Adaptive Learning in Regime-Switching Models*

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Abstract

We study adaptive learning in economic environments subject to recurring structural change. Stochastically evolving institutional and policymaking features can be described by regime-switching models with parameters which evolve according to finite state Markov processes. We demonstrate that in non-linear models of this form, the presence of sunspot equilibria implies two natural schemes for learning the conditional means of endogenous variables: under *mean value learning*, agents condition on a sunspot variable which captures the self-fulfilling serial correlation in the equilibrium; whereas, under *vector autoregression learning* (VAR learning), the self-fulfilling serial correlation must be learned. We show that an intuitive condition ensures convergence to a regime-switching rational expectations equilibrium. However, the stability of sunspot equilibria, when they exist, depends on whether agents adopt mean value or VAR learning: coordinating on sunspot equilibria via a VAR learning rule is not possible. To illustrate these phenomena, we develop results for an overlapping generations model and a New Keynesian model.

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1 Introduction

A given forward-looking macroeconomic model may admit classes of rational expectations equilibria that differ in terms of the set of state variables that agents use when forming expectations. For example, standard linear stochastic rational expectations models have solutions that depend only on the minimal set of state variables, and may also have solutions that depend on extrinsic random variables (i.e. sunspots). Existence and uniqueness of equilibria are well understood in linear models with constant parameters; however, in a growing area of research that focuses on models with changing parameters, these issues are re-emerging.[1] In regime-switching models, which constitute the focus of this paper, parameters evolve according to finite state Markov processes.[2] The non-linear structure of regime-switching rational expectations models prevents a complete characterization of the full class of solutions, though recent papers by Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009) suggest that multiplicity of equilibria – some depending on extrinsic sunspot or bubbles processes – can arise in models of interest to applied economists and policymakers.

There has long been an interest in indeterminacy, that is, the possibility that an economy can be driven by inefficient, self-fulfilling expectations, i.e. "sunspots". For example, an extensive literature studies whether monetary policy can be designed to prevent coordination on sunspots; and a separate literature examines the extent to which "rational bubbles" equilibria can account for observed movements in asset prices. In this paper, we are interested in a distinct but related question: in non-linear-regime switching models, which, if any, equilibria are attainable by agents using an adaptive learning rule, such as least squares? Equilibria which are attainable in this manner are said to be "learnable" or "stable under learning."

In regime-switching models, assessing stability under learning is complicated by the fact that the number and nature of the rational expectations equilibria depend on whether agents condition their expectations only on the current realization of the Markov chain capturing the time-varying nature of the model's parameters, or also on the history of its realizations. If agents condition their expectations only on the current regime, we say the economy is in a *regime dependent equilibrium* (RDE); if agents also condition their expectations on past regimes, we say the economy is in a *history dependent equilibrium* (HDE). It is well-known that conditions guaranteeing uniqueness within the class of regime dependent equilibria may not impinge on the

[1]Some examples of work in this area include Leeper and Zha (2003), Andolfatto and Gomme (2003), Benhabib (2009),Davig (2004), Zampolli (2006), Chung, Davig and Leeper (2007), Davig and Leeper (2007), Farmer, Waggoner, and Zha (2009), and Svensson and Williams (2007). Brainard (1967) is an early example of work on parameter instability.

[2]There has been an extensive empirical literature modeling the economy as following a regime-switching process (see, Hamilton (1989), Kim and Nelson (1999), Sims and Zha (2006)).

existence of history dependent equilibria: a model may have a unique RDE and also have multiple HDE. We note that history dependent equilibria correspond to an indeterminacy in the model since the dependence on past regimes is self-fulfilling.

The presence of multiple equilibria in our model leads naturally to the question of equilibrium selection, which is the main topic of our paper; and, as mentioned, we propose using stability under adaptive learning as the equilibrium selection mechanism. Following Lucas (1986), we maintain that stability under adaptive learning is a useful metric for identifying empirically relevant equilibria.[3] An equilibrium is plausible or reasonable if, whenever rational expectations are replaced with a standard adaptive learning rule, agents' beliefs converge to those consistent with the rational expectations equilibrium. Based on this assumption, we assess whether regime-switching equilibria are learnable.

Our viewpoint is informed by a large and growing literature that replaces rational expectations with learning rules where agents are modeled as professional econometricians, that is, they hold forecasting models that share a reduced-form with a rational expectations equilibrium, and they adjust the parameters of their model in light of new data. The advantage to this approach is that it places economist and agent on equal footing and avoids the cognitive dissonance inherent in rational expectations models. This approach is particularly compelling in regime-switching models because of the co-existence of equilibria in the regime and history dependent classes.

Among the stochastic properties of the sunspot equilibria associated to linear models with constant or time-varying parameters is self-fulfilling serial correlation, that is, serial correlation that is present in the equilibrium process only because agents believe it is present. Whether agents know and condition on this serial correlation, or instead must learn about it, implies two distinct learning rules, which may lead to distinct stability outcomes. Under "mean value learning," agents regress on a sunspot variable that includes in its stochastic structure the equilibrium's self-fulfilling serial correlation, whereas under "VAR learning" agents must learn the self-fulfilling serial correlation from the data by employing a (first order) vector autoregression model.

Our primary results are surprising. When the conditions for a unique RDE are satisfied, the associated equilibrium is stable under learning. Moreover, this condition also governs the stability of HDE under mean value learning; on the other hand, the HDE are not attainable under VAR learning.

The paper is organized as follows: Section 2 provides results for a simple univariate model; Section 3 generalizes the model and defines the classes of equilibria; Section 4 provides the main stability analysis; Section 5 presents applications to an over-lapping generations model with switching preferences, and to a New Keynesian model with switching monetary regimes; Section 6 concludes.

[3] See, in addition, Evans (1985), Bray and Savin (1986), Marcet and Sargent (1989).

2 The Univariate Case

We begin with a general, reduced-form, univariate, regime-switching model

$$y_t = \beta(s_t) E_t y_{t+1} + \gamma(s_t) r_t. \tag{1}$$

where r_t is a stationary, exogenous process, and $s_t \in \{1, 2\}$ is a two-state Markov process, independent of r_t, with transition matrix $P = (p_{ij})$. Although we assume that s_t takes on two values the analysis provided holds for any finite number of states.

We note that the expectational difference equation (1) might arise from the Fisherian model of inflation in Davig and Leeper (2007). The Fisherian model is given by

$$\begin{aligned} i_t &= E_t \pi_{t+1} + r_t \\ i_t &= \alpha(s_t) \pi_t. \end{aligned}$$

The first equation is a (log-linearized) Fisher equation, where i_t is the nominal interest rate, π_t is the inflation rate, and r_t is the real interest rate taken to be a stationary, exogenous process. The second equation is the policy rule pursued by the central bank in setting nominal interest rates, and it indicates the potentially time-varying nature of policy. By combining equations and redefining variables, we see that the Fisherian model fits the more general model (1). If (1) captures the Fisherian model then $\beta(s_t) > 0$, however, in general $\beta(s_t)$ can be any real number.

2.1 Rational Expectations Equilibria

A rational expectations equilibrium of the regime-switching model is any bounded solution to (1).[4] Because the model (1) is a non-linear expectational difference equation, it is not, in general, possible to identify the entire collection of rational expectations equilibria; however, it is possible to identify several natural classes of equilibria depending on the conditioning behavior of the agents.

