Adaptive Learning in Regime-Switching Models*

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Abstract

This paper studies adaptive learning in economic environments subject to recurring structural change. Stochastically evolving institutional and policy-making features can be described by regime-switching rational expectations models whose parameters evolve according to a finite state Markov process. We demonstrate that in non-linear models of this form, two natural schemes emerge for learning the conditional means of endogenous variables: under *mean value learning*, the equilibrium's lag structure is assumed exogenous and therefore known to agents; whereas, under *vector autoregression learning* (VAR learning), the equilibrium lag structure depends endogenously on agents' beliefs and must be learned. We show that an intuitive condition, analogous to the 'Long-run Taylor Principle' of Davig and Leeper (2007), ensures convergence to a regime-switching rational expectations equilibrium. However, the stability of sunspot equilibria, when they exist, depends on whether agents adopt mean value or VAR learning. Coordinating on sunspot equilibria via a VAR learning rule is not possible. These results show that, when assessing the plausibility of rational expectations equilibria in non-linear models, out of equilibrium behavior is important.

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1 Introduction

A given forward-looking macroeconomic model may admit different classes of rational expectations equilibria. Solutions can differ in terms of the set of state variables that agents use when forming expectations. For example, standard linear stochastic rational expectations models have solutions that depend only on the minimal set of state variables, but may also have solutions that depend on extrinsic random variables (i.e. sunspots). The existence of equilibria are well understood in linear models with constant parameters; however, in a growing area of research that focuses on models with changing parameters, these issues are re-emerging.[1] In regime-switching models, which constitute the focus of this paper, parameters evolve according to a finite state Markov process. The non-linear structure of regime-switching rational expectations models prevents a complete characterization of the full class of solutions. Recent papers by Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009b,a) each focus on a particular class of equilibria in the context of a standard New Keynesian model with a monetary policy rule whose coefficients are subject to occasional regime change.

Davig and Leeper (2007) consider the class of equilibrium processes that explicitly depend on current, but not lagged, regimes. In the context of a New Keynesian model with Markov switching monetary regimes, these authors introduce a condition known as the *long run Taylor Principle*, which generalizes the well known "Taylor Principle" that monetary policy should raise nominal interest rates more than one-for-one with inflation. Davig and Leeper show that the long run Taylor Principle ensures a unique equilibrium within the class under consideration. The central economic insight of the long run Taylor Principle is that when private-sector expectations build in the possibility of regime-change to, say, a policy that responds passively to inflation, this places restrictions on policy in the other regimes in order to remain consistent with a Taylor Principle for monetary policy

Farmer, Waggoner, and Zha (2009b) consider a New Keynesian model with Markov switching monetary policy, but these authors allow their equilibrium processes to depend also on lagged policy regimes. They construct equilibria that depend on sunspot shocks even when the long run Taylor Principle holds, thus establishing the existence of multiple equilibria when there is a unique equilibrium within the class considered by Davig and Leeper (2007).

To facilitate discussion and analysis of the relationship between solutions in Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009b,a), we begin this pa-

[1]Some examples of work in this area include Leeper and Zha (2003), Andolfatto and Gomme (2003), Benhabib (2009),Davig (2004), Zampolli (2006), Chung, Davig and Leeper (2007), Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009b,a), and Svensson and Williams (2007). Brainard (1967) is an early example of work on parameter instability.

per by precisely defining the equilibrium classes of interest within a general modeling framework that includes the New Keynesian model with Markov switching monetary regimes as a special case. Because the equilibrium processes studied by Davig and Leeper depend only on current regimes, we define the associated class as *Regime-Dependent Equilibria* (RDE); and because the equilibrium processes studied by Farmer, Waggoner and Zha also allow for explicit dependence on lagged regimes, we denote the associated class as *History-Dependent Equilibria* (HDE). Our first result is a generalization of the long run Taylor principle to our model: we establish conditions necessary and sufficient to guarantee a unique regime dependent equilibrium. We also find, consistent with the work of Farmer, Waggoner and Zha, that even if our condition is met, there may also exist history-dependent equilibria.

The results above distinguish regime-switching models from their constant parameter counterparts. In particular, conditions that establish uniqueness of equilibria within a certain natural class may fail to preclude the existence of other types of equilibria. The presence of multiple equilibria in our model leads naturally to the question of equilibrium selection, which is the main topic of our paper; and we propose using stability under adaptive learning as the equilibrium selection mechanism. Following Lucas (1986), we maintain that stability under adaptive learning is a useful metric for identifying empirically relevant equilibria.[2] An equilibrium is plausible or reasonable if, whenever rational expectations are replaced with a standard adaptive learning rule, agents' beliefs converge to the rational expectations equilibrium values. Based on this assumption, we assess whether regime-switching equilibria are learnable.

Our viewpoint is informed by a large and growing literature that replaces rational expectations with learning rules where agents are modeled as professional econometricians, that is, they hold forecasting models that share a reduced-form with a rational expectations equilibrium, and adjust the parameters of their model in light of new data. The advantage to this approach is that it places economist and agent on equal footing and avoids the cognitive dissonance inherent in rational expectations models. This approach is particularly compelling in regime-switching models because of the co-existence of equilibria in the regime and history dependent classes.

In this paper, we endow agents with a forecasting model, similar in reduced-form to the equilibria identified by Davig and Leeper (RDE) and Farmer, Waggoner and Zha (HDE), and we study the stability of the associated equilibria. It is well known that in models with multiple equilibria, different learning rules (each *a priori* plausible) may lead to distinct stability outcomes.[3] Sticking with the statistical learning approach outlined above, two natural learning models emerge for history dependent equilibria. These learning processes differ based on basic informational assumptions. In the first formulation – "mean value learning" – agents know the lag structure of the economy's

[2] See, in addition, Evans (1986), Bray and Savin (1986), Marcet and Sargent (1989).
[3] See Lucas (1986), Woodford (1990).

endogenous variables when estimating the conditional mean. The second natural learning process – "VAR learning" – agents try also to uncover the lag structure from the data by employing a (first order) vector autoregression (VAR(1)) model. Within a rational expectations equilibrium the two formulations are equivalent, but out of equilibrium they lead to different conclusions about the learnability of HDE.

Our primary result is that when the conditions for a unique RDE are satisfied (i.e. the Long-run Taylor Principle in a New Keynesian model), then the equilibrium is stable under learning. Moreover, this condition also governs the stability of HDE under mean value learning, that is, when agents know the lag structure of the model, but estimate in real-time the mean. On the other hand, the HDE are not attainable under VAR learning, that is, when agents must also learn the (endogenous) lag structure.

The results of this paper provide a clear illustration of the usefulness of expectational stability as an equilibrium selection criterion. As the results of Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009b) demonstrate, local determinacy is not particularly useful in these settings for selecting equilibria and designing policy. This paper shows that the conditions leading to a unique RDE ensures that a rational expectations equilibrium can be attained as the limiting outcome of a reasonable learning process. Moreover, if agents behave as econometricians and try to uncover the lag structure of the endogenous state variables, then these conditions guarantee the existence of a unique stable equilibrium.

The paper is organized as follows : Section 2 introduces the framework of Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009b); Section 3 generalizes the model and defines the classes of equilibria; Section 4 provides the main stability analysis; Section 5 presents results for a univariate and a New Keynesian model; and, Section 6 concludes.

2 A New Keynesian Model with Recurring Policy Change

There is extensive empirical evidence of regime change in monetary policymaking. For example, there is a breakpoint in the parameters of a Taylor-type nominal interest rate rule in Clarida, Gali, and Gertler (1999), or shifting policymaker preferences in Bernanke (2004) and Dennis (2006). These findings motivate models that build regime-switching directly into rational expectations frameworks since whenever the systematic nature of monetary policymaking has changed in the past, then it is reasonable that agents might anticipate future policy changes.

As an example, Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009b) construct rational expectations solutions to the standard New Keynesian model closed with a nominal interest rate rule whose coefficients are subject to occasional regime change. The New Keynesian model is given by (linearized) reduced-form equations for inflation, π, and the output gap, x, such as

$$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + g_t \quad (1)$$
$$x_t = E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) + u_t \quad (2)$$

where g_t are aggregate demand shocks and u_t are aggregate supply shocks, typically taken to be exogenous (stationary) AR(1) processes. The first equation (1) is the New Keynesian Phillips curve. It is derived from a first-order approximation to a monopolistically competitive firm's price-setting decision when there is the possibility that future prices will remain fixed (e.g. Calvo price-setting). The second equation (2) is the New Keynesian IS equation that represents the demand side of the economy. It is derived as a linear approximation to a representative household's Euler equation. In the standard formulation of the New Keynesian model the g_t typically arise as a combination of productivity, preference, and government spending shocks, while the u_t shocks arise from variations in market-power from, e.g. shocks to the elasticity of substitution.

