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Consistent expectations and misspecification in stochastic non-linear economies

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Abstract

This paper generalizes existence results on first-order Stochastic Consistent Expectations Equilibria (SCEE) obtained by Hommes et al. (Learning to Believe in Linearity in an Unknown Nonlinear Stochastic Economy, 2002). We present a stochastic non-linear self-referential model in which expectations are based on linear perceptions. In an SCEE the sample mean and correlation coefficients of the true and perceived processes coincide. We provide conditions on the non-linear maps governing the stochastic process that are sufficient to establish existence of SCEE. Our approach defines a map that takes linear perceptions to actual outcomes in such a way that fixed points of this map are SCEE; by establishing existence of fixed points, we are able to demonstrate existence of SCEE. Stability of SCEE under real-time learning is analyzed numerically.

JEL classification: C62; D83; D84; E30

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1. Introduction

The rational expectations hypothesis, while still the benchmark expectations formation mechanism, is often criticized for the assumption that agents know the true distributions of the economy's variables. Instead, some researchers adopt an equilibrium concept based on misspecification in the agents' forecasting models: see, for example, Evans et al. (1993), Evans and Honkapohja (2001), Sargent (1999), and Branch and Evans (2004). In these papers, an equilibrium occurs in a self-referential model when agents are unable to detect their misspecification. Along these lines, an interesting and new equilibrium concept called a Stochastic Consistent Expectations Equilibrium (SCEE) has been developed by Hommes and Sorger (1998) and Hommes et al. (2002). In an SCEE the true process is defined by a (unknown) non-linear self-referential map. Agents, though, make forecasts via a linear perceived law of motion (PLM). An SCEE obtains when the sample mean and correlation coefficients of the non-linear stochastic process coincide with those predicted by the agents.

Motivation for the notion of an SCEE comes from a desire to instill agents with feasible, as well as in some sense optimal, forecasting mechanisms. Many economic models such as overlapping generations models (OLG), asset pricing models, and stochastic general equilibrium models of the business cycle, follow non-linear laws of motion. But, without first linearizing the relevant model, it is very difficult to obtain representations of the associated rational expectations equilibria that are useful for making optimal forecasts. The idea put forth by Hommes and Sorger (1998) and Hommes et al. (2002) is to assume that agents make forecasts using linear models, and then ask whether it is possible for these linear models to be consistent with the data produced by the underlying non-linear system.

An economy in a SCEE has a number of desirable features. For example, in an SCEE, agents' forecast errors are serially uncorrelated. Thus this concept satisfies one of the primary criteria cited in favor of rational expectations – indeed testing for the presence of serial correlation in the forecast errors is a standard method of empirically testing the rational expectations hypothesis. Furthermore, in an SCEE agents' forecast errors are consistent with their linear model. In this sense, agents are unable to detect their misspecification as simple econometric tests would reinforce their linear perceptions, and they would thus have no reason to alter their behavior. For a more complete discussion of the notion of SCEE, see Hommes et al. (2002).

It should be noted that SCEE fit into a larger class of equilibria called Restricted Perceptions Equilibria (RPE). RPE were first considered by Evans et al. (1993) and generalized in Evans and Honkapohja (2001). In an RPE, agents misspecify, in some dimension, but form their beliefs optimally given this misspecification. For example, in Evans and Honkapohja (2001) and Branch and Evans (2004) agents under-parameterize their perceived law of motion. In Evans et al. (1993) a fraction of agents mistakenly assume that the true non-linear model is actually linear.[1]

[1]The model in Evans et al. (1993) is based on the same OLG model of Grandmont (1985) that is considered in Hommes et al. (2002). In Evans, Honkapohja, and Sargent, however, the interest is whether cycles can be preserved if a fraction of agents believe the model is linear.

Optimality of these misspecified beliefs is imposed via an orthogonality condition. An SCEE is similar to an RPE in that all agents form forecasts using a linear model, even though the true model is non-linear. Expectations are consistent given this misspecification since the correlation coefficients of their perceived model exactly coincide with the non-linear model.

Our modeling framework closely follows Hommes et al. (2002). In that paper they specify dynamics that follow a non-linear self-referential map

$$x_t = G(x_{t+1}^e, \eta_t), \tag{1}$$

where x_{t+1}^e are expectations of x_{t+1} formed at time t, and η_t is an iid zero-mean process. Hommes et al. (2002) depart from the rational expectations hypothesis and instead suppose beliefs are formed based on the linear law of motion

$$x_t = c + b(x_{t-1} - c) + \varepsilon_t.$$

These authors then define an equilibrium concept that may be informally described as follows: an SCEE is a stationary process x_t satisfying the system (1), and so that $E(x_t) = c$, and $b^j = corr(x_t, x_{t-j})$.[2]

Hommes et al. (2002) consider the existence of SCEE. They show that if G is bi-linear with slope-zero then an SCEE exists with non-trivial b.[3] However, this functional form does not fit the OLG model of Grandmont (1985), the desired laboratory, and so Hommes et al. (2002) resort to numerical evidence to show that SCEE exist in an OLG model. In our paper, by relaxing the notion of equilibrium to what Hommes et al. (2002) call a first-order SCEE – an equilibrium concept that requires only the mean and first correlation coefficient to be consistent – we are able to provide an existence argument with considerably weaker restrictions on the functional form of G.[4]

We also examine the stability of these SCEE under real-time learning. We show that if agents do not have a constant in their perceived model, then the non-trivial SCEE will be stable under learning. On the other hand, if agents' beliefs incorporate a constant term the non-trivial SCEE will be unstable and beliefs will converge to the trivial SCEE. This type of result is standard in the learning literature, as stability often hinges on the constant term.