Our first class of equilibria corresponds to those studied by Davig and Leeper (2007), and are closely associated to what Davig-Leeper call "the minimal state vari-

[4] The relevant notion of boundedness will be made formal in the next section.

able solution" (MSV) of (1), which takes the form[5]

$$y_t = B(s_t)r_t. \tag{2}$$

The dependence of y_t on s_t (and not lags of s_t), and the independence of s_t and r_t, suggests that we consider solutions to (1) of the form $y_t = y_{it} \Leftrightarrow s_t = i$, where y_{1t}, y_{2t} are stochastic processes independent of s_{t-n} for all n. We call such a solution a *regime dependent equilibrium* (RDE), and note that the MSV is an RDE.[6]

In an RDE, the dependence of y_t on s_t allows for simple state-contingent expectations formation so that the y_{it} solve

$$\begin{aligned} y_{1t} &= \beta_1 \left(p_{11}E_t y_{1t+1} + p_{12}E_t y_{2t+1}\right) + \gamma_1 r_t \\ y_{2t} &= \beta_2 \left(p_{21}E_t y_{1t+1} + p_{22}E_t y_{2t+1}\right) + \gamma_2 r_t, \end{aligned} \tag{3}$$

where, here and in the sequel, we use the notation $\beta_i = \beta(i)$ and $\gamma_i = \gamma(i)$. We refer to (3) as the *stacked system* and conclude that an RDE must satisfy (3). We also note that the MSV-solution (2) identified by Davig and Leeper (2007) corresponds to the unique MSV-solution to (3) in the sense of McCallum (1983).[7]

The stacked system (3) shows that by conditioning expectations on current s_t, the univariate non-linear model (1) is recast into a multivariate linear model, which can then be solved using standard techniques, e.g. Blanchard and Khan (1980); and, in particular, the number and nature of the RDE can be determined. Recall that a linear model is determinate if there is a unique REE and indeterminate if there are multiple REE. Standard determinacy analysis implies that there is a unique non-explosive rational expectations equilibrium to (3), and therefore a unique RDE to (1), provided that the matrix

$$\begin{pmatrix} \beta_1 p_{11} & \beta_1 p_{12} \\ \beta_2 p_{21} & \beta_2 p_{22} \end{pmatrix} \tag{4}$$

has eigenvalues inside the unit circle. We refer to the determinacy condition in (4) as the Conditionally Linear Determinacy Condition (CLDC); and it is analogous to the Long Run Taylor Principle in Davig and Leeper (2007).[8]

[5] A minimal state variable solution of a system of expectational difference equations is a notion introduced by McCallum (1983), and corresponds to a solution that conditions on the smallest possible collection of exogenous and predetermined variables. McCallum (1983) provides an additional "limiting" criterion to select among multiple MSV-like solutions. Within the context of the models studied by McCallum, his limiting criterion identifies a unique MSV solution. In this paper, we will simply refer to the equilibrium of the form (2) as the MSV-solution. The multivariate counterpart in Section 3 has the same form.

[6] The precise definition of an RDE will be given in the next section.

[7] A more complete discussion of the stacked system is given in Section 3.1.

[8] Davig and Leeper (2007) showed that, in the univariate case, provided that $|\beta_1| < 1 < |\beta_2|$, the CLDC is equivalent to

$$p_{11}\beta_1(1-\beta_2) + p_{22}\beta_2(1-\beta_1) + \beta_1\beta_2 < 1.$$

Because the underlying model (1) is non-linear, the CLDC does not necessarily guarantee uniqueness when agents also condition their expectations on s_{t-n} for $n > 0$. This point was made by Farmer, Waggoner, and Zha (2009), who showed that while the CLDC implies uniqueness for the univariate model in case of positive feedback across regimes (i.e. for all i, $\beta_i > 0$), if some regimes exhibit negative feedback then multiple equilibria may exist even when the CLDC is satisfied. To illustrate, we first simplify the model by setting $\gamma_i = 0$. In this case, the MSV solution is particularly simple: $y_t = 0$. Now note that a rational expectations equilibrium y_t may be associated to a martingale difference sequence (mds) ξ_t, i.e. $E_{t-1}\xi_t = 0$, so that

$$y_t = \beta_{t-1}^{-1} y_{t-1} + \xi_t, \tag{5}$$

and further, any mds ξ_t identifies an REE via (5), provided that the implied process for y_t is bounded. We now construct an mds ξ_t so that the boundedness criterion is met. Assume $|\beta_1| < 1 < |\beta_2|$ (this corresponds to the case in which "one regime is determinate and one regime is indeterminate"). Non-degeneracy requires that regimes are not absorbing; thus let $p_{22} > 0$. Define

$$\xi_t = \begin{cases} -\beta_1^{-1} y_{t-1} + \delta_{11}\varepsilon_t & (s_{t-1}, s_t) = (1,1) \\ \frac{p_{11}}{p_{12}}\beta_1^{-1} y_{t-1} + \delta_{12}\varepsilon_t & (s_{t-1}, s_t) = (1,2) \\ -\beta_2^{-1} y_{t-1} + \delta_{21}\varepsilon_t & (s_{t-1}, s_t) = (2,1) \\ \frac{p_{21}}{p_{22}}\beta_2^{-1} y_{t-1} + \delta_{22}\varepsilon_t & (s_{t-1}, s_t) = (2,2) \end{cases}$$

where $\delta_{ij} \in \mathbb{R}$ is arbitrary, and ε_t is any martingale difference sequence with uniformly bounded support. Then ξ_t is an mds.[9] The dynamics for y_t implied by plugging ξ_t into (5) follow

$$y_t = \begin{cases} \delta_{11}\varepsilon_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{p_{12}}\beta_1^{-1} y_{t-1} + \delta_{12}\varepsilon_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21}\varepsilon_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{p_{22}}\beta_2^{-1} y_{t-1} + \delta_{22}\varepsilon_t & (s_{t-1}, s_t) = (2,2) \end{cases} \tag{6}$$

It is straightforward to verify that (6) is a solution to the model by stepping y_t forward one period, taking conditional expectations, and plugging into (1). To show that it is an REE, we have to demonstrate boundedness. We have the following result[10]:

Lemma 1 *The process y_t, as given by (6), is uniformly bounded if and only if $|\beta_2 p_{22}| > 1$.*

Intuitively, provided that $|\beta_2 p_{22}| > 1$, the only explosive "state" occurs when $(s_{t-1}, s_t) = (1,2)$, but then $(s_t, s_{t+1}) = (2,1)$ or $(s_t, s_{t+1}) = (2,2)$: either way, the divergence is halted.

[9] It is straightforward to verify that ξ_t is an mds by taking one step ahead expectations conditional on the observable states $s_{t-n}, n > 0$.

[10] The proof is in the Appendix, Section 7.1.

It is straightforward to find β_i so that the CLDC is satisfied and (6) is uniformly bounded, and a micro-founded example based on an overlapping generations model is provided in Section 5.1. We note that if the CLDC is satisfied and $|\beta_2 p_{22}| > 1$ then β_1 must be positive.[11] Thus, in order for multiple equilibria to exist when the CLDC is satisfied it must be the case that $\beta_2 < 0$ so that the univariate model exhibits positive feedback in the determinate regime and negative feedback in the indeterminate regime. This observation will be important when constructing the example in Section 5.1.

When $|\beta_2 p_{22}| > 1$, we refer to the process given by (6) as a *history dependent equilibrium*, since dynamics explicitly depend on s_t and s_{t-1}. Notice that the indeterminacy of regime 2 spills over across regimes so that there is sunspot dependence in both regimes. This is in contrast to regime-dependent equilibria, where y_t switches between two stochastic processes that are independent of the underlying Markov state. In the history dependent equilibria the value of y_t depends on the current state s_t and also explicitly on the Markov state in the previous period. This dependence is self-fulfilling in the sense that it exists only because of agents' expectations.

2.2 Digression: Expectational Stability in Constant Parameter Models

To establish the language and techniques of learning, we begin with the constant parameter version of (1),

$$y_t = \beta E_t^* y_{t+1} + \gamma r_t \tag{7}$$

now written with a (possibly) boundedly rational expectations operator E^* and where we now allow dependence on the exogenous process

$$r_t = \rho r_{t-1} + \nu_t,$$

where $0 < \rho < 1$ and ν_t is white noise.

The model is determinate provided $|\beta| < 1$. In this case there exists a unique equilibrium that has the form $y_t = br_t$. To analyze stability under learning, we posit that agents hold a perceived law of motion (PLM, i.e. a forecasting model) whose functional form is consistent with the equilibrium representation:

$$y_t = A + Br_t.$$

While there is no constant in the equilibrium representation $y_t = br_t$, it is standard to allow agents to consider the possibility that there may be a constant term, i.e. to learn the steady-state values of y as well.

[11]This follows from algebra, with the help of Mathematica.