It is typical to close a New Keynesian model with a nominal interest rate targeting rule along the lines proposed by Taylor (1993). A New Keynesian model with recurring policy change assumes a nominal interest rate rule with time-varying parameters

$$i_t = \alpha_t \pi_t + \gamma_t x_t \quad (3)$$

To capture recurrent regime change, Davig and Leeper (2007), assume that the parameters α_t, γ_t in (3) follow a two state Markov chain:

$$\alpha_t = \begin{cases} \alpha_1 \text{ for } s_t = 1 \\ \alpha_2 \text{ for } s_t = 2 \end{cases}$$

and

$$\gamma_t = \begin{cases} \gamma_1 \text{ for } s_t = 1 \\ \gamma_2 \text{ for } s_t = 2 \end{cases}$$

The random variable s_t follows a finite-state Markov chain with transition probabilities $p_{ij} \equiv \Pr[s_t = j | s_{t-1} = i]$ for $i, j = 1, 2$.

The Taylor Principle dictates that, in a model with constant policy coefficients α, γ, nominal interest rates rise more than one for one with inflation, that is $\alpha > 1$. Policy that satisfies the Taylor Principle leads to a model with a unique rational expectations equilibrium, while when $\alpha < 1$ it is possible for there to exist multiple equilibria that exhibit inefficiently high volatility. Davig and Leeper (2007) generalize the Taylor Principle to a setting where policy coefficients follow the two state Markov chain assumed above. The central insight of their generalization, the Long Run Taylor Principle, is that private-sector expectations build in the possibility of future passive monetary policy and that this places a restriction on how active, i.e the extent to which $\alpha > 1$, policy must be to ensure determinacy.

The definition of a rational expectations equilibrium in regime-switching models is analogous to the concept in constant parameter models. In particular, a *rational expectations equilibrium* is any uniformly bounded solution to (1)-(3). Because the Markov chain enters the model multiplicatively, the model is inherently non-linear, preventing a characterization of all solutions to the model. Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009b) propose two classes of solutions. Farmer, Waggoner, and Zha (2009b) show that it is possible for there to exist a continuum of solutions in one class even under conditions that guarantee uniqueness in the other class. This raises the question of which equilibria are plausible. The intuition for why there might exist multiple classes of equilibria in regime-switching models is clear. When the parameters evolve stochastically this introduces additional exogenous noise into the model. The number and nature of rational expectations equilibria then depends critically on the manner in which agents incorporate the additional exogenous variables into their expectations. This provides an additional motivation for studying adaptive learning as an equilibrium selection device.

3 Equilibria In Regime Switching Models

We focus on models whose reduced form consists of a system of non-linear expectational difference equations such as

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t, \tag{4}$$
$$r_t = \rho r_{t-1} + \varepsilon_t, \tag{5}$$

where y_t is an $(n \times 1)$ vector of random variables, β_t and γ_t are conformable matrices that follow an m state Markov process with $(\beta_t = \beta_i, \gamma_t = \gamma_i) \Leftrightarrow s_t = i, i = 1, 2, ..., m$, and r_t is a $(k \times 1)$ exogenous stationary VAR(1) process independent of s_j for all j. The stochastic matrix P governs the evolution of the state, s_t, and contains elements

$$p_{ij} \equiv \Pr\left[s_t = j | s_{t-1} = i\right],$$

for $i, j \in \{1, 2, ..., m\}$. P is taken to be recurrent and aperiodic, so that it has a unique stationary distribution Π. For simplicity, β_i is taken to be invertible for all i. Davig and Leeper (2007) consider a version of this model in the context of a univariate monetary model and a bivariate New Keynesian model. Most macroeconomic models feature expectational structures similar to (4) – albeit with constant parameters – making (4) a natural laboratory to study the existence and stability of rational expectations equilibria in regime-switching models. In Section 5, we present a univariate example and return to the New Keynesian example.

A rational expectations equilibrium of the model is a solution to (4) that also satisfies a boundary condition. Often the definition of the boundary condition is somewhat vague, given as "non-explosiveness" and justified by appealing to a transversality condition, even though the usual transversality condition implies that solutions not explode "too quickly."[4]

We focus on processes satisfying the following property:

Definition. A stochastic process y_t, with initial condition y_0 is *uniformly bounded* (almost everywhere) or UB if $\exists M(y_0)$ so that $\sup_t ||y_t||_\infty < M(y_0)$, where $||\cdot||_\infty$ is the L^∞ or "essential supremum" norm.

With this definition available, we may define a rational expectations equilibrium:

Definition. A *Rational Expectations Equilibrium* is any UB stochastic process satisfying (4).

While uniformly bounded (UB) may appear to be an *a priori* strong notion of boundedness, it is common in the linear rational expectations literature. In linear models with constant parameters, uniform boundedness is consistent with the usual notion of model determinacy, such as in Blanchard and Kahn (1980). Also, UB "bounds the paths" of all endogenous variables and is often desirable when using a first-order approximation to a nonlinear model around a fixed point, such as a steady state.

An important difference that arises in regime-switching rational expectations models, versus constant-parameter models, is that agents incorporate the probability of a regime change into their expectations. The resulting non-linear structure prevents characterizing the full class of rational expectations equilibria. However, several classes naturally emerge, which we define as *Regime-Dependent Equilibria* (RDE), *Stacked System Equilibria*, and *History-Dependent Equilibria* (HDE).

3.1 Regime-Dependent Equilibria

The first class focuses on state-contingent solutions that allow the current realization of the regime, s_t, to enter the state vector, but are otherwise independent of its history. The state vector also includes current realizations of the exogenous shocks, as well as

[4] As an alternative, sometimes the boundary condition requires the paths of variables in a rational expectations equilibrium remain conditionally uniformly bounded, such as in Evans and McGough (2005).

(possibly) sunspot variables. Formally, the definition for an RDE is as follows:

Definition. Let s_t be the Markov process governed by P and taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (4). Then y_t is a *Regime Dependent Equilibrium* (RDE) if it is uniformly bounded and there exist uniformly bounded stochastic processes $y_{1t}, y_{2t}, ..., y_{mt}$, with y_{it} independent of s_{t+j} for all integers j, such that $y_t = y_{it} \Leftrightarrow s_t = i$.

In an RDE, depending on the realization of s_t, y_t takes on values from one of m stochastic processes, with each process being independent of the Markov state. Conditioning (4) on each regime leads to the following system

$$\begin{aligned}
y_{1t} &= \beta_1 p_{11} E_t y_{1t+1} + \beta_1 p_{12} E_t y_{2t+1} + \cdots + \beta_1 p_{1m} E_t y_{mt+1} + \gamma_1 r_t, \\
y_{2t} &= \beta_2 p_{21} E_t y_{1t+1} + \beta_2 p_{22} E_t y_{2t+1} + \cdots + \beta_2 p_{2m} E_t y_{mt+1} + \gamma_2 r_t, \\
\vdots &\ \vdots \qquad\qquad\qquad\qquad \vdots \\
y_{mt} &= \beta_m p_{m1} E_t y_{1t+1} + \beta_m p_{m2} E_t y_{2t+1} + \cdots + \beta_m p_{mm} E_t y_{mt+1} + \gamma_m r_t,
\end{aligned}$$

which governs dynamics for y_{it} for $i = 1, 2, ..., m$. We note that this is a linear system.

3.2 Stacked System Equilibria

The linear system above can be recast in the form of a 'stacked system', which has a more compact representation. Stacked System Equilibria are rational expectations solutions to the conditionally linear system, without the restriction of independence from s_{t+n} for all n that is explicitly imposed on RDE.

Definition. The *stacked system* associated with the switching model (4) is the system of multivariate linear expectational difference equations

$$\hat{y}_t = (\oplus_{j=1}^m \beta_j)(P \otimes I_n) E_t \hat{y}_{t+1} + \gamma r_t \tag{6}$$

where $\widehat{y_t} = [y'_{1t}, y'_{2t}, ..., y'_{mt}]'$ and $\gamma' = (\gamma'_1, \ldots, \gamma'_m)'$.[5]

Definition. Let s_t be the Markov process governed by P and taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (4). Let $\hat{y}_t$ be a uniformly bounded solution to the Stacked System (6). Then y_t is a *Stacked System Equilibrium* (SSE) of the original model (4) if $y_t = \hat{y}_{it} \Leftrightarrow s_t = i$.

The stacked system (6) is a multivariate linear rational expectations model. The number and nature of solutions to (6) is well-known. We are particularly interested in conditions under which there exists a unique UB solution to the stacked system, as this condition will also govern the expectational stability of solutions to the economic system (4). We summarize this in the following remark.

[5] Throughout, $\oplus$ denotes the direct sum operator.

Remark. A necessary and sufficient condition for the existence of a unique uniformly bounded solution to (6) is that the eigenvalues of $(\oplus_{j=1}^{m}\beta_j)(P \otimes I_n)$ lie inside the unit circle. In this case, we say that the *Conditionally Linear Determinacy Condition* (CLDC) is satisfied.