Finally, though intuition suggests SCEE should exist for decreasing G as well, for technical reasons our existence results only pertain to increasing G. Further, results from the learning literature suggest that SCEE in case of decreasing G are more likely to be stable under learning. We consider these issues numerically by specifying a decreasing G and running simulations. We find strong numerical support for both hypotheses: non-trivial SCEE exist in case G is decreasing, and further, these SCEE are stable under learning even when a constant term is included in the PLM.

The restrictions on G required for our existence arguments to hold are satisfied by applications such as the Increasing Returns model of Evans and Honkapohja (2001).

[2]We provide a more formal definition below.

[3]That is, with non-zero autocorrelation.

[4]In Tuinstra (2003) first-order consistent expectations equilibria are numerically analyzed in a deterministic version of the OLG model.

We present the results in a general context, though, in order to outline an approach for establishing existence of SCEE in other economic models, such as OLG, asset pricing, real business cycle models, and New-Keynesian general equilibrium models. We are optimistic that the methods established here can be extended to incorporate these other functional specifications.

This paper proceeds as follows. In Section 2 we present the existence argument. Section 3 considers real-time learning. Section 4 concludes.

2. Existence of SCEE

In this section we use a T-map – a map that takes perceptions to implications – to show the existence of order one SCEE in a simple non-linear forward looking model. The model is given by

$$x_t = G(x_{t+1}^e) + \eta_t, \tag{2}$$

where x_t is univariate, $G : \mathbb{R} \to \mathbb{R}$, and η_t is zero mean iid with full support, taking on values in $[-\overline{\eta}, \overline{\eta}]$. Models of this form are prevalent in dynamic macroeconomics: for example, Grandmont (1985) and Guesnerie and Woodford (1991) consider OLG models of the form (2). In particular, the increasing returns model in Evans and Honkapohja (2001) closely fits the framework (2).

The notation x_{t+1}^e captures agents' expectations of x_{t+1} formed at time $t-1$. We impose that agents form these expectations using the perceived law of motion

$$x_t = c + b(x_{t-1} - c) + \varepsilon_t \tag{3}$$

and ε_t is not assumed known. Thus $x_{t+1}^e = c + b^2(x_{t-1} - c)$, and so the resulting dynamic system, called the actual law of motion, is given by

$$x_t = G(c + b^2(x_{t-1} - c)) + \eta_t. \tag{4}$$

Linear beliefs, such as those we assume in (3), can be justified when the non-linear environment (2) is unknown. Agents may believe there is some type of non-linearity present, but because they do not know the form of this non-linearity, they are unable to exploit it for the purpose of forecasts. In these instances, agents may behave like econometricians and form optimal linear beliefs. Indeed, as we will see below, in an SCEE, agents are unable to detect their misspecification within the context of their perceived model.

The authors Hommes et al. (2002) pose the following question: If agents have linear beliefs (3) and if the state variable follows the non-linear reduced form model (2), do there exist belief parameters c and b that are linearly consistent with the associated stochastic process x_t? To state carefully a precise definition of the equilibrium notion they propose, we require the following notation: for any initial distribution λ_0 on $[-a, a]$, with the initial condition x_0 chosen with respect to this distribution, and for $t \geqslant 1$, let $\lambda_t(\lambda_0)$ be the unconditional distribution of x_t, and let $\Lambda_t(\lambda_0)$ be the unconditional joint distribution of (x_t, x_{t-1}), as determined by the recursion (4).

Definition 1. The triple $(\{x_t\}, c, b)$ is a (first-order) stochastic consistent expectations equilibrium (SCEE) provided the following hold:

1. x_t is generated by the recursion (4);
2. there exists a unique distribution λ so that for initial distribution λ_0, the distribution $\lambda_t(\lambda_0)$ converges weakly to λ;
3. for any λ_0, $\lim_{t\to\infty} \mathrm{E}_{\lambda_t(\lambda_0)}(x_t) = c$ and $\lim_{t\to\infty} corr_{\Lambda_t(\lambda_0)}(x_t, x_{t-1}) = b$.

Property (2) says that there is a unique distribution to which x_t converges weakly, regardless of initial condition, and property (3) says that the asymptotic mean and correlation coincide with the beliefs of agents.

Showing existence of this type of equilibrium is not difficult: if α is a fixed point of G then the parameter pair $(\alpha, 0)$ characterizes an SCEE. We say an SCEE with zero autocorrelation is trivial. It is more difficult to show existence of non-trivial SCEE because we need details of the asymptotic behavior of a non-linear stochastic process. To obtain analytic results, some restrictions on the function G are required. We will find it convenient to first change coordinates. We impose that G has a fixed point α and define $F(x) \equiv G(x+\alpha) - \alpha$. For convenience, we proceed by specifying restrictions directly on F (hence, indirectly on G). Assume F has the following properties:

A.1 F is twice continuously differentiable with $F' > 0$ and $sgn(F''(x)) = -sgn(x)$.
A.2 F is symmetric about the origin: $F(-x) = -F(x)$.
A.3 If $F'(0) > 1$ then there exists $x^* > 0$ so that $x > x^* \Rightarrow F(x) < x$.
A.4 If $F'(0) > 1$ then $a' = \inf\{x > x^* | x - F(x) > \overline{\eta}\}$ exists. If $F'(0) < 1$ then $a' = \inf\{x > 0 | x - F(x) > \overline{\eta}\}$ exists.
A.5 If $F'(0) > 1$ then $\overline{\eta} > \sup_{0 \leqslant x \leqslant x^*} F(x) - x$.

A brief description of a function F satisfying the above criteria may be useful. Thus, we note that any F which is symmetric about the origin, concave down for positive x (i.e. $x > 0 \Rightarrow F''(x) < 0$) and having horizontal asymptotes has the assumed properties. An example of a function with the desired properties is $F(x) = \alpha \tan^{-1}(x)$, with $\alpha > 0$. Fig. 1 illustrates this example for $\alpha = 2$.

