)\theta z_{t-1}. \tag{6}$$

The associated rational model is now

$$z_t = \gamma E_t z_{t+1} + \delta z_{t-1}, \tag{7}$$

where $\gamma = \beta\alpha$ and $\delta = \beta\theta(1 - \alpha)$. This model has one predetermined variable (z_{t-1}) and one free variable (z_t). Any solution to (7) must satisfy

$$z_t = \gamma^{-1} z_{t-1} - \gamma^{-1}\delta z_{t-2} + \varepsilon_t$$

where ε_t is a martingale difference sequence. Whether this ARE-model is determinate, indeterminate, or explosive depends on the parameters γ and δ as noted in the Appendix.

For clarity, we emphasize that in the sequel, the *HE-model* refers to the reduced form (5) with $\hat{E}$ given by (6), the *ARE-model* refers to the reduced form (7), and the *RE-model* refers to (5) with $\hat{E} = E$.

In case of naive agents ($\theta = 1$), we have the following proposition:

Proposition 2 *If $\theta = 1$ the following hold.*

1. *If $\alpha > 1/2$ then indeterminacy in the RE-model implies that the HE-model is indeterminate.*

2. *If $\alpha < 1/2$, indeterminacy in the RE-model implies the HE-model is explosive.*

3. *If the RE-model is determinate then the HE-model is determinate for all $0 \leq \alpha \leq 1$.*

The proofs of all propositions may be found in the Appendix.

We conclude that this type of heterogeneity does not alter the region of indeterminacy if the proportion of naive agents is not too large. However, when the naive agents form a majority then the HE-model may be explosive even though the RE-model is indeterminate. This result signals a cautionary note to modelers: mistakenly assuming rational expectations in a model may result in modeling choices which bring about a model with no stationary equilibria. When considering existence of equilibria in a rational expectations model the existence in a heterogeneous expectations model should also be considered. In our simple example, the result depends on the proportion of naive agents. However, recent empirical evidence suggests that the proportion of adaptive agents may be high.[8]

The intuition for this result is straightforward. Naive expectations in this model are reflective and either dampen or reinforce past shocks. Which effect naive agents have on the model depends on whether the self-referential feature of the model is dampening (i.e. $\beta < 1$) and on how many naive agents there are. In this first result, the combination of the self-referential feature $\beta > 1$ and naive expectations will tend to make equilibrium sequences explosive. The condition on the self-referential parameter β creates indeterminacy in the rational expectations case because, via backward induction, any belief process can be supported as an REE. However, when expectations are purely backwards looking the self-referential feature causes past shocks to be amplified. Hence, when there are a majority of naive agents, i.e. $\alpha < 1/2$, the model tends to be explosive. When there are a majority of rational agents the anticipatory behavior of rational agents dominates and the HE-model is indeterminate. Suppose now that $0 < \theta < 1$. We have the following proposition:

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[8]See, for example, (Branch 2004) and (Carroll 2003).

Proposition 3 *Let $0 < \theta < 1$.*

1. *If the RE-model is determinate then the HE-model is determinate.*

2. *If $\theta < (1 - \alpha\beta) / (\beta(1 - \alpha))$ then indeterminacy in the RE-model implies determinacy in the HE-model.*

Proposition 3 provides parameter values for which the RE-model is indeterminate and the associated HE-model is determinate. We conclude that heterogeneity of this type may have a stabilizing effect on the economy (here stabilizing is in the sense that there is a unique HEE).[9] And for some model parameterizations, only a small fraction of adaptive agents are required; in particular, as $\beta \to 1$, the required proportion of adaptors goes to zero.

The intuition for this result is identical to Proposition 2. When $\theta < 1$ the adaptive agents tend to dampen past realizations. Whether this dampening is sufficient to make the steady-state saddle-path stable depends on the size of β and α relative to θ. Proposition 3 shows that if the dampening is strong enough, i.e. θ is small enough, then the model is non-explosive. Proposition 3 shows how this dampening behavior of adaptive agents interacts with the forward-looking behavior of rational agents and the self-referential properties of the model.

The results of proposition 3 are significant. A sizeable literature considers the effects of small deviations from rationality.[10] We are unaware of any paper, however, which examines the effect of departures from rationality on the existence of multiple equilibrium sequences. Proposition 3 presents a new result. Models which have sunspot equilibria exhibit an infinite number of possible sequences on which agents may coordinate. Under some conditions, a small number of adaptive agents may preclude such coordination. This indicates that sunspot equilibria may be less robust than previously thought.