The following proposition summarizes the relationship between solutions to the stacked system (6), RDE, and UB solutions to (4).

Proposition 1 *Let U be the collection of all UB solutions to (4). Let R, S denote the collection of all Regime Dependent Equilibria and all UB solutions to the Stacked System (6), respectively. The following relations among the sets of equilibria hold.*

1. *$R \subset S \subset U$.*
2. *If the CLDC holds, then $R = S$ and R has one element.*
3. *If the CLDC does not hold, then R has a continuum of elements and $R \subsetneq S$*

All proofs are contained in the Appendix.

In the context of monetary policy analysis, Davig and Leeper (2007) refer to the CLDC as the *Long Run Taylor Principle* (LRTP), which, as mentioned above, is a generalization of the Taylor Principle. In this respect, Davig and Leeper (2007) completely characterize the uniqueness conditions for RDE in standard monetary models. Davig and Leeper, however, restrict attention to models with positive feedback from monetary policy so that the eigenvalues of $\oplus\beta_j$ are positive. The results in Proposition 1 establish that the necessary and sufficient condition for existence of a unique RDE is the CLDC. Importantly, however, when the CLDC is not satisfied there may exist other equilibria that are not independent of past realizations of the Markov state. Below we call such equilibria *History Dependent Equilibria.* Farmer, Waggoner, and Zha (2009b) show that the CLDC does not imply uniqueness in the economic model (4). In the next Section, we propose expectational stability as a device for selecting equilibria when both RDE and HDE exist.

Subsequent sections show a close connection between the conditions for unique RDE and E-stable rational expectations equilibria, and so the CLDC takes on added importance below.

3.3 History Dependent Equilibria

Proposition 1 shows that the collection of uniformly bounded solutions to (4) super cedes the set of regime-dependent equilibria. This section characterizes another class of equilibria to (4). This definition is related to the work of Farmer, Waggoner, and Zha (2009b) who illustrate that in a New Keynesian model, it is possible for there to exist sunspot equilibria even when the RDE is unique. In this case, conditions ensuring uniqueness within the class of RDE, such as the CLDC, does not ensure equilibrium determinacy in the regime-switching rational expectations framework. The central element in Farmer, Waggoner, and Zha (2009b,a), is they allow lagged states to enter the state vector. That is, FWZ have agents conditioning their expectations on an expanded state vector that includes s_{t-1}. For this reason, we call the class of solutions *History Dependent Equilibria* (HDE). By assuming agents condition on current and past realizations of the state variable s_t, this class of equilibria includes solutions that depend on arbitrary sunspot variables.

Definition. Let s_t be the Markov process governed by P, taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (4). Then y_t is a *History Dependent Equilibrium* (HDE) if it is uniformly bounded and its distribution conditional on s_t differs from its distribution conditional on s_t and s_{t-1}; that is, $y_t | s_t \not\sim y_t | (s_t, s_{t-1})$.

Remark. The definition of an HDE restricts solutions to the class of uniformly bounded stochastic processes whose conditional density exhibits dependence on s_t and s_{t-1}. Notice that if y_t is an RDE then $y_t | s_t \sim y_t | (s_t, s_{t-1})$. However, by Proposition 1 when the CLDC is not satisfied, then there may exist solutions to the stacked system that are not RDE. In particular, when the matrix $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$ has n_s eigenvalues inside the unit circle then for each n_s-dimensional martingale difference sequence ξ_t there is a martingale difference sequence $\tilde{\xi}_t$ and an SSE $\hat{y}_t$ with a representation given by

$$\hat{y}_t = b\hat{y}_{t-1} + cr_{t-1} + d(s_{t-1}, s_t)\tilde{\xi}_t,$$

where d is any function of s_{t-1} and s_t.

HDE solutions to the stacked system have constant parameters except for the coefficient on the sunspot shock $\tilde{\xi}_t$. It is natural to wonder whether there exist HDE with time-varying coefficients on the lagged endogenous variable. To consider this, we assume $\gamma_t = 0$ for all t, as the presence of exogenous shocks does not alter the results and distracts from the presentation. Note that if ξ_t is any martingale difference sequence, then $y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t$ is a solution to (4). Farmer, Waggoner, and Zha (2009b) show that there exist multiple uniformly bounded HDE that have the following representation

$$y_t = \left(\frac{c_{s_{t-1}}}{v'_{s_{t-1}} v_{s_{t-1}}} v_{s_t} v'_{s_{t-1}} \right) y_{t-1} + v_{s_t} \xi_t, \tag{7}$$

provided there exists $c_1, \ldots, c_m$ and $v = (v'_1, \ldots, v'_m)' \neq 0$ so that $|c_j| \leq 1$ and c and v solve

$$\left[\left(\oplus_{j=1}^m \beta_j \right)^{-1} - \left(\left(\oplus_{j=1}^m c_j \right) P \right) \otimes I_n \right] v = 0. \tag{8}$$

Here ξ_t is independent of s_{t+n} for all n. The condition (8) is essentially derived from the method of undetermined coefficients. When (8) is satisfied, solutions to the representation (7) are solutions to (4).[6] The construction of the autoregressive parameter in the representation (7) is chosen so that, regardless of the history of realizations of s_t, these parameters are bounded in matrix norm and, hence, the solutions are uniformly bounded.

Farmer, Waggoner, and Zha (2009b) write (7) in alternative form

$$y_t = \eta_t \tag{9}$$

where

$$\eta_t = \left(\frac{c_{s_{t-1}}}{v'_{s_{t-1}} v_{s_{t-1}}} v_{s_t} v'_{s_{t-1}} \right) \eta_{t-1} + v_{s_t} \xi_t.$$

The stochastic properties of (9) are equivalent to (7); however, as we will see below, during out of equilibrium learning dynamics, these two representations imply different informational assumptions and distinct stability results. These observations lead to two natural learning rules: a "mean value learning" formulation where agents use a forecasting model consistent with (9) by *conditioning* on η_t and try to learn the endogenous variable's state-contingent constant term (which, in this case, is zero); and a "VAR learning" formulation where agents estimate a forecasting model consistent with (7) by *conditioning* on a state-contingent constant and on lagged y – in this case, agents must also learn the endogenous variable's lagged coefficients. Importantly, under mean value learning, the lag structure is exogenous while it is determined endogenously under VAR learning. This provides a crucial distinction for the stability results presented below.

By defining HDE as rational expectations equilibria that exhibit conditional dependence on both s_t and s_{t-1}, it is possible to identify a more general class of equilibria than those represented by (9). Assume HDE take the form[7]

$$y_t = B(s_{t-1}, s_t) y_{t-1} + C(s_{t-1}, s_t) \xi_t. \tag{10}$$

where the coefficients must satisfy

$$\left(I_n - \beta_j \left(\sum_{k=1}^{m} p_{jk} B(j,k) \right) \right) B(i,j) = 0 \tag{11}$$

$$\left(I_n - \beta_j \left(\sum_{k=1}^{m} p_{jk} B(j,k) \right) \right) C(i,j) = 0 \tag{12}$$

[6] If one were to literally use the method of undetermined coefficients, the v in (8) would be y_t. However, if v is taken to be a vector of initial conditions chosen to lie on the stable manifold, and if (8) is satisfied at $t = 1$, then it will be satisfied for all t.

[7] Adopting the earlier notation, since $\gamma_t = 0$ it follows that $\tilde{\xi}_t = \xi_t$.

Notice that provided non-zero $B(i,j)$ satisfy (11), the $C(i,j)$ are arbitrary. It is straightforward to verify that (7) is a solution to (10).

4 Equilibrium Selection in Regime-Switching Models: Expectational Stability

Although rational expectations solutions to regime-switching models are of interest to applied economists and policymakers, the technical details leave important practical issues unsettled. First, as illustrated in the above section, the concepts of determinacy and uniqueness of rational expectations equilibria in non-linear models are not readily available. The RE hypothesis, in reduced-form models, is silent about which class of equilibria is most reasonable. Second, by imposing rational expectations, the modeler makes strong assumptions that require private-sector agents know the true distribution generating the data, even though the model is self-referential. Applied economists and professional forecasters typically formulate reduced-form models, inspired by rational expectations equilibria, that they estimate based on available data and update as new data becomes available. It is reasonable to expect that private-sector agents behave similarly.