Most of the restrictions above exist in many non-linear economic models. The symmetry restriction A.2, however, requires altering the functional form of most models. Take, as an example, the Increasing Returns model of Evans and Honkapohja (2001). This model is an OLG model with production externalities. The law of motion for the economy may have multiple fixed points. If the production function is assumed symmetric about a constant, then it is easily verified that the resulting law of motion for Evans and Honkapohja's Increasing Returns model fits the restrictions of this paper.[5] We note, however, that the symmetry restriction may be awkward for some economic models even though it is necessary for our analytic

[5]Production functions which are symmetric about a constant arise in behavioral models where a certain proportion of labor is a wasted input. For instance, it is sometimes argued that e-mail or internet use in the workplace leads to wasted productivity.

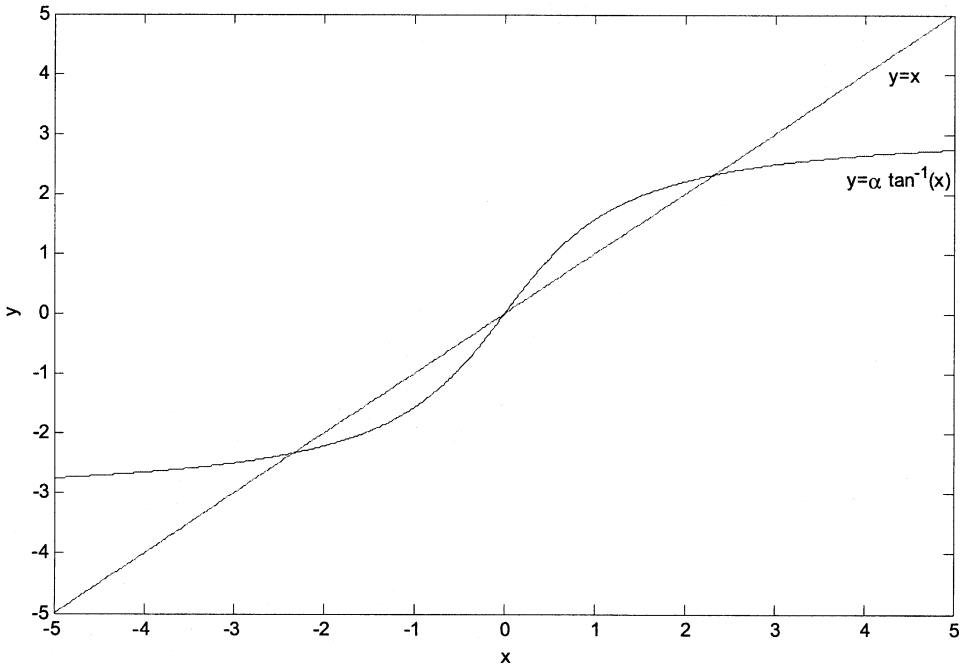

Fig. 1. Plot of $F(x) = y = \alpha \tan^{-1}(x)$, $\alpha = 2$.

results. This paper, though, is a first step in providing analytic existence of SCEE. Future research will examine other applications which do not include symmetric laws of motion.

A discussion of the intuition behind these restrictions is warranted. Assumption A.1 simplifies the dynamics by restricting the number of fixed points to at most 3. This restriction allows us to focus attention around fixed points where the dynamics of x may linger. Symmetry about the origin, A.2, simplifies analysis of the asymptotic mean of the system, and, in particular, implies that $F(0) = 0$. Assumption A.3, together with A.4 and the bounded support of the noise term, restricts the range of x and thus facilitates mathematical arguments which require this range to be compact.[6] Finally, assumption A.5 guarantees the noise term is sufficient to keep the process from remaining near a stable fixed point. Otherwise, only trivial SCEE would exist.

The primary contribution of this paper is to provide general analytical results on the existence of SCEE. Our approach is sufficiently general so to be useful in applications to a variety of economic models. Our proof of existence proceeds as follows. We first consider the special case in which $\alpha = 0$ and $c = 0$, and hence G and F coincide. We show that for any $b \in [0, 1]$, the process (4), with $c = 0$, is weakly

[6]Whether the results hold in case η_t has unbounded support is unknown, though we can think of no intuitive reason why unbounded support would result in failure of SCEE to exist. This intuition is strongly supported by the results in Section 3.3.

convergent to a unique distribution, and thus asymptotically stationary.[7] Then, given b, we may then define $T(b)$ to be the asymptotic correlation between x_t and x_{t-1}. The map T takes perceptions to realized outcomes. Its role in the model is important because an equilibrium occurs when perceptions are reinforced by the actual process – hence, a fixed point of T. The next step is to show T is continuous and takes the interval $[0, 1]$ into itself. That a fixed point exists is then guaranteed by Brouwer's theorem, but, as previously mentioned, it is clear that $b = 0$ is a fixed point, and so more work is required. We show non-trivial fixed points of the T-map exist given a further restriction on the function F. Existence in the special case that $\alpha = 0$ may then be used to prove the more general result by applying the coordinate change emphasized at the beginning of the section.[8] This general case is the central result of the paper.

2.1. The special case: $\alpha = 0$

In this section, we assume that G is symmetric about the origin and so is equal to its translation F. This allows us to set $c = 0$ and specify a univariate T-map, which makes analytic results easier to obtain.

2.1.1. Asymptotic stationarity

Let $a = a' + \overline{\eta}$ and notice that if x_t is generated by the recursion (4), $b \in [0, 1]$, and if $x_{t-1} \in [-a, a]$ then $x_t \in [-a, a]$.[9] In the sequel, with the exception of Section 3.3, we assume $b \in [0, 1]$, and we will refer to a as the boundary on the range of x.