The parameters A and B capture agents' perceptions of the relationship between y and r, and may be estimated using, for example, recursive least squares. Let A_t and B_t be the respective estimates using data up to time t. Agents form forecasts using the PLM to obtain[12]

$$E_t^* y_{t+1} = A_{t-1} + B_{t-1}\rho r_t.$$

Plugging these forecasts into (7) leads to the actual law of motion (ALM)

$$y_t = \beta A_{t-1} + (\beta B_{t-1}\rho + \gamma) r_t.$$

Here we assume that agents know the true process governing r_t. The actual law of motion illustrates the manner in which time t endogenous variables are determined by perceptions (A_{t-1}, B_{t-1}) and realizations of r_t. Given new data on y_t, agents update the forecasting model to obtain (A_t, B_t). The unique rational expectations equilibrium $y_t = br_t$ is stable under learning if $(A_t, B_t) \to (0, b)$ almost surely. Stability under learning is non-trivial because of the self-referential nature of rational expectations models.

While assessing the asymptotic behavior of the non-linear stochastic process (A_t, B_t) is quite difficult, it turns out that the technical requirements for convergence often reduce to a fairly simple and intuitive condition known as E-stability: see Evans and Honkapohja (2001). To illustrate, suppose agents hold generic beliefs (A, B). The actual law of motion then defines a map $T : \mathbb{R}^2 \to \mathbb{R}^2$ that takes perceived coefficients to actual coefficients

$$T(A, B) = (\beta A, \beta B\rho + \gamma).$$

Notice that the fixed point of the T-map identifies the unique rational expectations equilibrium of the model. The rational expectations equilibrium is said to be E-stable if it is a locally asymptotically stable fixed point of the ordinary differential equation (o.d.e.)

$$\frac{d(A, B)}{d\tau} = T(A, B) - (A, B).$$

The E-stability Principle states that if agents use recursive least squares – or, similar reasonable learning algorithms – then E-stable rational expectations equilibria are locally stable under learning.[13] In this simple example, if $(0, b)$ is a locally asymptotically stable fixed point of the o.d.e. then $(A_t, B_t) \to (0, b)$ almost surely.

The economic intuition behind the E-stability principle is simple: reasonable learning algorithms dictate that agents update their parameter estimates in the direction

[12] We assume that learning agents may condition on lagged endogenous variables and current exogenous variables, but not on current endogenous variables. This is a standard timing convention in the learning literature. Alternative informational assumptions are available and can affect stability analysis: see Evans and McGough (2005b) for details.

[13] The connection between E-stability of a rational expectations equilibrium and its stability under real time learning is quite deep: see Evans and Honkapohja (2001) for details.

indicated by the forecast errors. This is evident in the o.d.e., as $T(A,B)-(A,B)$ is, in a sense, a forecast error. If the resting point of the o.d.e. is stable then adjusting parameters in the direction indicated by the forecast error will lead the parameters toward the rational expectations equilibrium. Conveniently, conditions for local asymptotic stability are easily computed by examining the eigenvalues of the Jacobian matrix DT. If all eigenvalues of DT have real parts less than one then the rational expectations equilibrium is E-stable. For the case at hand, the derivatives are given by β and $\rho\beta$.[14] Since the model is determinate by assumption (i.e. $|\beta|<1$), the rational expectations equilibrium is stable under learning.

If the model is indeterminate, that is, if $|\beta|>1$, then there exists a continuum of equilibria. To fix ideas we assume the model is non-stochastic ($r_t=0$), and let y_t be an REE corresponding to (7). Then there exists an mds ξ_t so that $y_t=\beta^{-1}y_{t-1}+\xi_t$: we call this recursion the "general form representation" of y_t. Also, there exists a serially correlated process given by $\eta_t=\beta^{-1}\eta_{t-1}+\xi_t$ so that $y_t=\eta_t$: we call this (trivial) recursion the "common factor representation" of y_t.[15] To analyze stability under learning, we must take a stand on the information available to agents, that is, which representation to use when specifying the agents' forecasting model.

We first assume that agents engage in "mean value learning," that is, they observe the extrinsic process η_t, and thus have only the conditional mean of y to estimate. Specifically, agents condition on η_t and so compute expectations using a forecasting model of the form

$$y_t=A+B\eta_t.$$

Computing the T-map provides $DT_A=\beta, DT_B=1$, so that the sunspot equilibria are E-stable provided that $\beta<-1$.[16]

Another natural learning process is "VAR learning" where agents estimate both the mean and the lag structure of the endogenous variables. Agents condition their forecast on the martingale difference sequence sunspot ξ_t, as well as a constant and lagged y:

$$y_t=A+By_{t-1}+C\xi_t.$$

The primary difference between VAR and mean value learning is that the latter

[14]Here, and below, we exploit that when the T-map decouples, we can compute derivatives separately.

[15]In the model (7), because there is no lag dependence, the serial correlation in the equilibrium y_t is present only because agents expect it to be: thus we call the serial correlation "self-fulfilling." If the reduced form model (7) also included lagged terms (i.e. y_{t-n} for some $n>0$) then the sunspot equilibrium would have both self-fulfilling and fundamental serial correlation. In a linear model, the common factor sunspot η_t is required to capture the self-fulfilling part of the model's serial correlation. This requirement is known as the resonance frequency condition – see Evans and Honkapohja (2003) and Evans and McGough (2005b) for details.

[16]That $DT_B=1$ reflects the fact that $T(B)=B$: if η_t is an appropriately serial correlated sunspot, then so too is $B\eta_t$ for any B. For more on this issue, see Evans and McGough (2005b).

assumes agents coordinate on the serially correlated sunspot η_t, while the former postulates that agents try to detect the self-fulfilling lag structure from the data. Computing the T-map provides the following derivatives

$$\begin{aligned} DT_A &= \beta(1+B) \\ DT_B &= 2\beta B \\ DT_C &= \beta B. \end{aligned}$$

Since, at the REE, $B = \beta^{-1}$, it follows that that $DT_B = 2$: so if agents employ VAR learning, then the sunspot equilibria are never stable.

This example illustrates that the stability of sunspot equilibria depends on agents' conditioning set. By incorporating the serial correlation into η_t – which only arises in the model because of self-fulfilling expectations – the agents can coordinate on a sunspot equilibrium. If, however, they are trying to learn the mean and the self-fulfilling lag structure, coordination via learning is not possible.

2.3 E-stability in univariate Regime Switching Model

We now extend the analysis in the previous subsection to the regime switching model (1). We analyze the stability of equilibria in each class identified above.

Without a loss of generality, we continue to assume $\gamma_i = 0, i = 1, 2$. Recall that, in case the CLDC holds, there is a unique regime-dependent equilibrium of the form $y_t = 0$. To analyze stability under learning, given the regime switching structure, we provide agents with a perceived law of motion that, while functionally consistent with the RDE, allows for regime-dependent learning:

$$y_t = A(s_t).$$

According to this PLM, the learning agent believes that if the state of the world is given by $s_t = 1$ then $y_t = A(1)$ and if the state of the world is given by $s_t = 2$ then $y_t = A(2)$.

Stepping this equation forward leads to state contingent expectations

$$E_t(y_{t+1}|s_t = j) = p_{j1}A(1) + p_{j2}A(2).$$

Thus, there is a state-contingent T-map given by

$$A(j) \rightarrow \beta_j(p_{j1}A(1) + p_{j2}A(2)).$$

Differentiating leads to the Jacobian matrix that governs E-stability:

$$DT = \begin{pmatrix} \beta_1 p_{11} & \beta_1 p_{12} \\ \beta_2 p_{21} & \beta_2 p_{22} \end{pmatrix}.$$

The condition for E-stability of regime-dependent equilibria are that the eigenvalues of DT have real parts less than one. But now notice that the matrix DT coincides with the matrix (4) governing uniqueness of RDE: if the CLDC is satisfied, then the unique regime-dependent equilibrium is E-stable. This connection between E-stability of the MSV solution, and uniqueness within the class of regime-dependent equilibria, is a central result of this paper.