A somewhat recent literature on adaptive learning in macroeconomics studies the plausibility of rational expectations equilibria by insisting on logical consistency between professional forecasters (or econometricians) and private-sector agents. Rather than rational expectations, this literature assumes that agents behave as econometricians who formulate forecasting models and update the parameters of their model in real-time. Because the data generating process depends on these recursively updated forecasting models, the convergence to, and stability of, rational expectations equilibria is a non-trivial problem. Woodford (1990), Marcet and Sargent (1989), Evans and Honkapohja (2001), Bullard and Mitra (2002) argue that stability under learning is a reasonable equilibrium selection mechanism. We adopt this viewpoint and study the stability under learning of regime-switching rational expectations equilibria. Our primary result is that the condition governing uniqueness in the class of regime dependent equilibria, namely, the CLDC, may also be the condition governing expectational stability. Crucially, though, this result depends on the assumed information structure. Evans and Honkapohja (2008) argue that indeterminacy need not concern policymakers when the fundamentals rational expectations equilibrium is the only equilibrium stable under learning. Adapting this viewpoint to a regime-switching framework suggests the CLDC, or the Long-run Taylor Principle of Davig and Leeper (2007), can form a sensible policy prescription since it may select the desired (unique) RDE.

4.1 E-stability in Constant Parameter Models

To fix ideas, we review the expectational stability approach in a constant parameter version of (4),

$$y_t = \beta E_t^* y_{t+1} + \gamma r_t \tag{13}$$

now written with a (possibly) boundedly rational expectations operator E^*. We first consider the case where the model is determinate and then, below, we examine the indeterminate case.

When the model is determinate, there exists a unique equilibrium that has the form $y_t = br_t$. Agents hold a perceived law of motion (i.e. a forecasting model) whose functional form is consistent with the equilibrium representation

$$y_t = A + Br_t. \tag{14}$$

While there is no constant in the equilibrium representation $y_t = br_t$, it is standard to allow agents to consider the possibility that there may be a constant term, i.e. to learn the steady-state values of y as well.

The parameters A and B capture agents' perceptions of the relationship between y and r and may be estimated using, for example, recursive least squares. Let A_t and B_t be the respective estimates using data up to time t. Agents form forecasts using the perceived law of motion $E_t^* y_{t+1} = A_{t-1} + B_{t-1}\rho r_t$. Plugging these forecasts into (13) leads to the actual law of motion

$$y_t = \beta A_{t-1} + (\beta B_{t-1}\rho + \gamma) r_t.$$

Here we assume that agents know the true process governing r_t. The actual law of motion illustrates the manner in which time t endogenous variables are determined by perceptions (A_{t-1}, B_{t-1}) and realizations of r_t. Given new data on y_t agents then update the forecasting model to obtain (A_t, B_t). The unique rational expectations equilibrium $y_t = br_t$ is stable under learning if $(A_t, B_t) \to (0, b)$ almost surely. Stability under learning is non-trivial precisely because of the self-referential nature of rational expectations models. That is, the actual law of motion depends on the perceptions A_{t-1}, B_{t-1} and convergence is not obvious.

While assessing the asymptotic behavior of the non-linear stochastic process (A_t, B_t) is quite difficult, it turns out that the technical requirements for convergence often reduce to a fairly simple and intuitive condition known as E-stability, see Evans and Honkapohja (2001). To illustrate, suppose agents hold generic beliefs (A, B). The actual law of motion then defines a map $T : \mathbb{R}^n \oplus \mathbb{R}^{n\times k} \to \mathbb{R}^n \oplus \mathbb{R}^{n\times k}$ that takes perceived coefficients to actual coefficients

$$T(A, B) = (\beta A, \beta B\rho + \gamma).$$

Notice that the fixed point of the T-map identifies the unique rational expectations equilibrium of the model. The rational expectations equilibrium is said to be E-stable if it is a locally asymptotically stable fixed point of the ordinary differential equation (o.d.e.)

$$\frac{d(A,B)}{d\tau} = T(A,B) - (A,B). \tag{15}$$

The E-stability Principle states that if agents use recursive least squares – or, similar reasonable learning algorithms – then E-stable rational expectations equilibria are locally stable under learning.[8] In this simple example, if $(0, b)$ is a locally asymptotically stable fixed point of (15) then $(A_t, B_t) \to (0, b)$ almost surely.

The economic intuition behind the E-stability principle is that reasonable learning algorithms dictate that agents update their parameter estimates in the direction of forecast errors. This is evident in (15), as $T(A,B) - (A,B)$ is, in a sense, a forecast error. If the resting point of the o.d.e. is stable then adjusting parameters in the direction of the forecast error will lead the parameters toward the rational expectations equilibrium. Conveniently, conditions for local asymptotic stability are easily computed by examining the eigenvalues of the Jacobian matrix DT. If all eigenvalues of DT have real parts less than one then the rational expectations equilibrium is E-stable. For the case at hand, the derivatives are given by β and $\rho' \otimes \beta$.[9] Since the model is determinate by assumption, the eigenvalues of β are inside the unit circle and so the rational expectations equilibrium is stable under learning.

If the model is indeterminate then there exists a continuum of equilibria. To analyze stability under learning, we must take a stand on the information available to agents. To fix ideas and avoid unnecessary complications, assume that the model (13) is univariate and non-stochastic ($r_t = 0$). We first assume that agents engage in "mean value learning," that is, they have knowledge of the endogenous variable's lag structure, and thus have only the mean to estimate.[10] Specifically, agents condition on the extrinsic process $\eta_t = \beta^{-1}\eta_{t-1} + \xi_t$, where ξ_t is a martingale difference sequence capturing fluctuations in forecast error. The extrinsic noise process η_t captures the serial correlation that arises as a self-fulfilling outcome. By conditioning on η_t the lag structure is imposed exogenously. Agents form expectations using a forecasting model of the form

$$y_t = A + B\eta_t.$$

Computing the T-map provides $DT_A = \beta, DT_B = I$, so that the sunspot equilibria are E-stable provided the eigenvalues of β are less than -1.

[8]The connection between E-stability of a rational expectations equilibrium and its stability under real time learning is quite deep: see Evans and Honkapohja (2001) for details.

[9]Here, and below, we exploit that when the T-map decouples, we can compute derivatives separately. Also, recall that the eigenvalues of the Kronecker product are the products of the eigenvalues.

[10]This method of learning is closely related to common factor representations, see Evans and McGough (2005) for details.

Another natural learning process is "VAR learning" where agents estimate both the mean and the lag structure of the endogenous variables. Specifically, agents condition their forecast on the martingale difference sequence sunspot ξ_t, as well as a constant and lagged y

$$y_t = A + By_{t-1} + C\xi_t.$$

The primary difference between VAR and mean value learning is that the latter assumes agents identically coordinate on the serially correlated sunspot η_t, while the former postulates that agents try to detect the appropriate lag structure from the data. Under VAR learning, then, the lag structure is determined endogenously. Computing the T-map provides the following derivatives

$$\begin{aligned} DT_A &= \beta(1+b) \\ DT_B &= 2bB \\ DT_C &= \beta b. \end{aligned}$$

Since $\bar{b} = \beta^{-1}$ it follows that that $DT_B = 2$. So if agents employ VAR learning, then the sunspot equilibria are never stable.

This example illustrates that the stability of sunspot equilibria depends on agents' conditioning set. By incorporating the serial correlation into η_t – which only arises in the model because of self-fulfilling expectations – the agents can coordinate on a sunspot equilibrium. If, however, they are trying to learn the mean and (endogenously determined) lag structure, coordination via learning is not possible.

4.2 E-stability in Regime-Switching Models

We now consider the stability properties of regime-switching equilibria.

4.2.1 E-stability and the CLDC

This Section demonstrates that the CLDC governs E-stability of RDE. If the CLDC is satisfied, then the unique RDE will have the following minimal state variable representation

$$y_t = B(s_t)r_t. \tag{16}$$

To solve for $B(s_t)$ for $s_t \in \{1, 2, ..., m\}$, use the stacked system and set

$$B = (B(1)', ..., B(m)')',$$

which yields $\hat{y}_t = Br_t$, where

$$\text{vec}(B) = \left(I_{nm} - \rho' \otimes \left(\oplus_{j=1}^m \beta_j\right)(P \otimes I_n)\right)^{-1} \text{vec}(\gamma).$$

It is worth remarking at this point that the class of RDE includes the MSV solution to the regime-switching model, but is larger than the set of MSV solutions since the RDE may also include a sunspot shock.

Using the representation (16) as our guide to specifying a perceived law of motion, we now turn to the stability of RDE under learning. Throughout, we assume that agents observe the current state s_t and know the true transition probabilities. This is consistent with the conventions of the adaptive learning literature that assumes agents observe contemporaneous exogenous variables, but not current values of endogenous variables.

