Dynamic Predictor Selection in a New Keynesian Model with Heterogeneous Expectations*

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Abstract

This paper introduces dynamic predictor selection into a New Keynesian model with heterogeneous expectations and examines its implications for monetary policy. We extend Branch and McGough (2009) by incorporating endogenous time-varying predictor proportions along the lines of Brock and Hommes (1997). We find that periodic orbits and complex dynamics may arise even if the model under rational expectations has a unique stationary solution. The qualitative nature of the non-linear dynamics turns on the interaction between hawkishness of the government's policy and the extrapolative behavior of non-rational agents.

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1 Introduction

Among the standard assumptions of the New Keynesian model is the macroeconomic benchmark of (homogeneous) rational expectations (RE). Recent empirical analysis, however, casts some doubt on this assumption's validity. Using survey data, Branch

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(2004, 2007), Carroll (2003), and Mankiw, Reis, and Wolfers (2003) provide evidence that economic forecasters (both consumers and professional economists) have heterogeneous expectations; and, importantly, the distribution of heterogeneity evolves over time in response to economic volatility. Branch (2004), in particular, provides evidence that survey respondents in the Michigan survey of consumers are distributed across rational and adaptive expectations and these proportions evolve over time as a reaction to past mean square forecast errors: for instance, in periods of high economic volatility such as the 1970's a higher proportion of agents used rational expectations than during periods of relatively low volatility.

In light of the empirical evidence, Branch and McGough (2009), relaxing the assumption of rational expectations, incorporate heterogeneous boundedly rational agents into the micro-foundations of a New Keynesian model. The primary contribution of Branch and McGough (2009) is an aggregation result based on linear approximations to agents' optimal decision rules which, themselves, depend on heterogeneous expectations operators. Specifically, under fairly general assumptions on how agents form expectations, it was shown that aggregate outcomes satisfy IS and AS equations which have the same functional form as those in the standard model except that the homogeneous expectations operator is replaced with a convex combination of heterogeneous expectations operators. This extension of the basic model has important implications for the dynamics of the economy. As a concrete example, Branch and McGough (2009) assume agents are (exogenously) split between rational and adaptive expectations and monetary policy follows a Taylor-type rule. In this special case, the dynamic properties of the heterogeneous expectations model depend crucially on the distribution of agents across forecasting models and, in particular, its dynamic properties differ from those implied by the standard RE model.

In Branch and McGough (2009), we took the distribution of heterogeneous expectations as fixed and exogenous. The empirical evidence cited above and the results from our previous paper, suggest that this assumption is overly restrictive. In this paper, we follow Brock and Hommes (1997) and assume that the degree of heterogeneity is allowed to vary over time in response to past forecast errors (net of a fixed cost) thereby coupling predictor choice with the dynamics of inflation and output. As in Brock and Hommes (1997), our predictor choice stems from a discrete choice framework that has a venerable history in economics, e.g. Manski and McFadden (1981).

The primary interest of this paper is to study the dynamics of a monetary economy with heterogeneous expectations and dynamic predictor selection. We assume agents choose between using a rational predictor (for a cost) and using an adaptive forecasting model. We find that for sufficiently low costs to using the rational predictor, the model's steady state is stable. For higher costs, however, the steady state may destabilize and the dynamic system may bifurcate. Whether this bifurcation leads to bounded complex dynamics depends on the coefficients in the monetary policy rule and the degree to which the adaptive agents extrapolate from past data. We find two different cases under which bounded complex dynamics may obtain, depending on the stance of monetary policy: first, if policy is passive, in the sense of adjusting nominal interest rates less than one for one with inflation, or, second, if monetary policy is active and the adaptive agents extrapolate from past data.

The intuition behind the onset of complicated dynamics in the heterogeneous New Keynesian model may be obtained as follows. Suppose that agents have a choice of using a rational predictor or an adaptive predictor, and that they seek to minimize their mean square forecast error net of a cost, C, to using the rational predictor. Suppose also that monetary policy follows a Taylor-type rule that adheres to the 'Taylor principle' by adjusting nominal interest rates more than one for one when inflation deviates from a target. This is the standard advice for setting monetary policy in New Keynesian models. As will be evident below, the heterogeneous expectations model with dynamic predictor selection can be represented as a dynamical system of the form

$$x_t = M(n_{t-1})x_{t-1},$$

where x is a vector consisting of aggregate output and inflation, and n is the fraction of rational agents. For n appropriately large, but less than one, the eigenvalues of M lie inside the unit circle. Now consider what happens to an economy that begins with a fraction of rational agents close to one. Since the eigenvalues of M will have modulus less than one, the economy will contract toward the steady state and the relative advantage of rational over adaptive expectations will diminish. As a result, a growing proportion of agents will not want to pay the fixed cost to being rational. The proportion of rational agents n will decrease until an eigenvalue of M again has modulus greater than one, causing the economy to repel from the steady state. This attracting/repelling feature of dynamic predictor selection is what makes bounded complex dynamics exist even in the case that monetary policy adheres to the Taylor principle.[1]