We now turn to the stability of the univariate history dependent equilibrium given in (6), and we consider VAR learning. HDE may depend on a constant, lagged y, and an extrinsic sunspot ξ, and the nature of these dependences may vary depending on the state of the world yesterday and today. Consistent with this, we provide agents with a perceived law of motion of the form

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t,$$

where ξ_t is an m.d.s, and the parameters A, B, C in the PLM are elements of the real line. We assume that s_t but not contemporaneous y_t is in the information set. Computing conditional forecasts using this PLM, we obtain the following T-map for B:

$$B(i,j) \longrightarrow \beta_j \left(p_{j1}B(j,1) + p_{j2}B(j,2)\right) B(i,j).$$

Ignoring the boundedness requirement, a fixed point of this map identifies an HDE. The only restrictions, then, are the following:

$$1 = \beta_1 \left(p_{11}B(1,1) + p_{12}B(1,2)\right) = \beta_2 \left(p_{21}B(2,1) + p_{22}B(2,2)\right).$$

In particular, there is a two dimensional continuum of coefficients on lagged y providing fixed points.

Following Farmer, Waggoner, and Zha (2009), we focus on particular fixed points, given by

$$B(1,1) = B(2,1) = 0, \;\; B(1,2) = \frac{\beta_1^{-1}}{p_{12}}, \;\; B(2,2) = \frac{\beta_2^{-1}}{p_{22}}.$$

To analyze stability we compute the eigenvalues of DT. The T-map for the coefficients B decouples and provides the following Jacobian:

$$DT_B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \beta_1^{-1}\beta_2 p_{21}/p_{12} & \beta_1^{-1}\beta_2 p_{22}/p_{21} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & p_{21}/p_{22} & 2 \end{pmatrix}.$$

The Jacobian has an eigenvalue of 2, which implies that under learning, HDE, when represented this way, are E-unstable.

We now explore the stability of HDE under mean value learning. Agents are assumed to know the self-fulfilling serial correlation, but are still required to learn the mean. This assumption implies a PLM of the form:

$$y_t = A(s_{t-1}, s_t) + B\eta_t, \tag{8}$$

where η_t is an extrinsic process given by

$$\eta_t = \begin{cases} \delta_{11}\varepsilon_t & (s_{t-1}, s_t) = (1,1) \\ \frac{1}{p_{12}}\beta_1^{-1}\eta_{t-1} + \delta_{12}\varepsilon_t & (s_{t-1}, s_t) = (1,2) \\ \delta_{21}\varepsilon_t & (s_{t-1}, s_t) = (2,1) \\ \frac{1}{p_{22}}\beta_2^{-1}\eta_{t-1} + \delta_{22}\varepsilon_t & (s_{t-1}, s_t) = (2,2) \end{cases}$$

That agents condition on η_t captures the assumption that they know the relevant serial correlation. Notice that agents' beliefs concerning the mean of y_t are still state contingent.

The T-map is given by

$$A(i,j) \to \beta_j(p_{j1}A(j,1) + p_{j2}A(j,2))$$

and $B \to B$; this yields the Jacobian matrix

$$DT_A = \begin{pmatrix} \beta_1 p_{11} & \beta_1(1-p_{11}) & 0 & 0 \\ 0 & 0 & \beta_2(1-p_{22}) & \beta_2 p_{22} \\ \beta_1 p_{11} & \beta_1(1-p_{11}) & 0 & 0 \\ 0 & 0 & \beta_2(1-p_{22}) & \beta_2 p_{22} \end{pmatrix},$$

which has the same eigenvalues as the Jacobian matrix for the regime-dependent equilibria plus a pair of zero eigenvalues.[17] Therefore, when there is a unique regime-dependent equilibrium, and agents form their forecasts via the mean-value perceived law of motion (8), then it is possible for agents to coordinate on either the regime-dependent equilibrium or the history-dependent equilibrium.

Whether HDE exist when the CLDC is satisfied depends on the specific economic model. It is possible to verify that in the univariate case, when β_1, β_2 are restricted to take non-negative values, that the CLDC implies $\beta_2 p_{22} < 1$; thus in the Fisherian model, if there is a unique RDE then HDE do not exist: the CLDC is sufficient to preclude the presence of history dependent equilibria. HDE can exist when the CLDC is not satisfied, for example when $\beta_1 < 1, \beta_2 > 1$ and $\beta_2 p_{22} > 1$; however, they are not stable under learning even if agents use mean value learning.

Moving outside the context of the Fisherian interpretation, in a model with negative coefficients, HDE can exist even when the CLDC holds. In Section 5, an OLG model is presented which naturally leads to $\beta_1 > 0, \beta_2 < 0$, and in which the CLDC is satisfied; then, whenever $|\beta_2 p_{22}| > 1$, it follows that HDE exist and are stable under mean value learning.

[17]That the T-map fixes B is a standard result: if η_t is a self-fulfilling sunspot then $B\eta_t$ is as well.

3 Regime-switching Equilibria: The Multivariate Case

Having introduced our results within a univariate model, we now formalize and generalize our findings. We focus on models whose reduced form consists of a system of non-linear expectational difference equations of the form

$$y_t = \beta(s_t) E_t y_{t+1} + \gamma(s_t) r_t, \tag{9}$$

where y_t is an $(n \times 1)$ vector of random variables, and $\beta(s_t)$ and $\gamma(s_t)$ are conformable matrices that depend on s_t, an m-state Markov process taking on values in $\{1, \ldots, m\}$. As before, we use the notation $\beta(s_t) = \beta_i, \gamma(s_t) = \gamma_i \Leftrightarrow s_t = i, i = 1, 2, ..., m$. The stochastic matrix P governing the evolution of s_t is taken to be recurrent and aperiodic, so that it has a unique stationary distribution. For simplicity, β_i is taken to be invertible for all i. Finally, $r_t = \rho r_{t-1} + \hat{\varepsilon}_t$ is a $(k \times 1)$ exogenous stationary VAR(1) process independent of s_j for all j.

A rational expectations equilibrium of the model is a solution to (9) that also satisfies a boundary condition. We focus on processes satisfying the following property:

Definition. A stochastic process y_t with initial condition y_0 is *uniformly bounded* (almost everywhere) or UB if $\exists M(y_0)$ so that $\sup_t ||y_t||_\infty < M(y_0)$, where $|| \cdot ||_\infty$ is the L^∞ or "essential supremum" norm.

With this definition available, we may define a rational expectations equilibrium:

Definition. A *Rational Expectations Equilibrium* is any UB stochastic process satisfying (9).

While uniformly bounded (UB) may appear to be an *a priori* strong notion of boundedness, it is common in the linear rational expectations literature.[18] In linear models with constant parameters, uniform boundedness is consistent with the usual notion of model determinacy, such as in Blanchard and Kahn (1980). Also, UB "bounds the paths" of all endogenous variables and is often desirable when using a first-order approximation to a nonlinear model around a fixed point, such as a steady state. Other existence/uniqueness results may arise with an alternative definition for rational expectations equilibria.

[18] As an alternative, sometimes the boundary condition requires the paths of variables in a rational expectations equilibrium remain conditionally uniformly bounded, such as in Evans and McGough (2005b).

3.1 Regime-Dependent Equilibria

The minimal state variable solution to the model (9) takes the form[19]

$$y_t = B(s_t)r_t, \tag{10}$$

and, as before, this functional form guides our definition of regime dependent equilibria.

Definition. Let s_t be the Markov process governed by P and taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (9). Then y_t is a *Regime Dependent Equilibrium* (RDE) if it is uniformly bounded and there exist uniformly bounded stochastic processes $y_{1t}, y_{2t}, ..., y_{mt}$, with y_{it} independent of s_{t+j} for all integers j, such that $y_t = y_{it} \Leftrightarrow s_t = i$. Note that the MSV-solution is an RDE.

In an RDE, depending on the realization of s_t, y_t takes on values from one of m stochastic processes, with each process being independent of the Markov state. Conditioning (9) on each regime leads to the following system

$$\begin{array}{rcl} y_{1t} & = & \beta_1 p_{11} E_t y_{1t+1} + \beta_1 p_{12} E_t y_{2t+1} + \cdots + \beta_1 p_{1m} E_t y_{mt+1} + \gamma_1 r_t, \\ y_{2t} & = & \beta_2 p_{21} E_t y_{1t+1} + \beta_2 p_{22} E_t y_{2t+1} + \cdots + \beta_2 p_{2m} E_t y_{mt+1} + \gamma_2 r_t, \\ \vdots & \vdots & \qquad\qquad\qquad\qquad \vdots \\ y_{mt} & = & \beta_m p_{m1} E_t y_{1t+1} + \beta_m p_{m2} E_t y_{2t+1} + \cdots + \beta_m p_{mm} E_t y_{mt+1} + \gamma_m r_t, \end{array}$$

which governs dynamics for y_{it} for $i = 1, 2, ..., m$. We note that this is a linear system that can be written compactly as

$$\hat{y}_t = M E_t \hat{y}_{t+1} + \gamma r_t, \tag{11}$$

where $M = (\oplus_{j=1}^m \beta_j)(P \otimes I_n)$, $\widehat{y}_t = (y'_{1t}, y'_{2t}, ..., y'_{mt})'$ and $\gamma' = (\gamma'_1, \ldots, \gamma'_m)'$, and where we use the direct sum notation: $\oplus_{j=1}^m \beta_j = diag(\beta_1, \beta_2, \cdots, \beta_m)$.