Given that the CLDC is satisfied, the eigenvalues of $\left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)$ are inside the unit circle. Agents have a perceived law of motion (PLM) of the following form, which is consistent with the MSV solution,

$$y_t = A(s_t) + B(s_t)r_t \tag{17}$$

where $A(j)$ is $(n\times 1)$, and $B(j)$ is $(n\times k)$. Notice that we assume that agents do not know that in equilibrium the $A_i = 0$.[11]

Given the PLM in (17), expectations are state contingent, where $s_t = j$ implies

$$\begin{aligned} E_t\left[y_{t+1}|s_t = j\right] &= p_{j1}A(1) + p_{j2}A(2) + ... + p_{jm}A(m) + && (18)\\ & \quad \left(p_{j1}B(1) + p_{j2}B(2) + ... + p_{jm}B(m)\right)\rho r_t. && (19)\end{aligned}$$

This produces a state-contingent actual law of motion, or, equivalently, a state-contingent T-map

$$\begin{aligned} A(j) &\rightarrow \beta_j\left(p_{j1}A(1) + p_{j2}A(2) + ... + p_{jm}A(m)\right)\\ B(j) &\rightarrow \beta_j\left(p_{j1}B(1) + p_{j2}B(2) + ... + p_{jm}B(m)\right)\rho + \gamma_j.\end{aligned}$$

Conveniently, this state-contingent T-map may be stacked, and becomes the T-map associated to the stacked system under the PLM $\hat{y}_t = A + Br_t$, where, as before, $B = (B(1)', ..., B(m)')'$, and also $A = (A(1)', ..., A(m)')'$. The T-map is given by

$$T(A,B)' = \left(\left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)A, \left(\oplus_{j=1}^{m}\beta_j\right)(P\otimes I_n)B\rho + \gamma\right),$$

and the RDE is a fixed point of $T(A,B)$. Here $T:\mathbb{R}^{(nm\times 1)}\oplus\mathbb{R}^{(nm\times k)}\rightarrow\mathbb{R}^{(nm\times 1)}\oplus\mathbb{R}^{(nm\times k)}$.

[11] In the univariate case below, we consider real time learning based on an alternative perceived law of motion of the following form

$$y_t = A + \hat{A}(s_t - 1) + Br_t + \hat{B}(s_t - 1)r_t,$$

where s_t acts as a dummy variable. While this formulation is more natural for real time learning, the E-stability results are identical in both cases and so we focus on the more parsimonious (17). As an additional alternative, agents can have a PLM with the same form as the stacked system.

The eigenvalues of the Jacobian matrices

$$\begin{aligned} DT_A &= \left(\oplus_{j=1}^m \beta_j\right)(P \otimes I_n) \\ DT_B &= \rho' \otimes \left[\left(\oplus_{j=1}^m \beta_j\right)(P \otimes I_n)\right] \end{aligned}$$

govern E-stability. Thus, we obtain the following result:

Proposition 2 *If the eigenvalues of $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$ are inside the unit circle (i.e. the CLDC holds), then the unique RDE is E-stable.*

This result states that an economy described by the main expectational difference equation (4), with expectations formed using (18) and updated using least squares, will converge to the unique RDE.

4.2.2 E-stability and Indeterminacy

Now we examine the stability of HDE, and again, for simplicity, and without loss of generality, we set $\gamma_t = 0$. We begin by considering VAR learning. In this case, the PLM takes the following form:

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t \tag{20}$$

where ξ_t is the m.d.s. sunspot variable independent of the Markov states. The PLM makes clear the primary distinction of HDE from the class of RDE solutions, since coefficients depend explicitly on s_t and s_{t-1}, whereas coefficients in the PLM for the RDE only depend on s_t.

Taking expectations conditional on the PLM given by (20) and values of (s_{t-1}, s_t) yields

$$\begin{aligned} E_t(y_{t+1}|s_{t-1} &= i, s_t = j) = \sum_{k=1}^m p_{jk} A(j,k) + \\ & \left(\sum_{k=1}^m p_{jk} B(j,k)\right)(A(i,j) + B(i,j)y_{t-1} + C(i,j)\xi_t). \end{aligned}$$

The T-map is given by

$$A(i,j) \rightarrow \beta_j \left[\sum_{k=1}^m p_{jk} A(j,k) + \left(\sum_{k=1}^m p_{jk} B(j,k)\right) A(i,j)\right], \tag{21}$$

$$B(i,j) \rightarrow \beta_j \left(\sum_{k=1}^m p_{jk} B(j,k)\right) B(i,j), \tag{22}$$

$$C(i,j) \rightarrow \beta_j \left(\sum_{k=1}^m p_{jk} B(j,k)\right) C(i,j). \tag{23}$$

E-stability is determined by the Jacobian $DT(A, B, C)$, where A, B, C are the HDE parameters found by solving (11)-(12).

General results on the stability of HDE are not available because the set of all fixed point solutions to (11)-(12) has not been characterized. For example, in the univariate example presented below there is a continuum of fixed points to the T-map. There is, however, a subclass of fixed points, corresponding to those studied by Farmer, Waggoner and Zha (2009), whose E-stability properties can, in general, be characterized. Recall the model given by

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t, \tag{24}$$

$$r_t = \rho r_{t-1} + \varepsilon_t. \tag{25}$$

Assume the underlying Markov process is 2-state, and assume that $\det(\beta_i) \neq 0$ for $i = 1, 2$. If the eigenvalues of β_2 are larger than $1/p_{22}$ in modulus, then there exists a continuum of HDE associated to the following values of B:

$$B(1,1) = B(2,1) = 0, B(1,2) = \frac{\beta_1^{-1}}{p_{12}}, B(2,2) = \frac{\beta_2^{-1}}{p_{22}}. \tag{26}$$

The restriction that the eigenvalues of β_2 are larger, in modulus, than $1/p_{22}$ ensures that the resulting process is uniformly bounded. Moreover, it is straightforward to see that the coefficients B in (26) are a fixed point to the T-map and, hence, constitute an HDE. We have the following result:

Proposition 3 *HDE of the form (26) are never stable under VAR learning.*

The proof is contained in the Appendix. The next subsection discusses, in further detail, the intuition and significance of this result.

Proposition 3 applies to the sunspot equilibria examined in the New Keynesian model by Farmer, Waggonner and Zha (2009). The model is given by (1), (2), and (3), and is repeated here for convenience:

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + g_t \\ x_t &= E_t x_{t+1} - \sigma^{-1} (i_t - E_t \pi_{t+1}) + u_t \\ i_t &= \alpha_t \pi_t + \gamma_t x_t. \end{aligned}$$

Simplification places this system into the form (24), (25), with β_i invertible. Because Farmer, Waggonner and Zha (2009) study equilibria corresponding to conditions (26), we may conclude that the sunspot equilibria they identify are not stable under VAR learning.

Now consider mean value learning. In this case, agents are assumed to understand the endogenous lagged dependence in the model and only estimate the state-dependent mean. Specifically, assume agents observe and understand the stochastic structure of the extrinsic noise process, η_t, where

$$\eta_t = \phi(s_{t-1}, s_t)\eta_{t-1} + \theta(s_{t-1}, s_t)\xi_t$$

for appropriately defined $\phi(i,j), \theta(i,j)$. Then we take our agent's forecasting model as

$$y_t = A(s_{t-1}, s_t) + B\eta_t$$

The corresponding T-map is

$$A(i,j) \to \beta_j \sum_{k=1}^{m} p_{jk} A(j,k)$$

and $T(B) = B$.[12]

Proposition 4 *Assume HDE exist. If there exists a unique E-stable RDE, then the HDE is stable under mean value learning. Furthermore, the CLDC governs stability of HDE.*

Proposition 4 also applies to the sunspot equilibria examined in the New Keynesian model by Farmer, Waggonner and Zha (2009). Because they showed that, under their calibration, the CLDC holds, it follows that the sunspot equilibria they identify are stable under mean value learning.

4.3 Further Discussion

The intuition for the instability of HDE under VAR learning can most easily be seen in the constant parameter example presented in Section 4.1. In that analysis, it was shown that when agents try to learn the lag coefficient their beliefs are driven further and further away from the rational expectations value. The proof to Proposition 3 illustrates that the T-map in the regime-switching model exhibits the same dynamic feature. Since under mean-value learning this lag structure is imposed exogenously, and known to the agents, it does not have the same unstable learning dynamic.

It is somewhat surprising that HDE can be stable under mean value learning. In a similar New Keynesian model, Evans and McGough (2005) studied the stability under mean-value learning of rational expectations equilibria and found that, under

[12]The result $T(B) = B$ is standard in models with sunspots and reflects the fact that multiples of sunspots are also sunspots.

a constant parameter policy rule of the same form as considered in this paper, the common factor representation of a sunspot equilibrium is E-unstable. It is interesting to note that the regime-switching formulation of the model implies that the HDE can be E-stable under mean-value learning.