The potential for complex dynamics have important implications for monetary policy. A wide and established literature appears to agree on one essential ingredient of sound monetary policy: policy should be set to act aggressively against inflation (e.g. Taylor (1999), Clarida, Gali and Gertler (2000), Bernanke and Woodford (1997), Svensson and Woodford (2003), and Woodford (2003)). A basis for this finding is that adherence to an active monetary policy rule (a variant on the 'Taylor principle') results in a determinate model and thus a unique rational expectations equilibrium.[2] However, in case of heterogeneous expectations, we find that even an active rule may

[1] This intuition for switching between stable and unstable dynamics was first discovered in a cobweb model by Brock and Hommes (1997).

[2] A determinate model is sometimes called "stable" because, due to its unique equilibrium, the economy is not subject to excessive volatility that can arise when agents' beliefs are driven by self-fulfilling prophecies (e.g. sunspots).

result in complex behavior and thus possibly excess volatility. In fact, not only does the interaction between monetary policy and expectations formation in part dictate the ensuing dynamic behavior, we even find that these complicated dynamics may arise when monetary policy is set to guarantee determinacy under rational expectations. To most convincingly illustrate this point, we specify a policy rule that yields determinacy under RE, and we assume that there is a fixed cost to deviating from rational expectations – precisely the setting assumed for, and implied by, standard monetary policy advice: even in this case the economy may exhibit bounded complex dynamics. Our results suggest that, in the presence of heterogeneous agents, determinacy under RE may not be a robust criterion for policy advice. The complex dynamics produced by our model are not outcomes limited to unusual calibrations or *a priori* poor policy choices: complex behavior appears to be an almost ubiquitous feature of a time-varying heterogeneous expectations New Keynesian model.

That determinacy of a steady state may not be sufficient to guard against instability has been demonstrated elsewhere. Benhabib, Schmitt-Grohe, and Uribe (2001) show that a determinate steady state may be surrounded by bounded complex dynamics when nominal interest rates have a zero lower bound, even under the assumption of rationality. Benhabib and Eusepi (2004) conclude that a New Keynesian model extended to include capital may possess chaotic dynamics. Bullard and Mitra (2002) show that determinacy is neither necessary or sufficient for a rational expectations equilibrium to be stable under adaptive learning. Gali, Lopez-Salido, and Valles (2004) derive a model with a proportion of rule of thumb consumers and demonstrate that the determinacy properties of the model are sensitive to the presence of these agents. Levin and Williams (2003) stress the importance of policy being robust across potential model specifications. DeGrauwe (2008) studies how heterogeneity and monetary policy can interact to lead to endogenous dynamics. Finally, Anufriev, et al. (2009) demonstrate that in a stylized macro model with heterogeneous expectations multiple steady states may arise even when the Taylor principle holds.

This paper is organized as follows. Section 2 presents an overview of the New Keynesian model with heterogeneous expectations and introduces dynamic predictor selection into the model. Section 3 presents the analysis and results while Section 4 concludes.

2 A New Keynesian Model with Heterogeneous Expectations

In Branch and McGough (2009), we derive a New Keynesian model with heterogeneous expectations where aggregate output and inflation are governed by the following equations

$$y_t = \hat{E}_t y_{t+1} - \sigma^{-1}(i_t - \hat{E}_t \pi_{t+1}) \quad (1)$$
$$\pi_t = \beta \hat{E}_t \pi_{t+1} + \lambda y_t. \quad (2)$$

Here y_t is aggregate output gap, π_t is the inflation rate, and $\hat{E}_t$ is a heterogeneous expectations operator defined as a convex combination of boundedly rational (and possibly rational) forecasting models. Below we make explicit assumptions on $\hat{E}$. Note that under rational expectations, i.e. $\hat{E} = E$, the unique steady state for the economy is $y = \pi = 0$.

The form of (1)-(2) is a New Keynesian model in which conditional expectations have been replaced by a heterogeneous expectations operator $\hat{E}_t$. Branch and McGough (2009) derive these reduced-form equations from linear approximations to the optimal decision rules in a Yeoman-farmer economy extended to include two types of agents, differing in their forecasting mechanism. The first equation (1) represents the demand side of the economy. Under homogeneous expectations, it is derived as a log-linear approximation to the representative agent's Euler equation. With heterogeneous agents, the IS equation (1) is found by aggregating the Euler equations across heterogeneous agents. The parameter σ^{-1} is the usual real interest elasticity of output.