Finally, suppose $\theta > 1$; we interpret this somewhat unlikely operator as suggesting that a proportion of agents have explosive expectations. Although unlikely, $\theta > 1$ is not unreasonable. It is considered, for instance, in (Pesaran 1987). If agents have trend-expectations so that they expect past trends to accelerate then this parameter restriction will fit. Trend followers, for example, seem to fit such a scheme. We have the following result:

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[9] This so-called saddle path stability is frequently argued as an important criteria for rational expectations models because it rules out bubble-solutions and makes coordination on equilibria easier. For a discussion of the debate over selection criteria for multiple equilibria models see (McCallum 1999). For a discussion of the connection between determinacy and stability of learning see (Bullard and Mitra 2001). A discussion of determinacy, coordination on sunspot equilibria, and learnability is addressed in (Gauthier 2000, 2002).

[10] See, for example, (Anderlini and Canning 2001).

Proposition 4 *Let $\theta > 1$. If the RE-model is indeterminate then*

1. *if $\alpha > \theta/(1+\theta)$ the HE-model is indeterminate.*

2. *if $\alpha < \theta/(1+\theta)$ the HE-model is explosive.*

If the RE-model is determinate then if $\alpha/(1-\alpha) > \theta$ and $1/(\theta(1-\alpha)+\alpha) < \beta$ the HE-model is indeterminate.

As before, how adaptive agents affect the number and nature of equilibria depends on the interaction between dampening and exploding forces. Proposition 4 shows that when $\theta > 1$ the number of rational agents needs to be high enough, relative to θ, to prevent explosiveness. Moreover, even when the self-referential feature leads to determinacy in the RE-model (i.e. $\beta < 1$) the HE-model can still be indeterminate if the expanding forces of adaptive agents (i.e. $\theta > 1$) is bounded relative to α and β.

For the case of a proportion of trend followers, the original RE-model may be determinate and the HE-model indeterminate. We conclude that heterogeneity of this type may have a de-stabilizing effect on the economy. Moreover, there exist equilibria which depend on extrinsic noise (i.e. sunspots) in the heterogeneous expectations model when the RE-model does not have any such equilibria. This result is interesting and suggests that if some agents are trend-followers it is possible for rational agents to coordinate on 'sunspots' when they would not under full-rationality.

3.3   Further Discussion

A brief note on how we treat heterogeneity is warranted. The degree of heterogeneity in the model is parameterized by the weight α. As we have shown, the number and nature of the HEE depend critically on the relative size of α. This suggests that a complete model of heterogenous expectations would determine α endogenously. We remark that this has been treated in the literature. (Brock and Hommes 1997) attach a fitness measure to the rational and naive predictors and assume that α is determined by a discrete choice model and may vary over time. They find that in a cobweb model with costly rational expectations and $\theta = 1$ the predictor proportion dynamics may be complicated.[11] The HEE model could be extended by incorporating a Brock-Hommes type mechanism. The approach in this paper is closest to the steady-state of (Brock and Hommes 1997) since we restrict attention to non-explosive solutions to the expectational difference equation.

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[11](Branch 2002) shows that for sufficiently small θ the dynamics may be stable.

There are a number of ways in which the HEE approach could be extended to incorporate the insight of (Brock and Hommes 1997). One could, for example, take a non-stochastic version of (3) and let α be time-varying and coupled to the expectational difference equation a la (Brock and Hommes 1997). Alternatively, (Branch and Evans 2004) provide guidance on how to find α endogenously as an equilibrium outcome in a Brock-Hommes framework that is stochastic. Heterogeneity is an empirically significant phenomena; there are many interesting alternatives to the approach advocated here and we plan to pursue them in future work.

4   Conclusion

The results of this paper show that the number and nature of equilibria are highly dependant on the specification of agents' expectations. The paper fills an important gap in the literature on uniqueness of equilibria in self-referential models. We showed that if expectations are heterogeneous then for certain parameterizations there is a correspondence between determinacy and indeterminacy in RE-models and HE-models. However, for other parameterizations the HE-model stabilizes an otherwise unstable RE-model. And finally, if expectations follow a trend-setting scheme then the economy may have sunspot equilibria under heterogeneous expectations when it would not under rational expectations.