The long run Taylor Principle of Davig and Leeper provides a natural generalization of the well understood Taylor Principle to an environment where policy occasionally shifts from being active to passive. However, as Farmer, Waggoner, and Zha clearly demonstrate the long run Taylor Principle may not rule out indeterminacy and the economy coordinating on an inefficiently volatile equilibrium. Stability under adaptive learning provides a natural equilibrium selection metric that policymakers can use to assess which, if any, equilibria the economy can be reasonably expected to coordinate on. The results of the previous section demonstrate that the CLDC ensures the existence of a unique E-stable regime dependent equilibrium. Even if there exists a unique E-stable RDE there may exist other inefficient equilibria, and whether the economy can coordinate on them depends crucially on what it is assumed agents know when forming their expectations. If agents know the economy's lag structure, which are then imposed exogenously onto the model, then provided the CLDC is satisfied the mean-value learning will converge to a history dependent equilibrium. If, however, the serial correlation arises endogenously through agents' self-referential beliefs, then the history dependent equilibria will be unstable under learning.

The primary reason why policymakers should set policy so that the economy does not coordinate on a sunspot equilibrium is the inefficient volatility and increased propogation of shocks that arise through private-sector agents' self-fulfilling beliefs. Thus, a history dependent equilibrium can be attainable under an adaptive learning rule only if the lag structure in forward-looking self-referential models is imposed exogenously. If the metric for equilibrium selection is whether agents can coordinate on a self-fulfilling equilibrium, then a natural way to model their learning process is the VAR-learning approach and, in this case, the economy will not coordinate on an HDE. Therefore, depending on the informational assumptions imposed on private-sector agents, adaptive learning can provide a sharp equilibrium selection result.

5 Examples

The previous section provided the conditions under which an equilibrium is learnable, and demonstrated that in the case of RDE these conditions reduce to the CLDC, the uniqueness condition within the class of RDE. In this section we illustrate the equilibrium selection results of this paper by examining in detail a simple univariate model, with particular focus on explicit construction and stability analysis of the HDE, and an analytic and numerical study of real time learning in case of RDE.

5.1 Univariate Model: RDE and HDE

To provide a concrete illustration of the class of solutions, we consider the special case where y_t is univariate, s_t takes values in $\{1, 2\}$, and $\gamma_t = 0$. Then

$$y_t = \beta_t E_t y_{t+1}. \tag{27}$$

Note that in this case, if there is a unique RDE, it is, trivially, $y_t = y_{it} \Leftrightarrow s_t = i$, where $y_{it} = 0$ for $i = 1, 2$.

To compute an HDE, recall that a rational expectations equilibrium is a process y_t such that

$$y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t \tag{28}$$

where ξ_t is an m.d.s. that satisfies $E_{t-1}\xi_t = 0$. Of particular interest is the case in which "one regime is determinate and one regime is indeterminate," or, formally, for example, $|\beta_1| < 1 < \beta_2$. In this case, non-degeneracy requires that regimes are not absorbing, so that $p_{22} > 0$. Define

$$\xi_t = \begin{cases} -\beta_1^{-1} y_{t-1} + \delta_{11} v_t & (s_{t-1}, s_t) = (1,1) \\ \frac{p_{11}}{p_{12}} \beta_1^{-1} y_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1,2) \\ -\beta_2^{-1} y_{t-1} + \delta_{21} v_t & (s_{t-1}, s_t) = (2,1) \\ \frac{p_{21}}{p_{22}} \beta_2^{-1} y_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2,2) \end{cases}$$

where $\delta_{ij} \in \mathbb{R}$ is arbitrary, and v_t is any martingale difference sequence with uniformly bounded support. The dynamics for y_t follow

$$y_t = \begin{cases} \delta_{11} v_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{p_{12}} \beta_1^{-1} y_{t-1} + \delta_{12} v_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21} v_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{p_{22}} \beta_2^{-1} y_{t-1} + \delta_{22} v_t & (s_{t-1}, s_t) = (2,2) \end{cases}, \tag{29}$$

Note that provided $|\beta_2 p_{22}| > 1$, y_t is UB. The process given by (29) is an HDE, since dynamics explicitly depend on s_t and s_{t-1}. Notice that the indeterminacy of region 2 spills over across regimes so that there is sunspot dependence in both regimes. It should be clear from this representation of an HDE that it is not possible to represent this class of equilibria in terms of a stacked system. In an RDE, y_t switches between two stochastic processes that are independent of the underlying Markov state. In an HDE the value of y_t depends on the current state s_t and also explicitly on the Markov state in the previous period. This dependence is self-fulfilling in the sense that it exists only because agents expect it.

We now turn to the stability of the univariate HDE given in (29), and we consider VAR learning. In this case, the parameters A, B, C in the PLM (20) are elements of the real line. Computing conditional forecasts using this PLM, we obtain the following T-map for B:

$$B(i,j) \longrightarrow \beta_j \left(p_{j1}B(j,1) + p_{j2}B(j,2)\right) B(i,j) \tag{30}$$

Ignoring the boundedness requirement, a fixed point of this map identifies an HDE. The only restrictions, then, are the following:

$$1 = \beta_1 \left(p_{11}B(1,1) + p_{12}B(1,2)\right) = \beta_2 \left(p_{21}B(2,1) + p_{22}B(2,2)\right). \tag{31}$$

In particular, there is a two dimensional continuum of coefficients on lagged y providing fixed points.

As noted in Section 4.2.2, Farmer, Waggoner, and Zha (2009b,a) focus on particular fixed points, given by

$$B(1,1) = B(2,1) = 0, \;\; B(1,2) = \frac{\beta_1^{-1}}{p_{12}}, \;\; B(2,2) = \frac{\beta_2^{-1}}{p_{22}},$$

and $d(i,j) = \delta_{ij}$.

To analyze stability we compute the eigenvalues of DT. The first four equations decouple, and, when evaluated at the fixed point, provide the following Jacobian:

$$DT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta_1^{-1}\beta_2 p_{21}/p_{12} & \beta_1^{-1}\beta_2 p_{21}/p_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & p_{21}/p_{22} & 2 \end{pmatrix}.$$

The Jacobian has an eigenvalue of 2, which implies that under VAR learning, HDE are E-unstable.

This instability result for HDE generalizes a finding in the sunspots literature. The instability of HDE arise here because of the difficulty in coordinating on a particular sunspot. In an HDE, agents' expectations have the additional propagation that arises through the lagged endogenous state variable. Out of equilibrium, if agents' hold beliefs close to the equilibrium, when agents' extrapolate these beliefs the actual parameters move further away from the equilibrium. Hence, because the additional propagation only exists because of self-fulfilling expectations, they fail to coordinate on these sunspot equilibria. If, however, this serial propagation is exogenously imposed into agents' information set via the sunspot, then the equilibria can be stable.

5.2 Real Time Learning of RDE

The connection between E-stability and stability under real-time learning is made in constant parameter models by Evans and Honkapohja (2001). However, it is not clear that the results in Evans and Honkapohja (2001) apply to the regime-switching framework. To address this issue, we present a real time learning formulation of regime dependent equilibria.

We again take y_t to be univariate, and assume s_t takes values in $\{1, 2\}$, but now we allow γ_t to be non-trivial. The model is given by

$$y_t = \beta_t E_t y_{t+1} + \gamma_t r_t. \tag{32}$$

Assume $(\beta_1 \oplus \beta_2)P$ has eigenvalues inside the unit circle, so that there is a unique RDE. To consider the stability under learning of this RDE, we provide agents with the following forecasting model

$$y_t = A + \hat{A}\hat{s}_t + Br_t + \hat{B}\hat{s}_t r_t,$$

where $\hat{s}_t = s_t - 1$. To simplify notation, let $\theta = \left(A, \hat{A}, B, \hat{B}\right)'$ and $X = (1, \hat{s}_t, r_t, \hat{s}_t r_t)'$. Agents estimate θ by regressing y_t on X_t. Letting θ_t be the time t estimate of θ, the recursive formulation of this estimation procedure is given by

$$\begin{aligned} \theta_t &= \theta_{t-1} + t^{-1} R_t^{-1} X_t \left(y_t - \theta'_{t-1} X_t \right) \\ R_t &= R_{t-1} + t^{-1} \left(X_t X'_t - R_{t-1} \right). \end{aligned} \tag{33}$$

The matrix R consists of the sample second moments of the regressors. The agents use these estimates, together with their forecasting model, to form expectations. These expectations are embedded into the expectational difference equation to obtain the actual law of motion and associated T-map, where the actual law of motion may then be written $y_t = T(\theta_{t-1})X_t$. The T-map is given by

$$\begin{aligned} A &\rightarrow \beta_1(A + \hat{A}(1 - p_{11})) \\ \hat{A} &\rightarrow \beta_2(A + \hat{A}p_{22}) - \beta_1(A + \hat{A}(1 - p_{11})) \\ B &\rightarrow \beta_1(B + \hat{B}(1 - p_{11}))\rho \\ \hat{B} &\rightarrow \beta_2(B + \hat{B}p_{22})\rho - \beta_1(B + \hat{B}(1 - p_{11}))\rho. \end{aligned}$$

Imposing this into the algorithm (33) identifies a dynamic system that can be analyzed using the theory of stochastic recursive algorithms. Letting θ^* be the fixed point of the T-map identifying an RDE, the learning question is, does θ_t converge to θ^* almost surely? We have the following proposition.