To prove stationarity results we rely on the theory developed in Stokey and Lucas (1989). Our notation is slightly different from theirs because of conflicts. The transition function P_b associated to the dynamic stochastic system (4) is defined by

$$P_b(x, A) = prob\{x_{t+1} \in A | x_t = x\}.$$

We may then define the two operators S and S^* (denoted T and T^* by Stokey and Lucas) as follows: for bounded measurable functions f, we have

$$Sf(z) = \int f(x) P_b(z, \mathrm{d}x);$$

and for measures λ we have

$$S^*\lambda(A) = \int P_b(x, A)\, \mathrm{d}\lambda(x).$$

[7] This proof relies on the monotonicity of the transition functions, which, in turn, relies on $F' > 0$. To our surprise, we are unable to find analogous theorems guaranteeing asymptotic stationarity in case $F' < 0$, though intuition and numerical results suggests it holds: see Section 3.3.

[8] We are grateful to a referee for suggesting this change of coordinates, and thus allowing us to obtain results in the more general case.

[9] Consider the case $x_{t-1} \geqslant 0$. If $x_{t-1} \leqslant x^*$ then $x_t \leqslant x^* + \overline{\eta} \leqslant a' + \overline{\eta}$; if $x_{t-1} \in [x^*, a']$ then $x_t \leqslant x_{t-1} + \overline{\eta} \leqslant a' + \overline{\eta}$ and finally if $x_{t-1} \geqslant a'$ then $x_t \leqslant x_{t-1}$. If $x_{t-1} < 0$ then $z = F(-b^2 x_{t-1}) - \eta_t < F(-b^2 x_{t-1}) + \overline{\eta} < a$, so that $-z > -a$. But $-z = -F(-b^2 x_{t-1}) + \eta_t = F(b^2 x_{t-1}) + \eta_t$ by the symmetry of $F(b^2 x)$.

The measure $S^*\lambda$ gives the distribution of x_{t+1} given that x_t has distribution λ. Notice that if $S^*\lambda = \lambda$, and for all initial distributions λ_0 we have $S^{*n}\lambda_0 \to \lambda$ weakly, then x_t is asymptotically stationary; thus we study the behavior of S^*.

A transition function is said to have the Feller property provided the associated operator S takes the set of bounded continuous functions to itself. Because F is continuous, it follows that our transition functions P_b have the Feller property (Stokey and Lucas, 1989, p. 237).

A transition function is said to be monotone if whenever f is bounded and increasing, so too is Sf. Stokey and Lucas (exercises 12.9 and 12.11) show that P_b is monotone provided

$$P_b(s, (-\infty, x]) \leqslant P_b(s', (-\infty, x])$$

whenever $s > s'$. In the Appendix we use this sufficient condition to show P_b is monotone.

Lemma 2. *The transition function P_b is monotone.*

The transition function P_b is said to satisfy the mixing axiom provided there is a $d \in [-a, a]$, $\varepsilon > 0$ and $N \geqslant 1$ so that

$$P_b^N(-a, [d, a]) > \varepsilon \quad \text{and} \quad P_b^N(a, [-a, d]) > \varepsilon.$$

In the Appendix, we show that property A.5 of F guarantees the mixing axiom is satisfied.

Lemma 3. *The transition function P_b satisfies the mixing axiom.*

Proposition 4. *The process x_t defined by* (4) *converges weakly to a unique limiting distribution.*

This follows from Lemmas (2) and (3), and that P_b satisfies the Feller property. See Theorem 12.12 in Stokey and Lucas. Of course, this unique limiting distribution is precisely the unique distribution invariate under the operator S^*.

2.1.2. Continuity

Above it was shown that for each $b \in [0, 1]$, the process x_t defined by (4) converges weakly to a unique limiting distribution, which we now denote λ_b. Now let $s_n \to s$ in $[-a, a]$ and $b_n \to b$.

Lemma 5. *As $s_n \to s$ and $b_n \to b$, $P_{b_n}(s_n, \cdot) \to P_b(s, \cdot)$ weakly.*

For proof, see Appendix.

Proposition 6. *If $b_n \to b$ then $\lambda_{b_n} \to \lambda_b$ weakly.*

This proposition follows directly from Lemma 5 and Theorem 12.13 in Stokey and Lucas.

We now define the map $T : [0, 1] \to \mathbb{R}$ as follows:

$$T(b) = \mathrm{E}_{\lambda_b}(x^2)^{-1}\mathrm{E}_{\lambda_b}(xF(b^2x)), \tag{5}$$

noting that $T(b) = \lim_{t\to\infty} corr(x_t, x_{t-1})$. The following proposition is a direct consequence of proposition 6.

Proposition 7. *The map T is continuous.*

2.1.3. Existence of SCEE

To prove existence, we first show that T has a fixed point.

Lemma 8.

1. If $b \geqslant 0$ *then* $T(b) \geqslant 0$.
2. If $b \leqslant 1$ *then* $T(b) \leqslant 1$.

For proof, see Appendix. It follows immediately from the above Lemma and Brouwer's fixed point theorem that T has a fixed point in $[0, 1]$, and thus demonstrates the existence of an SCEE. This shows that it is possible for agents' linear beliefs to be consistent with a non-linear process. Examination of (4) makes it clear why a fixed point to T is an equilibrium. Agents have linear beliefs parameterized by b. These beliefs, in turn, affect the stochastic process x_t and its correlation coefficient. An equilibrium occurs when these beliefs lead to a process whose correlation coefficient reinforces those beliefs. The sufficient conditions provided guarantee that such an equilibrium exists.

The above proposition guarantees the existence of an SCEE. But this existence can be established much more easily; the pair $(\eta_t, 0)$ is an SCEE, and, of course, zero is a fixed point of the T-map. The question remains, 'Do there exist non-trivial fixed points?'. The following results address this question.

Proposition 9. *If $0 < F' < 1$ then the only fixed point of the T-map is zero.*

For proof, see Appendix.

Proposition 10. *If there exists $\hat{x} > 0$ so that $a < F(\hat{x})^2/\hat{x}$ then there exists $b > 0$ so that $T(b) = b$.*

For proof, see Appendix.