The second equation (2) is the aggregate supply relation. Similar to the representative agent model, it is found by averaging the pricing decisions of firms in the economy. In this formulation, λ is the usual measure of output elasticity of inflation. The functional forms of these IS-AS relations are the same as those in the standard New Keynesian model. The key distinction is that because of the heterogeneity in beliefs, the equilibrium processes for aggregate output and inflation depend on the distribution of agents' expectations. In Branch and McGough (2009), we provide the axiomatic foundations that facilitate aggregating heterogeneous expectations into the tractable reduced form (1)-(2). This paper takes as given that these are the equations governing the economy and studies the dynamic implications of heterogeneous expectations. We remark, however, that the form of heterogeneity assumed in this paper is consistent with the theoretical foundations in Branch and McGough (2009).

We assume that monetary policy is specified by the following instrument rule:

$$i_t = \alpha_y \hat{E}_t y_{t+1} + \alpha_\pi \hat{E}_t \pi_{t+1}. \quad (3)$$

The form of (3) is what Evans and Honkapohja (2005), Evans and McGough (2005a,b), and Preston (2005b) call an 'expectations-based' rule. It is a simple implementable rule that takes advantage of a policymakers' observations of private sector expectations, and follows Bernanke (2004) in advocating for a policy that reacts aggressively to private-sector expectations. Implementation of such a rule is straightforward, even in an economy with heterogeneous agents, so long as the average forecast is observed by policymakers. In practice, this is a reasonable assumption as there are many market and survey based measures of the average, or consensus, inflation and output forecasts. None of the qualitative results in this paper, however, are sensitive to the form of the nominal interest rate targeting rule. To verify robustness we also considered a policy rule in which the government sets the instrument against the optimal forecasts of inflation and output, rather than the average of the agents' forecasts.[3] All qualitative results are robust to the alternate form of the instrument rule.

Policy rules with forms similar to (3) are often described as Taylor-type instrument rules. These rules are said to satisfy the Taylor principle if the response of nominal interest rates to the inflation metric is greater than one, i.e. $\alpha_\pi > 1$. This ensures that when the central bank adjusts the nominal interest rate it is also adjusting the real interest rate in the same direction. Below we will find that the qualitative features of the model's dynamics hinge on whether the policy rule satisfies the Taylor principle.

2.1 Expectations and predictor dynamics

To close the model we must specify the operator $\hat{E}_t$. For simplicity, we assume there are precisely two types of predictors available to agents: the type 1 predictor, which is called "rational," and the type 2 predictor, which is called "adaptive." Agents using type 1 predictors are assumed to be very good forecasters, which we capture by providing them perfect foresight when forming one-step-ahead forecasts (and this is why we call them "rational"): if $x = y$ or π then $E_t^1 x_{t+1} = x_{t+1}$.[4] Agents using type 2 predictors are accessing a less sophisticated technology, and are assumed to look backwards when forming forecasts (which is why we call them adaptive): if $x = y$ or π then $E_t^2 x_{t+1} = \theta_x^2 x_{t-1}$; this formulation is derived from a linear forecast rule of the form $x_t = \theta_x x_{t-1}$. Finally, we may set

$$\hat{E}_t x_{t+1} \quad = \quad n_{xt} x_{t+1} + (1 - n_{xt}) \theta_x^2 x_{t-1}, \tag{4}$$

where n_{xt} is the fraction of agents using rational predictors at time t. More details about the construction of the expectations operator $\hat{E}_t$ may be found in Branch and McGough (2009).

The form of heterogeneous expectations in (4) imposes that agents are heterogeneous in their forecasting of a particular aggregate outcome, rather than heterogeneous in their forecasts related to consumption (i.e. the IS equation) versus pricing

[3]Here, by "optimal forecasts," we mean forecasts that minimize mean square error. Because we are in a non-stochastic environment (and we are not considering the possibility of associated stochastic sunspot equilibria) optimal forecasts in our model correspond to agents having one-step ahead perfect foresight. These issues are more carefully addressed in Section 2.1 below.

[4]This is the version of rationality studied by Brock and Hommes (1997); we employ it here to approximate the notion that rational agents will minimize mean square forecast errors.

behavior (i.e. the AS equation). This assumption is consistent with the learning literature which models boundedly rational agents as econometric forecasters attempting to forecast aggregate output and inflation. That agents might want a different forecasting model for inflation and output is in line with the findings of Branch and Evans (2006a) who compute simple recursive forecasting models that are consistent with survey data on inflation and output expectations. However, the assumption is still somewhat *ad hoc* and we checked that our qualitative results were robust to imposing the heterogeneity in forecasting methods across IS-AS relations.