The stacked system (11) is a multivariate linear rational expectations model, and, as before, the MSV-solution (10) to the switching model (9) corresponds to the unique MSV-solution to the stacked system (11) in the sense of McCallum (1983). The number and nature of solutions to (11) are well-known. We are particularly interested in conditions under which there exists a unique UB solution to the stacked system, as this condition will also govern the expectational stability of solutions to the economic system (9). We summarize this in the following remark.

Remark. A necessary and sufficient condition for the existence of a unique uniformly bounded solution to (11) is that the eigenvalues of $(\oplus_{j=1}^m \beta_j)(P \otimes I_n)$ lie inside the unit circle. In this case, we say that the *Conditionally Linear Determinacy Condition* (CLDC) is satisfied.

[19] The coefficient matrices $B(i)$ are computed in the Appendix, Section 7.3.

Proposition 2 *If the CLDC holds then there is a unique RDE that corresponds to the MSV-solution.*

This proposition follows from the fact that any RDE solves the stacked system (11): see Appendix for proof. Subsequent sections show a close connection between the conditions for unique RDE and E-stable rational expectations equilibria, and so the CLDC takes on added importance below.

3.2 History Dependent Equilibria

This subsection formalizes the definition of History Dependent Equilibria, that is, equilibria where agents condition their expectations on an expanded state vector that includes s_{t-n}. We focus on the case $n = 1$.

Definition. Let s_t be the Markov process governed by P, taking values in $\{1, 2, ..., m\}$. Let y_t be a solution to (9). Then y_t is a *History Dependent Equilibrium* (HDE) if it is uniformly bounded and its distribution conditional on s_t differs from its distribution conditional on s_t and s_{t-1}: $y_t|s_t \not\sim y_t|(s_t, s_{t-1})$.

Sunspot solutions to the stacked system (which have constant parameters) can be HDE by having a history dependent sunspot shock: see the Remark in the Appendix (Section 7.4) for details. More generally, under conditions identified in the Appendix (Section 7.4), there exist HDE with time-varying coefficients on the lagged endogenous variable, and we turn to equilibria of this form now.

Assuming without loss of generality that $\gamma_t = 0$ for all t, these HDE may be represented as

$$y_t = \hat{B}(s_{t-1}, s_t)y_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t, \tag{12}$$

where ξ_t is a uniformly bounded mds, and detailed expressions for the coefficient matrices $\hat{B}, \hat{C}$ are provided in the Appendix, Section 7.4. Also, just as in the univariate case, it is possible to represent the same HDE in the following alternative form:

$$y_t = \eta_t \tag{13}$$

where

$$\eta_t = \hat{B}(s_{t-1}, s_t)\eta_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t.$$

The stochastic properties of (13) are equivalent to (12); however, these two representations imply different informational assumptions and distinct stability results. These observations lead to two natural learning rules analogous to those discussed in Section 2: a "mean value learning" formulation where agents use a forecasting model consistent with (13) by *conditioning* on η_t and trying to learn the endogenous variable's state-contingent constant term (which, in this case, is zero); and a "VAR learning" formulation where agents estimate a forecasting model consistent with (12) by *conditioning* on a state-contingent constant and on lagged y – in this case, agents must also learn the endogenous variable's lagged coefficients.

3.3 Equilibrium Representations

The representations of regime-switching rational expectations equilibria above, and in Section 2, form the basis for the forecasting rules used by boundedly rational agents when forming expectations. As emphasized in Evans and McGough (2005b), and as exemplified by (12) and (13) above, a given sunspot equilibrium may be respresented in a number of ways, thus giving rise to alternate specifications for agents' forecasting model. Stability under adaptive learning may depend on the functional form of the forecasting model, that is to say the representation, used by agents. Because of the important role played by representations when studying E-stability, we provide the following summary:

1. There exists an RDE capturing the minimal state variable (MSV) solution to (11), and its representation is given by
$$y_t = B(s_t)r_t.$$
2. HDE are sunspot equilibria which (setting $\gamma_t = 0$ for convenience) can be naturally represented in (at least) two ways: as $y_t = \hat{B}(s_{t-1}, s_t)y_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t$ where ξ_t is an arbitrary, uniformly bounded martingale difference sequence; or as $y_t = \eta_t$ where $\eta_t = \hat{B}(s_{t-1}, s_t)\eta_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t$.

4 Expectational Stability: The Multivariate Case

In this section we investigate equilibrium stability in our general regime switching model. Our primary result is that the condition governing uniqueness in the class of regime dependent equilibria, namely, the CLDC, may imply expectational stability. Crucially, though, this result depends on the assumed information structure as captured by the representation assigned to agents.

4.1 E-stability and the CLDC

This section demonstrates that the CLDC implies E-stability of RDE. Using the MSV-solution as our guide to specifying a perceived law of motion, we now turn to the stability of RDE under learning. Throughout, we assume that agents observe the current state s_t and know the true transition probabilities, but do not observe y_t. This is consistent with a convention of the adaptive learning literature that assumes agents observe contemporaneous exogenous variables, but not current values of endogenous variables.[20]

Agents have a perceived law of motion (PLM) of the following form:

$$y_t = A(s_t) + B(s_t)r_t,$$

where $A(j)$ is $(n \times 1)$, and $B(j)$ is $(n \times k)$. Notice we assume agents do not know that in equilibrium the $A_i = 0$.

Proposition 3 *If the CLDC holds, then the unique RDE is E-stable.*

The proof is in the Appendix. This result states that an economy described by the main expectational difference equation (9), with expectations formed from the MSV forecasting model and updated using least squares, will converge to the unique RDE.

4.2 E-stability and Indeterminacy

Now we examine the stability of HDE, and again, for simplicity, we set $\gamma_t = 0$. We begin by considering VAR learning. In this case, the PLM takes the following form:

$$y_t = A(s_{t-1}, s_t) + B(s_{t-1}, s_t)y_{t-1} + C(s_{t-1}, s_t)\xi_t, \tag{14}$$

where ξ_t is an mds, independent of the Markov states. The PLM makes clear the primary distinction between HDE and the class of RDE solutions, since, here, coefficients depend explicitly on s_t and s_{t-1}, whereas coefficients in the PLM for the RDE only depend on s_t.

General results on the stability of HDE are not available because the set of all fixed point solutions has not been characterized. If $m = 2$, there is, however, a subclass of fixed points that corresponds to multivariate analogues of the equilibria studied by Farmer, Waggoner and Zha (2009) in the univariate case. If the eigenvalues of β_2 are larger than $1/p_{22}$ in modulus, then there exists a continuum of HDE such that

$$B(1,1) = B(2,1) = 0, B(1,2) = \frac{\beta_1^{-1}}{p_{12}}, B(2,2) = \frac{\beta_2^{-1}}{p_{22}}. \tag{15}$$

[20] This may be a strong assumption, but it is the same assumption made under rational expectations. To study E-stability, there must be a consistency between the informational assumptions under rational expectations and adaptive learning. Relaxing this assumption is likely to have interesting implications that are beyond the scope of the present paper.

The restriction that the eigenvalues of β_2 are larger, in modulus, than $1/p_{22}$ ensures that the resulting process is uniformly bounded. We have the following result, which is proved in the Appendix.