Recall that a is the boundary on the range of x, that is, $x_t \in [-a, a]$ for all times t. The condition $a < F(\hat{x})^2/\hat{x}$ requires F to have a certain degree of steepness for x on some subinterval of $(0, x^*)$; this steepness criterion is used in the proof to guarantee the existence of some b for which $T(b) > b$. Intuitively, it requires that, as x goes to zero, the average value of F (i.e. $F(x)/x$) be increasing faster than the value of F is decreasing. The existence of F satisfying this condition is easy to establish: let $F(x) = x^{1/3}$. Also, it is shown in the Appendix that the existence of $\hat{x}$ guarantees $F'(0) > 1$, thus distinguishing this proposition from the previous one.[10]

[10]Whether the condition $a < F(\hat{x})^2/\hat{x}$ for some $\hat{x}$ is necessary is an open question as it is certainly stronger than the condition that $F'(0) > 1$.

2.2. The general case: $\alpha \neq 0$

Establishing existence in the general case is straightforward given the results of the previous section. Assume G is symmetric about the fixed point α, and assume agents have the PLM

$$x_t = \alpha + b(x_{t-1} - \alpha).$$

To facilitate the existence argument we impose agents' PLM has the actual asymptotic mean. The ALM is given by

$$x_t = G(\alpha + b^2(x_{t-1} - \alpha)) + \eta_t,$$

and is also symmetric about the fixed point α. Let $y_t = x_t - \alpha$ and notice that

$$y_t = F(b^2 y_{t-1}) + \eta_t.$$

The results from the previous section show that y_t is asymptotically stationary and, provided F satisfies the hypothesis of Proposition 10, there is a non-zero b^* so that $\lim_{t\to\infty} corr(y_t, y_{t-1}) = b^*$. Further, x_t is asymptotically stationary because y_t is, and by the symmetry of G, has asymptotic unconditional mean equal to α. It follows that $\lim_{t\to\infty} corr(x_t, x_{t-1}) = b^*$, and thus the pair (α, b^*) determines a non-trivial SCEE. We summarize this result in the following proposition, which is the main result of the paper.

Proposition 11. *If G has fixed point α, $F(x) = G(x + \alpha) - \alpha$ satisfies A.1–A.5, and there exists $\hat{x} > 0$ so that $a < F(\hat{x})^2/\hat{x}$ then there exists a non-trivial SCEE.*

To illustrate this equilibrium we present a simple example. It is convenient to first specify F and then define G as a translation of F along the 45° line. Set

$$F(x) = \begin{cases} x^{1/3} & \text{if } x \geqslant 0 \\ -(-x)^{1/3} & \text{if } x < 0 \end{cases} \tag{6}$$

and $G(x) = F(x - \alpha) + \alpha$.[11] Agents are assumed to have beliefs

$$x_t = \alpha + b(x_{t-1} - \alpha)$$

thus generating an actual law of motion

$$x_t = G(\alpha + b^2(x_{t-1} - \alpha)) + \eta_t.$$

Our goal is to numerically find a value of b so that, asymptotically, the correlation coefficient is given by b.[12] Computation of the asymptotic correlation coefficient is obtained by repeatedly simulating the actual law of motion long enough to eliminate transient behavior, recording the last two realizations, and obtaining sample means, variances, and covariances. This process is repeated for various values of b until it

[11] Such a map may arise in the symmetric extension of the Increasing Returns model for a particular parameterization of the CRRA utility function.

[12] Formally, we must also show that the asymptotic mean of the process is α, but this is clear by symmetry.

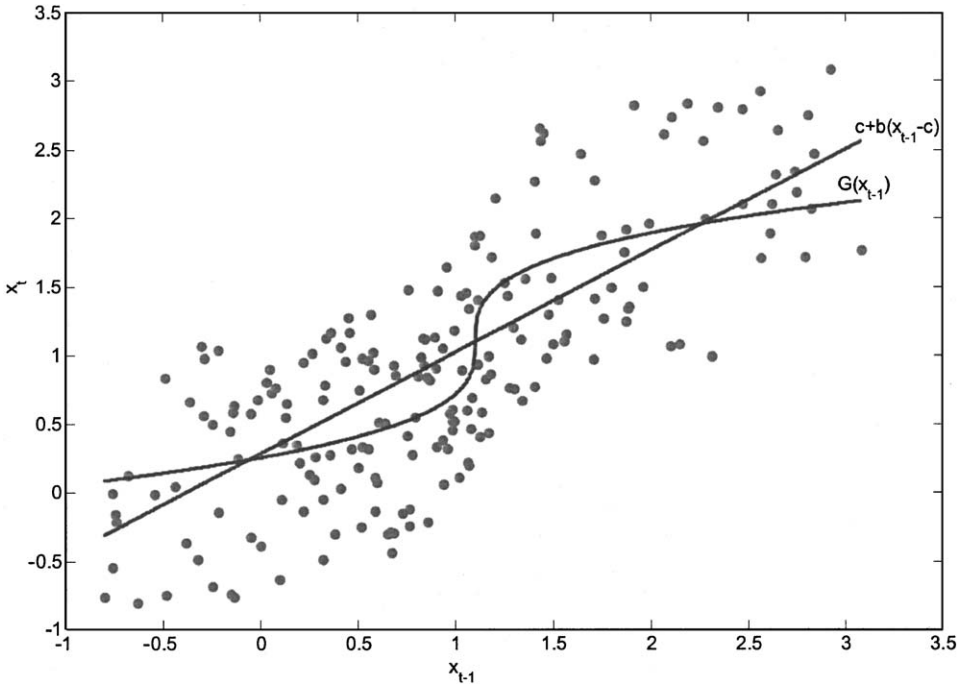

Fig. 2. Simulation and plots of map G with linear beliefs $c + b(x - c)$, where $c = 1.1$, $b = .74$, and $\alpha = 1.1$.

coincides with the asymptotic correlation.[13] For this particular exercise we specify $\alpha = 1.1$, $\bar{\eta} = 1$, and we ran 1000 simulations for 500 time periods. We find that $b \approx .74$ corresponds to an SCEE.[14]

Fig. 2 illustrates the SCEE obtained above. This figure consists of the map G, the linear beliefs, and realizations from one of the simulations. As the figure makes clear, the beliefs $(c, b) = (1.1, .74)$ combined with the map G leads to a stochastic process that has a trend line consistent with these linear beliefs. This demonstrates the nature of the SCEE; although agents have misspecified their model the resulting economic process produces realizations that appear consistent with this misspecification.