Our results suggest there is no generic relationship between the number of equilibria in a model under rational expectations and the same model under heterogeneous expectations. The inclusion of heterogeneity may have a stabilizing or destabilizing effect, or even preclude the existence of equilibria. We have presented a method for analyzing the effects of heterogeneity, and these results strongly suggest this analysis is important when working with forward looking models. Incorrectly assuming rational expectations may lead the modeler to predict outcomes far different from what the model actually exhibits.

The results here are particularly important for models in which the determinacy of RE-models depends on policy parameters. For example, in monetary models such as (Woodford 2003) it is often suggested that monetary policy should follow an active Taylor rule in order to rule out multiple rational expectations equilibria. Our results indicate that if policymakers unwittingly assume agents have rational expectations they may destabilize an already stable system. In future work we will examine this very issue in order to specify robustness criteria for monetary policy rules.

Appendix

The following result is well-known, and can be found, for example, in (Evans and Honkapohja, 2001).

Lemma 1 *The rational expectations model (7) is indeterminate if and only if the pair (γ, δ) satisfy either of the two conditions*

  1. $\gamma > 1/2$, $\delta > 1 - \gamma$, and $\delta < \gamma$.

  2. $\gamma < -1/2$, $\delta < -1 - \gamma$, and $\delta > \gamma$.

The model is determinate if and only if (γ, δ) satisfy both of the conditions

  1. $\delta < 1 - \gamma$

  2. $\delta > -\gamma - 1$

Proof of Proposition 2. The proof of this result and the others relies on the observation that $\delta = \theta(\beta - \gamma)$. Assume $\theta = 1$. To prove part 1, assume $\beta > 1$ so that the RE-model is indeterminate. Then $\alpha > 1/2$ implies $\gamma = \beta\alpha > 1/2$. Also, $\alpha > 1/2 \Rightarrow \gamma = \beta\alpha > \beta(1 - \alpha)$. Finally, $\beta > 1 \Rightarrow \delta = \beta - \gamma > 1 - \gamma$, so that the HE-model is indeterminate by Lemma 4. To prove part 2 we must show explosiveness, which, since $\beta > 0$, requires $\delta > \gamma$ and $\delta > 1 - \gamma$. Thus let $\beta > 1$ and $\alpha < 1/2$. Then $\beta > 1 \Rightarrow \delta = \beta - \gamma > 1 - \gamma$ and $\alpha < 1/2 \Rightarrow \delta = \beta(1 - \alpha) > \beta/2 > \gamma$. Finally, to prove part 3, let $\beta < 1$. To show determinacy, we must show $\delta, \gamma > 0$ and $\delta < 1 - \gamma$. But δ and γ are positive by construction and $\beta < 1 \Rightarrow \delta = \beta - \gamma < 1 - \gamma$. ∎

Proof of Proposition 3. Let $\theta < 1$, $\beta < 1$. Again δ and γ are positive by construction, so we must show that $\delta < 1 - \gamma$. But $\delta = \theta(\beta - \gamma) < \theta(1 - \gamma) < 1 - \gamma$. This proves part 1. Part 2. is simply algebra. ∎

Proof of Proposition 4. Let $\beta > 1$ and $\theta > 1$, and assume $\alpha > \theta/(1 + \theta)$. First notice $\delta = \theta(\beta - \gamma) > \beta - \gamma > 1 - \gamma$. Also, $\gamma > \beta\theta/(1 + \theta)$ so that $\gamma(1 + \theta) > \theta\beta$ or $\gamma > \theta(\beta - \gamma) = \delta$. Finally, $1 - \gamma < \delta < \gamma \Rightarrow \gamma > 1/2$. This proves part 1. To prove part 2, notice that $\delta = \theta(\beta - \gamma) > \beta - \gamma > 1 - \gamma$ still holds. Also, if $\alpha < \theta/(1 + \theta)$, the same argument as above with the inequalities reversed shows that $\gamma < \beta\theta/(1+\theta)$ implies $\gamma < \delta$. The last part of the proposition follows from algebra and the realization that the assumptions imply

$$\beta\alpha > \frac{\alpha}{\alpha + \theta(1 - \alpha)} = \frac{1}{1 + \frac{\theta(1-\alpha)}{\alpha}} > \frac{1}{2}. \quad \blacksquare$$

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