Proposition 4 *HDE of the form (15) are never stable under VAR learning.*

Now consider mean value learning. In this case, agents are assumed to condition on a sunspot that captures the self-fulfilling serial correlation in the endogenous vector, and thus only estimate the state-dependent mean. Specifically, assume agents observe and understand the stochastic structure of the extrinsic noise process, η_t, where

$$\eta_t = \hat{B}(s_{t-1}, s_t)\eta_{t-1} + \hat{C}(s_{t-1}, s_t)\xi_t.$$

Then we take our agent's forecasting model as

$$y_t = A(s_{t-1}, s_t) + B\eta_t.$$

We have the following result, which is proved in the Appendix.

Proposition 5 *Assume the model is parameterized so that HDE exist. If there exists a unique RDE, then there exist HDE which are stable under mean value learning.*

4.3 Discussion

A brief review of our results is in order. First, consider the univariate model. If both regimes exhibit positive feedback, i.e. $\beta_j > 0, j = 1, 2$, then the CLDC implies determinacy: there is a unique equilibrium corresponding to the MSV-solution that is stable under learning. HDE do not exist in this case. On the other hand, there exist models with positive feedback in the "determinate" regime, $\beta_j < 1$, and negative feedback in the "indeterminate" regime, $\beta_j < -1$, for which the CLDC does not rule out multiple equilibria. In these cases, HDE are shown to exist even when the CLDC holds, and these HDE are unstable under VAR learning and stable under mean value learning. An economic example that fits this case is the overlapping generations model presented in the next section.

The results for the multivariate case are quite similar to the univariate model. The CLDC guarantees a unique RDE, but might be insufficient, in general, to guarantee a unique equilibrium. Models for which the CLDC is satisfied may exhibit HDE which are unstable under VAR learning and stable under mean value learning. An economic example that fits this case is the New Keynesian model presented in the next section.

There are some questions left open by the preceding analysis. We have only studied stability of HDE of the form (15). Regardless of whether the CLDC holds, other HDE may exist, either as different fixed points to the same T-map implied by the PLM (14), or by conditioning on different states (i.e. s_{t-n} for some $n > 1$). We cannot comment on the existence or stability of such HDE.

Finally, we note that if the CLDC does not hold then there are mutliple RDE: indeed, if the CLDC does not hold then the stacked system (11) has multiple uniformly bounded solutions corresponding to sunspot equilibria, and provided that the sunspot shock associated to a given equilibrium is independent of the Markov state, that equilibrium will be an RDE.

5 Economic Examples

To gain some insight into the economics necessary to allow for stable sunspot equilibria, we consider two examples. The first is an overlapping generations model with preference switching, which allows for a reduced form consistent with our univariate model of Section 2; the second is the New Keynesian model with stochastically varying policy regimes as emphasized by Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009) .

5.1 An Overlapping Generations Model

This section presents a simple overlapping generations (OLG) model with money as storage extended to incorporate a preference parameter that follows a two-state Markov process. The basic set-up is standard: households live for two periods, they work n_t hours when they are young and consume c_{t+1} when they are old. The produce q_t units of a non-storable good according to the production function $q_t = n_t$. Finally, households can buy and sell goods in exchange for fiat money M at the price p_t.

The representative household solves the following problem:

$$\begin{array}{ll} \max & u(c_{t+1}) - V(q_t) \\ & p_t q_t = M_t = p_{t+1} c_{t+1} \end{array}$$

The household's FOC is given by

$$v'(n_t) = E_t \left(\frac{p_t}{p_{t+1}} u'(c_{t+1}) \right).$$

In equilibrium, $c_t = q_t$; and, assuming the money supply is constant, $p_{t+1}q_{t+1} = p_t q_t$. Combining these observations yields the equilibrium condition $v'(q_t)q_t = E_t u'(q_{t+1})q_{t+1}$.

Finally, we assume v is linear and u is CRRA, with relative risk aversion coefficient σ. Under these assumptions, the non-autarky steady state is $q = 1$ and the linearized model is

$$q_t = (1 - \sigma) E_t q_{t+1}, \tag{16}$$

where now variables are in deviation from steady-state form.

Our goal is to parameterize the model so that the CLDC is satisfied and HDE exist; then, using Proposition 5, we may conclude that there exist HDE which are stable under mean value learning. In Section 2, we learned that in the univariate model, if the CLDC is satisfied and HDE of the form (6) exist then the determinate regime must exhibit positive feedback, and the indeterminate regime must exhibit negative feedback. This guides us to assume that the risk aversion parameter follows a two-state Markov process: $\sigma(s_t) \in \{\sigma_1, \sigma_2\}$, where $0 < \sigma_1 < 1 < \sigma_2$. With this assumption, the model (16) fits the reduced-form structure (1); further, it is straightforward to choose parameter values so that the CLDC is satisfied and HDE of the form (6) exist: $\sigma_1 = .7, \sigma_2 = 2.5, p_{11} = .5, p_{22} = .7$ provides an example.

5.2 A New Keynesian Model

There is extensive empirical evidence of regime change in monetary policymaking. For example, there is a breakpoint in the parameters of a Taylor-type nominal interest rate rule in Clarida, Gali, and Gertler (1999), and shifting policymaker preferences in Bernanke (2004) and Dennis (2006). These findings motivate models that build regime-switching directly into rational expectations frameworks since whenever the systematic nature of monetary policymaking has changed in the past, then it is reasonable that agents might anticipate future policy changes.

As an example, Davig and Leeper (2007) and Farmer, Waggoner, and Zha (2009) construct rational expectations solutions to the standard New Keynesian model closed with a nominal interest rate rule whose coefficients are subject to occasional regime change. The New Keynesian model is given by (linearized) reduced-form equations for inflation, π, and the output gap, x, such as

$$\begin{aligned} \pi_t &= \beta E_t \pi_{t+1} + \kappa x_t + u_t \\ x_t &= E_t x_{t+1} - \sigma^{-1} \left(i_t - E_t \pi_{t+1} \right) + g_t \end{aligned}$$

where g_t captures shocks to aggregate demand and u_t represents aggregate supply shocks; they are typically taken to be exogenous (stationary) AR(1) processes. The first equation is the New Keynesian Phillips curve. It is derived from a first-order approximation to a monopolistically competitive firm's price-setting decision when there is the possibility that future prices will remain fixed (e.g. Calvo price-setting). The second equation is the New Keynesian IS equation that represents the demand side of the economy. It is derived as a linear approximation to a representative household's Euler equation. In the standard formulation of the New Keynesian model the g_t typically arise as a combination of productivity, preference, and government spending shocks, while the u_t shocks arise from variations in market-power from, e.g. shocks to the elasticity of substitution.

It is typical to close a New Keynesian model with a nominal interest rate targeting rule along the lines proposed by Taylor (1993). A New Keynesian model with recurring policy change assumes a nominal interest rate rule with time-varying parameters

$$i_t = \alpha(s_t)\pi_t + \gamma(s_t)x_t.$$

To capture recurrent regime change, Davig and Leeper (2007), assume that the parameters $\alpha(s_t), \gamma(s_t)$ in the policy rule follow a two state Markov chain:

$$\alpha(s_t) = \begin{cases} \alpha_1 \text{ for } s_t = 1 \\ \alpha_2 \text{ for } s_t = 2 \end{cases}$$

and

$$\gamma(s_t) = \begin{cases} \gamma_1 \text{ for } s_t = 1 \\ \gamma_2 \text{ for } s_t = 2 \end{cases}$$

The random variable s_t follows a finite-state Markov chain with transition probabilities $p_{ij} \equiv \Pr[s_t = j | s_{t-1} = i]$ for $i, j = 1, 2$.

The Taylor Principle dictates that, in a model with constant policy coefficients α, γ, nominal interest rates rise more than one for one with inflation, that is $\alpha > 1$. Policy that satisfies the Taylor Principle leads to a model with a unique rational expectations equilibrium, while when $\alpha < 1$ it is possible for there to exist multiple equilibria that exhibit inefficiently high volatility. With regime-switching policy rules, private-sector expectations build in the possibility of future passive monetary policy and this places a restriction on how active, i.e the extent to which $\alpha > 1$, policy must be to ensure determinacy.