3. Stability of SCEE

Having established existence of SCEE, we are naturally led to wonder whether agents can learn to coordinate on them. To address this question, we analyze the

[13]We are able to eliminate this search problem by computing the asymptotic behavior of the economy under learning; see Section 3.

[14]Intuition suggests that this is the only non-trivial SCEE, but we have no analytic results supporting that claim.

behavior of the economy under real time learning. Specifically, we assume agents recursively estimate the coefficients of their linear PLM and use these estimates to form expectations. These expectations generate new data via the reduced form model, and agents use these new data to again update their estimates. As usual, the relevant learning question is, 'Do agents' estimates converge to a fixed point of the T-map?'.

Analysis of the asymptotic behavior of recursive estimators typically depends on the theory of stochastic approximation; and there is a subset of the theory – that part dealing with non-conditionally linear Markovian state dynamics – that applies to our model. Unfortunately, little is known about the functional form of the T-map, and thus the relevant regularity conditions required for application of the theory can not be verified.[15]

Given the intractability of analytic results, we turn to simulations. We begin by specifying the same functional form for F as above; see Eq. (6), and further that G is obtained by translating F along the 45° line. We will consider two cases below: $\alpha = 0$, $\alpha \neq 0$. We treat each case independently as they produce distinct stability results.

3.1. Stability in the special case

We begin by considering the special case $\alpha = 0$, and, importantly, that agents know the constant term in their regression is zero.[16] We find that convergence to a non-trivial SCEE obtains in this special case.

By assuming $\alpha = 0$ we may identify G with F. Further, we let agents have a PLM of the form

$$x_t = bx_{t-1}$$

thus generating an actual law of motion

$$x_t = F(b^2 x_{t-1}) + \eta_t.$$

Agents learn their parameter b using real-time recursive least squares estimates

$$\begin{aligned} b_t &= b_{t-1} + t^{-1} R_t^{-1} x_{t-1}(x_t - bx_{t-1}), \\ R_t &= R_{t-1} + t^{-1}(x_{t-1}^2 - R_{t-1}), \end{aligned} \tag{7}$$

where R_t is the sample second moment of x_{t-1}.

We examine stability under learning by running simulations of 10 000 periods each for random initial conditions. Fig. 3 illustrates a typical trajectory with $\overline{\eta} = 1$. As can be seen the non-trivial SCEE of $b \approx .74$ appears stable under real-time learning.[17] Further, simulations suggest the trivial SCEE ($b = 0$) is unstable. Other calibrations of the model's parameters yield similar results.

[15] Another benchmark technique in the learning literature is the application of the E-stability principle. However, stability analysis using this principle again requires knowledge of the functional form of the T-map.

[16] It will become apparent in the next section why it is crucial that agents know the constant term is zero.

[17] Repeated experimentation suggests that this is the only stable non-trivial SCEE, which further supports the intuition that it is the unique non-trivial SCEE.

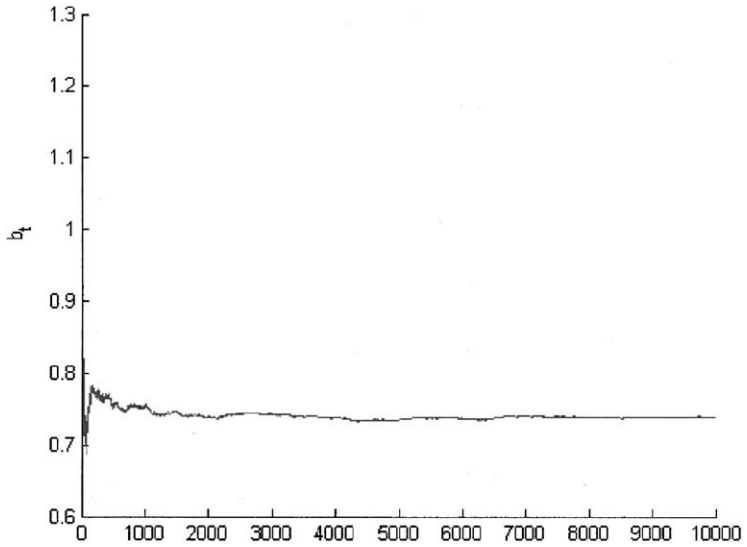

Fig. 3. Real-time learning when $\alpha = 0$ with no constant in the PLM.

3.2. Instability in the general case

This subsection considers the more general case in which $\alpha \neq 0$ and agents have a constant in their perceived law of motion. In this case, only the trivial SCEE are stable under learning.

We modify the functional form of agents' PLM to more easily incorporate least squares learning. We assume agents believe

$$x_t = c + bx_{t-1} + \varepsilon_t,$$

thus generating an actual law of motion of

$$x_t = G(c(1+b) + b^2 x_{t-1}) + \eta_t.$$

Agents are assumed to use a recursive least squares algorithm similar to (7) to estimate the parameters of their PLM. We use simulations to analyze whether these parameter estimates converged to a non-trivial SCEE. We find that all simulations failed to converge to a non-trivial SCEE. Moreover, even if we set our map to be symmetric about the origin ($\alpha = 0$), but assume agents do not know this and hence continue to include a constant in their regression model, then the non-trivial SCEE will be unstable. However, we find that, for both $\alpha = 0$ and $\alpha \neq 0$, each simulation converged to one of the two trivial SCEEs, thus suggesting that, if a constant term is included in the model then the trivial SCEEs are stable under learning.