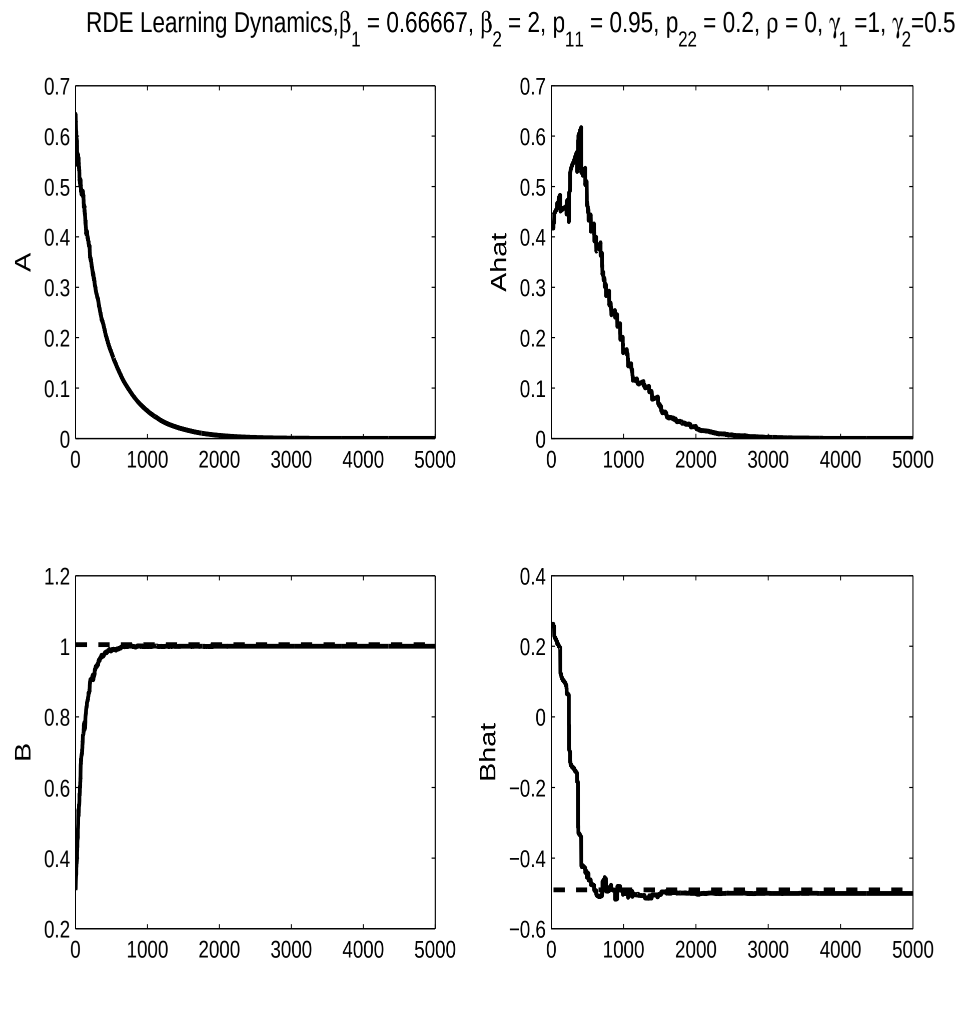
Proposition 5 *If $\gamma_t = 0$, then, locally, $\theta_t \rightarrow \theta^*$ almost surely.*[13]

[13] As is standard in the learning literature, in order to apply the theory of stochastic recursive algorithms requires imposing a "projection facility" on the recursive least squares algorithm. See Evans and Honkapohja (2001) for details.

The restriction γ_t is needed to simplify the proof, though we feel it is very likely that the proposition holds for $\gamma_t \neq 0$. The difficulty raised by non-zero γ reflects the fact that the state dynamics are no longer conditionally linear, a property that the convergence theorems typically rely upon.

To illustrate this result for $\gamma \neq 0$, we use simulations. We parameterize the model so that the CLDC is satisfied. This ensures the existence of a unique rational expectations equilibrium that is also an RDE. We set $\beta_1 = 1/1.5, \beta_2 = 2, p_{11} = .95, p_{22} = .2, \rho = 0, \gamma_1 = 1, \gamma_2 = .5$. For these parameter values the unique RDE coefficients are $A_1 = A_2 = 0, B_1 = 1, B_2 = .5$. We draw initial conditions for θ randomly and simulate the model for 5000 time periods. Figure 1 plots a typical simulation. As the figure makes clear, the RDE is stable under least squares learning.

Figure 1: Real time learning of an RDE.



6 Conclusion

This paper studies the existence and stability of two classes of rational expectations equilibria in a regime-switching rational expectations model under adaptive learning, extending the literature on learning to a non-linear framework. Building on the work of Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2006, 2009b), the two classes are:

- *Regime Dependent Equilibria*: An RDE is a uniformly bounded process that satisfies the regime-switching expectational difference equation and imposes the restriction that agents do not condition their expectations on lagged regimes (i.e. only the current regime enters the state vector).
- *History Dependent Equilibria*: An HDE is a process that satisfies the regime-switching expectational difference equation, where agents condition expectations on current and lagged values of the regime (i.e. current and past regimes enter the state vector).

The Conditionally Linear Determinacy Condition (CLDC), a generalization of the long run Taylor Principle of Davig and Leeper (2007), ensures the existence of a unique RDE that is also E-stable. When the CLDC is satisfied, there may still exist sunspot equilibria as demonstrated by Farmer, Waggoner, and Zha (2009b). However, we demonstrate that these HDE may not be learnable, depending on the conditioning set imposed on boundedly rational agents.

Our results have implications for monetary policy design and indicate that as long as monetary policy follows the long run Taylor Principle (i.e. the CLDC holds), then private agents that use adaptive learning to infer the mean and lag properties of the data will converge on the unique RDE, which is free of sunspot shocks.

7 Appendix

Proof of Proposition 1

To establish part (b.), let y_{it} identify an RDE. Denote by f_t the time t density functions; for example, $f_t\left(y, s|s_{t-1}=i, \Omega_{t-1}\right)$ is the joint density of y_t and s_t conditional on $s_{t-1}=i$ and on all other time $t-1$ information, not including current and past s_{t-1}, as captured by Ω_{t-1}. Also, let $f_t^i\left(y|\Omega_{t-1}\right)$ be the density for y_{it} conditional on Ω_{t-1}, and $f\left(s|s_{t-1}=i\right)$ be the conditional density of s_t given $s_{t-1}=i$ (course, $f\left(s=j|s_{t-1}=i\right)=P_{ij}$). With this notation, we may compute expectations as follows:

$$\begin{aligned} E(y_{t+1}|s_t = i, \Omega_t) &= \int\int y f_{t+1}\left(y, s|s_t = i, \Omega_t\right) dsdy \\ &= \int\int y f_{t+1}\left(y|s, s_t = i, \Omega_t\right) f(s|s_t = i)\, dsdy \\ &= \int\int y f^s_{t+1}\left(y|\Omega_t\right) f(s|s_t = i)\, dsdy \\ &= \sum_j P_{ij} E_t y_{jt+1}, \end{aligned}$$

where the third equality precisely follows from the facts that $y_t = y_{it} \Leftrightarrow s_t = i$ and that y_{it} is independent of s_{t+n} for all n. Now we may simply use this formula for the expectations of y_t to verify that the stacked system is satisfied.

The other parts of the proposition are established in the text in the Section on HDE.