As an example, we parameterize the model as in Davig and Leeper (2007): $\beta = 0.99, \kappa = 0.17, \sigma = 1$. Moreover, we follow Farmer, Waggoner, and Zha (2010) and set $\alpha_1 = 3.0, \alpha_2 = 0.92, \gamma_1 = \gamma_2 = 0$ and $p_{11} = 0.8, p_{22} = 0.95$. With these parameter values the CLDC is satisfied as the eigenvalues of the matrix M are $0.98, 0.81, 0.63, 0.63$. Thus, there is a unique E-stable RDE. Similarly, it is possible to compute the eigenvalues of the matrices $\beta_j, j = 1, 2$: eigenvalues of β_1 (in absolute value) are $0.81, 0.81$, and for β_2 they are $1.06, 0.81$. Notice that $\beta_2 > 1/p_{22} = 1.05$. Thus, there exist HDE that, from Proposition 5, are E-stable under mean-value learning.

It is surprising that HDE can be stable under mean value learning. In a constant parameters New Keynesian model, Evans and McGough (2005a) studied the stability under mean value learning of rational expectations equilibria and found that, under a constant parameter policy rule of the same form as considered in this paper, the "common factor" representation of a sunspot equilibrium is E-unstable.[21] Since the common factor representation is analogous to mean-value learning, ours results suggest that the conditions necessary for stable sunspot equilibria in a regime switching model are somewhat weaker than in a model with constant parameters.

6 Conclusion

We extend the literature on learning to a non-linear framework to allow for regime shifts and we study the stability under adaptive learning of two classes of equilibria:

- *Regime Dependent Equilibria*: An RDE is a uniformly bounded process that satisfies the regime-switching expectational difference equation and imposes the restriction that agents do not condition their expectations on lagged regimes (i.e. only the current regime enters the state vector).
- *History Dependent Equilibria*: An HDE is a process that satisfies the regime-switching expectational difference equation, where agents condition expectations on current and lagged values of the regime (i.e. current and past regimes enter the state vector).

The Conditionally Linear Determinacy Condition (CLDC) ensures the existence of a unique RDE, and further indicates that it is also E-stable. When the CLDC is satisfied, there may still exist sunspot equilibria and we demonstrate that these equilibria may be learnable, depending on the conditioning set imposed on boundedly rational agents.

We applied our results to three simple models. In the univariate Fisherian model, because feedback is positive there is a unique equilibrium, and it is selected by E-stability. On the other hand, an overlapping generations model allows for negative feedback regimes even in the univariate case, and therefore induces equilibrium multiplicity even when there is a unique RDE; finally, a benchmark version of the New Keynesian model yields the analogous result in higher dimensions and in a model of applied interest. When there is simulataneously a unique RDE and many HDE, we found that under VAR learning, E-stability selects a unique equilibrium; however, it is possible that if agents condition their expectations on the self-fulfilling serial correlation (e.g. sunspots) – a stronger requirement than VAR learning – then there may exist multiple E-stable equilibria.

[21] Evans and McGough (2005b) did find common factor sunspots could be E-stable for forward-looking rules.

7 Appendix

7.1 Proof of Lemma 1

Necessity is straightforward: if $|\beta_2 p_{22}| < 1$ then, for any $T > 0$ there is a positive probability – namely, p_{22}^T – that $s_t = s_2$ for T consecutive periods, thus causing y_t to get large in magnitude. For sufficiency, let $\rho = (p_{22}\beta_2)^{-1}$ and $M = (p_{12}\beta_1)^{-1}$. Also, let $\varepsilon \in \mathbb{R}$ be so that $\varepsilon > \delta_{ij}\varepsilon_t$ almost surely. Finally, begin with the assumption that $0 < \rho < 1$ and $M > 1$. Now notice that if $s_t = 1$ then $y_t \leq \varepsilon$. Also, if $s_t = 2$ and $s_{t-1} = 1$ then $y_t \leq (1 + M)\varepsilon < M\varepsilon + \frac{1}{1-\rho}\varepsilon$. Finally, if $s_t = 2$ and $s_{t-1} = 2$ then $y_t \leq \rho y_{t-1} + \varepsilon$. To examine this last case more closely, note that if $s_{t-2} = 1$, and keeping $s_t = 2$ and $s_{t-1} = 2$, we have

$$y_t \leq \rho M\varepsilon + (1 + \rho)\varepsilon < M\varepsilon + \frac{1}{1-\rho}\varepsilon,$$

and if $s_{t-2} = 2$ then $y_t \leq \rho^2 y_{t-2} + (1+\rho)\varepsilon$. Again, this last case requires examination, but the pattern is now clear: we have

$$y_t \leq \min\left\{M\varepsilon + \frac{1}{1-\rho}\varepsilon, y_0 + \frac{1}{1-\rho}\varepsilon\right\}, \tag{17}$$

which imparts a uniform upper bound, contingent on the initial condition. A lower bound is established analogously, and cases with $\rho < 0$ and $M < 0$ are dealt with in a similar fashion.

7.2 Proof of Proposition 2

It suffices to show that an RDE solves the stacked system (11). Thus let y_{it} identify an RDE. Denote by f_t the time t density functions; for example, $f_t\left(y, s|s_{t-1} = i, \Omega_{t-1}\right)$ is the joint density of y_t and s_t conditional on $s_{t-1} = i$ and on all other time $t-1$ information, not including current and past s_{t-1}, as captured by Ω_{t-1}. Also, let $f_t^i\left(y|\Omega_{t-1}\right)$ be the density for y_{it} conditional on Ω_{t-1}, and $f\left(s|s_{t-1} = i\right)$ be the conditional density of s_t given $s_{t-1} = i$ (course, $f\left(s = j|s_{t-1} = i\right) = P_{ij}$). With this notation, we may compute expectations as follows:

$$\begin{aligned}
E(y_{t+1}|s_t = i, \Omega_t) &= \int\int y f_{t+1}\left(y, s|s_t = i, \Omega_t\right) dsdy \\
&= \int\int y f_{t+1}\left(y|s, s_t = i, \Omega_t\right) f(s|s_t = i)\, dsdy \\
&= \int\int y f_{t+1}^s\left(y|\Omega_t\right) f(s|s_t = i)\, dsdy \\
&= \sum_j P_{ij} E_t y_{jt+1}.
\end{aligned}$$

We may simply use this formula for the expectations of y_t to verify that the stacked system is satisfied.

7.3 MSV solution of the main model

Following Davig and Leeper (2007), we take the MSV solution of (9) to have the form $y_t = B(s_t)r_t$, and note that this equilibrium coincides with the MSV solution (in the sense of McCallum (1983)) to the stacked system (11). Using this later insight, we may compute the MSV coefficients. We solve for $B(s_t)$ for $s_t \in \{1, 2, ..., m\}$ by using the stacked system: set

$$B = (B(1)', ..., B(m)')',$$

which yields $\hat{y}_t = Br_t$, where $\hat{y}_t' = (y_{1t}', \ldots, y_{nt}')$, and

$$\mathrm{vec}(B) = \left(I_{nm} - \rho' \otimes \left(\oplus_{j=1}^{m} \beta_j\right) (P \otimes I_n)\right)^{-1} \mathrm{vec}(\gamma).$$

It is worth remarking at this point that the class of RDE includes the MSV-solution to the regime-switching model, may also include a sunspot equilibria, provided that the sunspot shock is not correlated with the underlying Markov process s_t.

7.4 HDE details

Farmer, Waggoner, and Zha (2009) show that there exist multiple uniformly bounded HDE that have the following representation:

$$y_t = \left(\frac{c_{s_{t-1}}}{v_{s_{t-1}}' v_{s_{t-1}}} v_{s_t} v_{s_{t-1}}' \right) y_{t-1} + v_{s_t} \xi_t; \tag{18}$$

provided there exists $c_1, \ldots, c_m$ and $v = (v_1', \ldots, v_m')' \neq 0$ so that $|c_j| \leq 1$ and c and v solve

$$\left[\left(\oplus_{j=1}^{m} \beta_j\right)^{-1} - \left(\left(\oplus_{j=1}^{m} c_j\right) P\right) \otimes I_n \right] v = 0. \tag{19}$$

Here ξ_t is independent of s_{t+n} for all n. The condition (19) is essentially derived from the method of undetermined coefficients. When (19) is satisfied, solutions to the representation (18) are solutions to (9).[22] The construction of the autoregressive parameter in the representation (18) is chosen so that, regardless of the history of realizations of s_t, these parameters are bounded in matrix norm and, hence, the solutions are uniformly bounded.