Fig. 4 presents the results from two typical simulations of 10 000 periods each. We fix $\alpha = 1.1$ and $\overline{\eta} = 1$. We may numerically establish that $b = .74$ approximately corresponds to an SCEE. As the figure makes clear, this non-trivial SCEE is not stable, and further, even if other non-trivial SCEEs exist, they too are unstable; in all

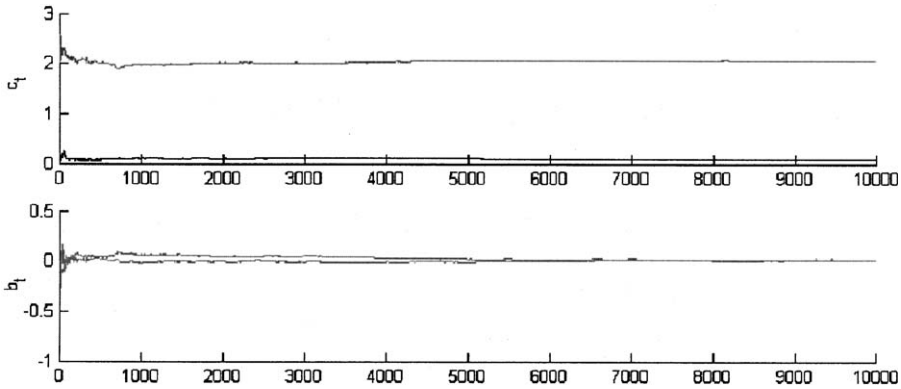

Fig. 4. Two real-time learning trajectories when $\alpha = 1.1$ and constant in the PLM.

simulations b_t converges to zero. The top panel illustrates, though, that the trivial SCEE corresponding to the two outside fixed points of the map G are stable. In one of the presented simulations the dynamics converge to the uppermost fixed point; in the other, they converge to the lower fixed point.

Hommes et al. (2002) also consider stability under learning, and to this end, they employ a slightly different updating algorithm. They assume agents update beliefs by computing the sample averages and sample autocorrelations. This method has the advantages of being simple (and thus consistent with the idea that agents are relatively unsophisticated) and of restricting the first-order autocorrelation estimate to lie in the interval $[-1, 1]$. For the same specification of G as identified above, and for the same parameter values, we analyzed the asymptotic behavior of their estimators; we found that, in all cases, the results were qualitatively identical.

3.3. Further results

The somewhat disappointing result that non-trivial SCEE are unstable in the general case is not surprising when considered in context of the learning literature. It is standard to find that equilibria associated to expectational difference equations of the form $y_t = \beta E_t y_{t+1}$ are unstable under learning if $\beta > 1$ and a constant term is present in the PLM. On the other hand, if $\beta < -1$, the associated equilibria may be stable under learning;[18] this suggests we might find that non-trivial SCEE are stable if $G' < 0$. And, while we have no existence results in case of negatively sloped G' – our proofs require a type of monotonicity in transition functions which is present only if $G' > 0$ – intuition suggests they should exist.

To test this intuition, as well as the possibility that if they exist, they are stable under learning, we again turn to simulations. Begin by reflecting F, as defined by (6), about the vertical axis, and then obtaining G by translating as usual. We then

[18]Provided a common factor representation is used; see Evans and McGough (2002).

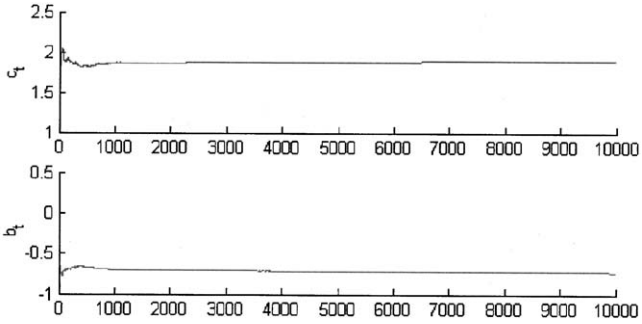

Fig. 5. Real time learning with a decreasing G, $\alpha = 1.1$, and a constant in the PLM.

consider real time learning using the algorithms described above, with $\alpha = 1.1$ and $\overline{\eta} = 1$. We find that the estimators appear to converge to $b = -0.74$ and $c = \alpha(1 - b) = 1.9$, which, as intuition suggests should be the case, is approximately obtained by reflecting the previous SCEE about the vertical line $x = \alpha$ (See Fig. 5.). These numerical results strongly suggest that non-trivial SCEE exist in case $G' < 0$, and further are stable under learning.

4. Conclusion

Non-linear stochastic models of the economy present a clear challenge to the rational expectations hypothesis; while a beneficial and important benchmark, the assumption that agents form forecasts with respect to distributions not obtainable by any reasonable means of computation needs to be carefully considered. Hommes et al. (2002) have put forth an interesting and natural alternative, that of a SCEE which only requires agents form forecasts via a consistent linear model; however, showing existence of this type of equilibrium at a sufficient level of generality has been difficult. In this paper, we have taken a step toward establishing SCEE as a contender in the realm of boundedly rational equilibria by proving existence for a general class of non-linear reduced forms, which include some well-known economic models.

The results from our learning analysis are as expected. We found that, without a constant term in the perceived law of motion, and provided G was symmetric about the origin, the associated non-trivial SCEE was stable. However, including the constant term in the perceived law of motion destabilized the non-trivial SCEE and the economy converged to the trivial SCEE. This is not an uncommon result in the learning literature; stability often hinges on the presence of a constant term. Furthermore, we found numerically that when the slope of G is negative, then as intuition suggests, non-trivial SCEE appear to exist, and, as the learning literature suggests, the non-trivial SCEE appear to be stable under learning.