Proof of Proposition 3 The block of the T-map associated to the perceived parameters B is given by

$$B(i,j) \to \beta_j\left(p_{j1}B(j,1) + p_{j2}B(j,2)\right) B(i,j).$$

Because this block decouples from the rest, showing that this block of the T-map is unstable is sufficient. The Jacobian is given by

$$\begin{pmatrix} DT^1_B & DT^2_B \\ DT^3_B & DT^4_B \end{pmatrix}$$

where

$$\begin{aligned} DT^1_B &= \begin{pmatrix} DT^1_B(1,1) & p_{12}B(1,1)' \otimes \beta_1 \\ 0 & I \otimes \beta_2\left(p_{21}B(2,1) + p_{22}B(2,2)\right) \end{pmatrix} \\ DT^2_B &= \begin{pmatrix} 0 & 0 \\ p_{21}B(1,2)' \otimes \beta_2 & p_{22}B(1,2)' \otimes \beta_2 \end{pmatrix} \\ DT^3_B &= \begin{pmatrix} p_{11}B(2,1)' \otimes \beta_1 & p_{12}B(2,1)' \otimes \beta_1 \\ 0 & 0 \end{pmatrix} \\ DT^4_B &= \begin{pmatrix} I \otimes \beta_1\left(p_{11}B(1,1) + p_{1,2}B(1,2)\right) & 0 \\ p_{21}B(2,2)' \otimes \beta_2 & DT^4_B(2,2) \end{pmatrix} \\ DT^1_B(1,1) &= p_{11}B(1,1)' \otimes \beta_1 + I \otimes \beta_1\left(p_{11}B(1,1) + p_{12}B(1,2)\right) \\ DT^4_B(2,2) &= p_{22}B(2,2)' \otimes \beta_2 + I \otimes \beta_2\left(p_{21}B(2,1) + p_{22}B(2,2)\right). \end{aligned}$$

Inserting

$$B(1,1) = B(2,1) = 0, \;\; B(1,2) = \frac{\beta_1^{-1}}{p_{12}}, \;\; B(2,2) = \frac{\beta_2^{-1}}{p_{22}}$$

yields repeated unit eigenvalues, plus the eigenvalues of

$$\beta_2^{-1} \otimes \beta_2 + I_2 \otimes I_2. \tag{34}$$

Now notice that for a given $n\times$ matrix A, if λ is an eigenvalue of A, then $\lambda + 1$ is an eigenvalue of $A + I_n$. Since the eigenvalues of $\beta_2^{-1} \otimes \beta_2$ are unity, we conclude that the eigenvalues of (34) are all equal to 2, thus implying instability.

Proof of Proposition 5

Using the notation from the body of the paper, we may write the recursive algorithm as

$$\begin{aligned} \theta_t &= \theta_{t-1} + t^{-1} S_{t-1}^{-1} X_t \left(y_t - \theta_{t-1}' X_t \right) \\ S_t &= S_{t-1} + t^{-1} \left(X_t X_t' - S_{t-1} \right) - \frac{1}{t^2} \frac{t}{t+1} \left(X_t X_t' - S_{t-1} \right), \end{aligned}$$

where $X_t = (1, \hat{s}_t, r_t, \hat{s}_t r_t)$ and $S_{t-1} = R_t$. If X_t could be written as a linear difference equation in i.i.d. noise conditional on values of θ and S, we could immediately apply the main results of the learning literature; however, X_t is not conditionally linear, so we must work harder: we must verify conditions M in Chapter 7.3 of Evans and Honkapohja (2001).

First, notice that the evolution of X_t is independent of θ and S, simplifying our task. Let $Q^n(x, \cdot)$ be the distribution of X_{t+n} given that $X_t = x$. We must demonstration the following:

1. For all n, m there exists K so that $\int \left(1 + \|y\|^m\right) Q^n(x, dy) \le K \left(1 + \|x\|^m\right)$
2. For all p there exist K and δ so that for all $g \in \text{Li}(p)$, for all n and for all x_1,x_2, we have

   $$\left| \int g(y) Q^n(x_1, dy) - \int g(y) Q^n(x_2, dy) \right| \le K \rho^n \|x_1 - x_2\| (1 + \|x_1\|^p + \|x_2\|^p)$$

   Here $\text{Li}(p)$ is a space of functions from $\mathbb{R}^2$ to itself, defined in Evans and Honkapohja (2001): it turns out, as will be seen shortly, the simplicity of our set-up allows us to ignore the special properties of $\text{Li}(p)$, so we may simply take any $g : \mathbb{R}^2 \to \mathbb{R}^2$.
3. For all $q \ge 1$ there exist n, $\alpha < 1$ and β so that for all x we have

   $$\int \|y\| Q^n(x, dy) \le \alpha \|x\|^q + \beta.$$

The simplicity of our state dynamics allows these items to be easily demonstrated. Indeed, X_t is uniformly bounded a.s. by some number M, so $\int (1 + \|y\|^m)\, Q^n(x, dy) \leq 1 + M^m$, thus demonstrating item 1. Item 2, which would be quite difficult to demonstrate if $\gamma_t \neq 0$ follows here because there only two states: the left-hand-side is

$$\left(\begin{array}{c} (1,0)P^n \begin{pmatrix} g_1(x_1) \\ g_1(x_2) \end{pmatrix} \\ (0,1)P^n \begin{pmatrix} g_2(x_1) \\ g_2(x_2) \end{pmatrix}, \end{array} \right)$$

which goes to zero exponentially because P is a stochastic matrix (here g_i is the $i-th$ coordinate of g, and it is because there are only a finite number of states that we do not have to worry about the special properties of g). Finally, item 3 follows in a fashion similar to item one because X_t is uniformly bounded.

Because the Markovian state dynamics satisfy the correct conditions, we may proceed as usual: stack the estimators S and θ into a matrix ϕ and write the recursive system as

$$\phi_t = \phi_{t-1} + \frac{1}{t} H(\phi_{t-1}, X_t) + \frac{1}{t^2} q(t, \phi_{t-1}, X_t)). \tag{35}$$

The linearity of the T-map makes it straight-forward to verify that this recursion satisfies the necessary properties. Now set

$$h(\phi) = \lim_t E\left(H(\phi, X_t)\right).$$

The possible convergence points of (35) are the locally asymptotically stable fixed points of the differential equation $\dot{\phi} = h(\phi)$. Computing $h(\phi)$ yields the decoupled system

$$\begin{aligned} \frac{d\theta}{dt} &= S^{-1} E(X_t X_t')(T(\theta) - \theta) \\ \frac{dS}{dt} &= E(X_t X_t') - S. \end{aligned}$$

We conclude that locally asymptotic stability obtains provided the eigenvalues of DT have negative real part. The proof is completed by noting that the eigenvalues of DT are the eigenvalues of $(\beta_1 \oplus \beta_2)P$.

References

Andolfatto, D., and P. Gomme (2003): "Monetary Policy Regimes and Beliefs," *International Economic Review*, 44(1), 1–30.

Benhabib, J. (2009): "A Note on Regime Switching, Monetary Policy and Multiple Equilibria," .

Bernanke, B. S. (2004): "The Great Moderation," Remarks by Governor Ben S. Bernanke at the Meetings of the Eastern Economic Association, Washington D.C., February 20, 2004.

Blanchard, O., and C. M. Kahn (1980): "The Solution of Linear Difference Models under Rational Expectations," *Econometrica*, 48(5), 1305–1312.

Brainard, W. C. (1967): "Uncertainty and the Effectiveness of Policy," *The American Economic Review*, 57(2), 411–425.

Bullard, J., and K. Mitra (2002): "Learning About Monetary Policy Rules," *Journal of Monetary Economics*, 49(6), 1103–1129.

Chung, H., T. Davig and E. M. Leeper (2007): "Monetary and Fiscal Policy Switching," *Journal of Money, Credit and Banking*, 39(4), 809–842.

Clarida, R., J. Gali, and M. Gertler (1999): "The Science of Monetary Policy : A New Keynesian Perspective," *Journal of Economic Literature*, 37, 1661–1707.

Davig, T. (2004): "Regime-Switching Debt and Taxation," *Journal of Monetary Economics*, 51(4), 837–859.

Davig, T., and E. M. Leeper (2007): "Generalizing the Taylor Principle," *American Economic Review*, 97(3), 607–635.

Dennis, R. (2006): "The Policy Preferences of the U.S. Federal Reserve," *Journal of Applied Econometrics*, 21(1), 55–77.

Evans, G. W. (1986): "Expectational Stability and the Multiple Equilibria Problem in Linear Rational Expectations Models," *Quarterly Journal of Economics*.

Evans, G. W., and S. Honkapohja (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press.

Evans, G. W., and B. McGough (2005): "Stable Sunspot Solutions in Models with Predetermined Variables," *Journal of Economic Dynamics and Control*, 29, 601–625.

Farmer, R. A., D. F. Waggoner, and T. Zha (2006): "Indeterminacy in a Forward Looking Regime Switching Model," Working paper, University of California - Los Angeles.

——— (2009a): "Generalizing the Taylor Principle: A Comment," *American Economic Review*.

——— (2009b): "Understanding the New Keynesian Model when Monetary Policy Switches Regime," NBER Working Paper No. 14710.

Leeper, E. M., and T. Zha (2003): "Modest Policy Interventions," *Journal of Monetary Economics*, 50, 1673–1700.

Lucas, R. E. (1986): "Adaptive Behavior and Economic Theory," *Journal of Business*, 59(4), S401–S426.

Marcet, A., and T. J. Sargent (1989): "Convergence of Least Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48(2), 337–368.

Svensson, L. E., and N. Williams (2007): "Monetary Policy with Model Uncertainty : Distributional Forecast Targeting," Princeton University, mimeo.

Woodford, M. (1990): "Learning to Believe in Sunspots," *Econometrica*, 58(2), 277–307.

Zampolli, F. (2006): "Optimal Monetary Policy in a Regime-Switching Economy : The Response to Abrupt Shifts in Exchange Rate Dynamics," *Journal of Economic Dynamics and Control*, (30), 1527–1567.