[22] If one were to literally use the method of undetermined coefficients, the v in (19) would be y_t. However, if v is taken to be a vector of initial conditions chosen to lie on the stable manifold, and if (19) is satisfied at $t = 1$, then it will be satisfied for all t.

Our definition of HDE – as rational expectations equilibria that exhibit conditional dependence on both s_t and s_{t-1} – allows for the identification of a more general class of equilibria than those represented by (18). Sticking with the case $\gamma = 0$, we consider equilibria of the form:

$$y_t = B(s_{t-1}, s_t) y_{t-1} + C(s_{t-1}, s_t)\xi_t.$$

The coefficients must satisfy

$$\left(I_n - \beta_j \left(\sum_{k=1}^{m} p_{jk} B(j,k) \right) \right) B(i,j) = 0 \tag{20}$$

$$\left(I_n - \beta_j \left(\sum_{k=1}^{m} p_{jk} B(j,k) \right) \right) C(i,j) = 0. \tag{21}$$

Notice that provided non-zero $B(i,j)$ satisfy (20), the $C(i,j)$ are arbitrary. It is straightforward to verify that (18) is a solution to (20), but as we saw in Section 2, there may be many others.[23]

Remark. The definition of an HDE restricts solutions to the class of uniformly bounded stochastic processes whose conditional density exhibits dependence on s_t and s_{t-1}. Notice that if y_t is an RDE then $y_t | s_t \sim y_t | (s_t, s_{t-1})$. However, by Proposition 2 when the CLDC is not satisfied, then there may exist solutions to the stacked system that are not RDE. In particular, when the matrix $(\oplus_{j=1}^{m} \beta_j)(P \otimes I_n)$ has n_s eigenvalues outside the unit circle then for each n_s-dimensional martingale difference sequence ξ_t of forecast errors independent of s_{t-n} for all n, there is a martingale difference sequence $\tilde{\xi}_t$ and an SSE $\hat{y}_t$ with a representation given by

$$\hat{y}_t = b\hat{y}_{t-1} + c r_{t-1} + d(s_{t-1}, s_t)\tilde{\xi}_t, \tag{22}$$

where d is any function of s_{t-1} and s_t.

[23] And the set of fixed points can be quite complicated. For example, note that if there is a non-trivial HDE with, say, $B(i,j) \neq 0$, then according to DT_B,

$$\beta_j \left(\sum_{k=1}^{m} p_{jk} B(j,k) \right) = 1$$

so that $C(i,j)$ can be any real number. In particular, the derivative of the T-map has a unit eigenvalue in this dimension. The effect of this unit eigenvalue on the link between E-stability and real-time learning (i.e. the E-stability Principle) is discussed in detail in Evans and Honkapohja (2001).

7.5 Proof of Proposition 3

Given the PLM $y_t = A(s_t) + B(s_t)r_t$, expectations are state contingent, where $s_t = j$ implies

$$\begin{aligned} E_t\left(y_{t+1}|s_t = j\right) &= p_{j1}A(1) + p_{j2}A(2) + ... + p_{jm}A(m) + \\ & \quad \left(p_{j1}B(1) + p_{j2}B(2) + ... + p_{jm}B(m)\right)\rho r_t. \end{aligned}$$

This produces a state-contingent actual law of motion, or, equivalently, a state-contingent T-map

$$\begin{aligned} A(j) &\to \beta_j\left(p_{j1}A(1) + p_{j2}A(2) + ... + p_{jm}A(m)\right) \\ B(j) &\to \beta_j\left(p_{j1}B(1) + p_{j2}B(2) + ... + p_{jm}B(m)\right)\rho + \gamma_j. \end{aligned}$$

Conveniently, this state-contingent T-map may be stacked, and becomes the T-map associated to the stacked system under the PLM $\hat{y}_t = A + Br_t$, where, as before, $B = (B(1)', ..., B(m)')'$, and also $A = (A(1)', ..., A(m)')'$. The T-map is given by

$$T(A, B)' = \left(\left(\oplus_{j=1}^{m}\beta_j\right)\left(P \otimes I_n\right)A, \left(\oplus_{j=1}^{m}\beta_j\right)\left(P \otimes I_n\right)B\rho + \gamma\right),$$

and the RDE is a fixed point of $T(A, B)$. Here $T : \mathbb{R}^{(nm\times 1)} \oplus \mathbb{R}^{(nm\times k)} \to \mathbb{R}^{(nm\times 1)} \oplus \mathbb{R}^{(nm\times k)}$.

The eigenvalues of the Jacobian matrices

$$\begin{aligned} DT_A &= \left(\oplus_{j=1}^{m}\beta_j\right)\left(P \otimes I_n\right) \\ DT_B &= \rho' \otimes \left[\left(\oplus_{j=1}^{m}\beta_j\right)\left(P \otimes I_n\right)\right] \end{aligned}$$

govern E-stability, i.e. E-stability requires real parts less than one, so that the E-stability condition is implied by the CLDC.

7.6 Proof of Proposition 4

The block of the T-map associated to the perceived parameters B is given by

$$B(i, j) \to \beta_j\left(p_{j1}B(j, 1) + p_{j2}B(j, 2)\right)B(i, j).$$

Because this block decouples from the rest, showing that this block of the T-map is unstable is sufficient. The Jacobian is given by

$$\begin{pmatrix} DT_B^1 & DT_B^2 \\ DT_B^3 & DT_B^4 \end{pmatrix}$$

where

$$
\begin{aligned}
DT_B^1 &= \begin{pmatrix} DT_B^1(1,1) & p_{12}B(1,1)'\otimes\beta_1 \\ 0 & I\otimes\beta_2\left(p_{21}B(2,1)+p_{22}B(2,2)\right) \end{pmatrix} \\
DT_B^2 &= \begin{pmatrix} 0 & 0 \\ p_{21}B(1,2)'\otimes\beta_2 & p_{22}B(1,2)'\otimes\beta_2 \end{pmatrix} \\
DT_B^3 &= \begin{pmatrix} p_{11}B(2,1)'\otimes\beta_1 & p_{12}B(2,1)'\otimes\beta_1 \\ 0 & 0 \end{pmatrix} \\
DT_B^4 &= \begin{pmatrix} I\otimes\beta_1\left(p_{11}B(1,1)+p_{12}B(1,2)\right) & 0 \\ p_{21}B(2,2)'\otimes\beta_2 & DT_B^4(2,2) \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
DT_B^1(1,1) &= p_{11}B(1,1)'\otimes\beta_1 + I\otimes\beta_1\left(p_{11}B(1,1)+p_{12}B(1,2)\right) \\
DT_B^4(2,2) &= p_{22}B(2,2)'\otimes\beta_2 + I\otimes\beta_2\left(p_{21}B(2,1)+p_{22}B(2,2)\right).
\end{aligned}
$$

Inserting

$$
B(1,1)=B(2,1)=0,\ \ B(1,2)=\frac{\beta_1^{-1}}{p_{12}},\ \ B(2,2)=\frac{\beta_2^{-1}}{p_{22}}
$$

yields repeated unit eigenvalues, plus the eigenvalues of

$$
\beta_2^{-1}\otimes\beta_2 + I\otimes I. \tag{23}
$$

Now notice that for a given $n\times$ matrix A, if λ is an eigenvalue of A, then $\lambda+1$ is an eigenvalue of $A+I_n$. Since the eigenvalues of $\beta_2^{-1}\otimes\beta_2$ are unity, we conclude that the eigenvalues of (23) are all equal to 2, thus implying instability.

7.7 Proof of Proposition 5

The corresponding T-map is

$$
A(i,j)\rightarrow\beta_j\sum_{k=1}^{m}p_{jk}A(j,k)
$$

and $T(B)=B$.[24] The result is immediate.

[24] The result $T(B)=B$ is standard in models with sunspots and reflects the fact that multiples of sunspots are also sunspots: if agents think $B\eta_t$ is the relevant sunspot vector, then, in a rational expectations equilibrium, $B\eta_t$ will be the relevant sunspot vector. Similar to the case of VAR learning, there is a continuum of fixed points to the T-map, each corresponding to a different HDE, and thus DT will have multiple unit eigenvalues.

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