In future research we intend to examine existence and stability of SCEE in other economic applications. In particular, we are interested in an extension to multivariate models such as the real business cycle model and the New-Keynesian monetary model. Both are models usually analyzed under linear approximations despite possessing well-known interesting global phenomena.

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Appendix A

This Appendix contains the proofs of most of the results of the paper.

Proof of Lemma 2. Let $s > s'$ and $I(z) = (-\infty, z]$. For real number w and set $A \subset \mathbb{R}$, define the notation

$$A - w = \{a - w | a \in A\}.$$

Note that $F' > 0$ implies $F(b^2 s) > F(b^2 s')$. Thus

$$\begin{aligned} P_b(s, I(z)) &= prob\{x_{t+1} \in I(z) | x_t = s\} \\ &= prob\{\eta_{t+1} \in I(z) - F(b^2 s)\} \\ &\leqslant prob\{\eta_{t+1} \in I(z) - F(b^2 s')\} \\ &= P_b(s', I(z)), \end{aligned}$$

where the inequality follows from the fact that $I(z) - F(b^2 s) \subset I(z) - F(b^2 s')$. □

Proof of Lemma 3. Let $d = 0$ and $x_0 = a$. Let $x_b^* \geqslant 0$ be the largest point such that $F(b^2 x_b^*) = x_b^*$. Since x_b^* is a stable fixed point of the non-stochastic system $x_t = F(b^2 x_{t-1})$ there exists N such that if $\eta_t \leqslant 0$ for N consecutive times then $x_N \leqslant x_b^*$. If $x_b^* = 0$ we are done. If not let

$$h = \sup_{0 \leqslant x \leqslant x^*} F(x) - x,$$

and recall by assumption that $\overline{\eta} > h$. Let $h' = -1/2(h + \overline{\eta})$. Then $prob\{\eta_t < h'\} = \delta > 0$. Now notice that if $\eta_t < h'$ then

$$\begin{aligned} x_t - x_{t-1} &= F(b^2 x_{t-1}) - x_{t-1} + \eta_t \\ &\leqslant F(x_{t-1}) - x_{t-1} + \eta_t \\ &\leqslant h + \eta_t < h + h' = 1/2(h - \overline{\eta}) < 0. \end{aligned}$$

Thus if $x_N \leqslant x_b^*$ then there exists M so that if $\eta_t < h'$ M consecutive times then $x_{N+M} \leqslant 0$. By symmetry an analogous proof holds for $x_0 = -a$. □

Proof of Lemma 5. It suffices to show the corresponding distribution functions converge pointwise; that is,

$$prob\{s' \in I(z)|s' = F(b_n^2 s_n) + \eta_t\} \to prob\{s' \in I(z)|s' = F(b^2 s) + \eta_t\}.$$

But this follows immediately from the fact that $F(b_n^2 s_n) \to F(b^2 s)$. Indeed

$$\begin{aligned} prob\{s' \in I(z)|s' = F(b_n^2 s_n) + \eta_t\} &= prob\{\eta_t \in I(z) - F(b_n^2 s_n)\} \\ &= \int_{-\infty}^{z-F(b_n^2 s_n)} f_\eta(x)\,\mathrm{d}x, \end{aligned}$$

and this integral is continuous in its limits of integration. □

Proof of Lemma 8. To prove statement one, it suffices to show that $\mathrm{E}(x_t x_{t-1}) \geqslant 0$. But

$$\begin{aligned} \mathrm{E}(x_t x_{t-1}) &= \mathrm{E}(x_{t-1} F(b^2 x_{t-1})) \\ &= \int_{-\infty}^{\infty} x F(b^2 x)\,\mathrm{d}\lambda_b(x) \geqslant 0, \end{aligned}$$

where the last inequality follows from the fact that the integrand is always positive. Statement 2 follows from the fact that $T(b)$ is a correlation coefficient. □

Proof of Proposition 9. It suffices to show that if $b > 0$ then $T(b) < b$. Let $m = F'(0)$ and notice that $|mb^2 x| > |F(b^2 x)|$. Let λ be the unconditional distribution of x_t. Then

$$\begin{aligned} T(b) &= \mathrm{E}(x_t^2)^{-1} \mathrm{E}(x_t x_{t-1}) \\ &= \mathrm{E}(x_t^2)^{-1} \int x_{t-1} F(b^2 x_{t-1})\,\mathrm{d}\lambda(x_{t-1}) \\ &\leqslant \mathrm{E}(x_t^2)^{-1} \int mb^2 x_{t-1}^2\,\mathrm{d}\lambda(x_{t-1}) = mb^2 < b. \qquad \square \end{aligned}$$

Proof of Proposition 10. It suffices to prove there is a $b > 0$ so that $T(b) \geqslant b$. Pick $\hat{x}$ as in the premise of the proposition and let $m = F(\hat{x})/\hat{x}$ and $b = 1/m$. We claim this b works. First notice that $\hat{x} \leqslant x^*$: this follows from the fact that $x > x^* \Rightarrow F(x)/x < 1 \Rightarrow F(x)^2/x < F(x) < x < a$. This shows that $b \leqslant 1$ as well as that $F'(0) > 1$ thus distinguishing the premise of this proposition from that of the previous. Next notice that $b^2 a = \hat{x}^2 a/F(\hat{x})^2 < \hat{x}$. Thus $x \in [-a, a]$ implies $|b^2 x| < \hat{x}$. We claim this shows $|mb^2 x| < |F(b^2 x)|$ for $x \neq 0$. First consider $0 < x < \hat{x}$. Then the line $y = mx$ lies under the graph of $F(x)$; indeed, F is concave down for positive x and the line mx joins the origin and the point $(\hat{x}, F(\hat{x}))$. A symmetric argument holds for $-\hat{x} < x < 0$. An argument analogous to the one given to prove Proposition 9 shows that $T(b) \geqslant mb^2 = b$. □

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