We call agents using type 1 predictors rational because they minimize their mean square forecast error. On the other hand, our agents make time t decisions based only on forecasts of time $t+1$ aggregate data, and in particular are not *ex-ante* concerned with meeting transversality conditions. In the sequel, we focus on equilibrium dynamics that remain uniformly bounded to remain consistent with agents' transversality conditions *ex-post*.[5] In this way we are modeling our type 1 agents in a manner similar to Euler equation learning – a common approach in the learning literature, particularly in the context of New Keynesian monetary models: See Honkapohja, Mitra, and Evans (2003) for further discussion. The Euler equation approach dictates that households' decisions satisfy their *ex ante* first-order optimality conditions and only satisfy the transversality conditions *ex post*. In a sense, then, our perfect foresight agents are really good myopic forecasters. An interesting alternative that explicitly accounts for infinite horizon planning is developed by Preston (2005a), and it would be quite natural to reconsider the questions addressed here using a model consistent with his method.

Agents with type 2 predictors use a fairly standard form of adaptive expectations. Such expectations can be thought of as arising from a simple linear perceived law of motion of the form $x_t = \theta_x x_{t-1}$. In many models, real-time estimates of θ_x converge to their REE minimal state variable (MSV) values. Here we take θ as fixed, though an extension with real-time learning and dynamic predictor selection is a topic of current research.[6]

When $\theta < 1$ adaptive agents dampen past data in forming expectations; when $\theta > 1$ agents have extrapolative expectations. Adaptive expectations of this form were assumed in Branch and McGough (2009) as well as in Brock and Hommes (1997, 1998), Branch (2002), Branch and McGough (2005), and Pesaran (1987). When $\theta = 1$ the adaptive predictor is usually called 'naive' expectations, and was the case emphasized by Brock and Hommes (1997). The $\theta > 1$ case was given particular emphasis by Brock and Hommes (1998). It is straightforward to extend the adaptive predictor to incorporate more lags. We anticipate that such an extension would not alter the qualitative results of this paper but would alter the quantitative details.

[5] As we remark below, for some parametric specifications, the equilibrium dynamics do not remain uniformly bounded.

[6] Branch and Evans (2006a) and Guse (2008) have made some progress on this issue.

Despite this simple form of adaptive expectations there is evidence in survey data that agents are distributed across rational expectations and a simple univariate forecasting model (see Branch (2004)). The contribution of this paper is to demonstrate in a simple monetary model that time-varying, endogenously determined distributions of heterogeneous expectations may significantly alter the equilibrium implications for a given monetary policy.

Having specified the predictors available to agents, it remains to determine the proportion of agents using a particular predictor. There is growing empirical evidence that the distribution of agents' heterogeneity is time-varying. For example, Mankiw, Reis, and Wolfers (2003) study various surveys of inflation expectations and show a wide, time-varying dispersion in beliefs. Branch (2004, 2007) documents time-varying distributions of agents across discrete predictors. In each case, the nature of the variation in the distributions appears structural: in Branch (2004, 2007) volatility causes more agents to adopt rational expectations, and in Mankiw, Reis, and Wolfers (2003) volatility causes dispersion to shrink. Given that the results from Branch and McGough (2009) suggest that the dynamic properties of the economy are highly sensitive to the fraction of rational agents, and given that there is empirical evidence of time-varying fractions, we turn to an endogenous dynamic predictor selection version of the heterogeneous expectations model.[7]

With dynamic predictor selection $n_{xt}, x = y,\ \pi$ are assumed to follow

$$n_{xt} = \frac{\exp\left[\omega U_{jt}^{x}\right]}{\exp\left[\omega U_{jt}^{x}\right] + \exp\left[\omega U_{j't}^{x}\right]} \quad j' \neq j. \tag{5}$$

Here U_{jt}^{x} is a predictor fitness measure to be specified below. This is a multinomial logit law of motion and was employed by Brock and Hommes (1997), and then extended to a stochastic setting by Branch and Evans (2006a). The parameter ω is called the 'intensity of choice'; it governs how strongly agents react to past forecast errors. Brock and de Fountnouvelle (2000) and Brazier, Harrison, King, and Yates (2008) adopt a discrete choice setting in monetary models. Brazier, et al., assume that $\omega < \infty$ proxies for measurement error in calculating forecast errors. In this setting, ω is inversely related to the variance of those errors.[8]

Brock and Hommes (1997), who develop their notion of predictor selection in the context of a univariate linear cobweb model, show that for large, but finite, values

[7] The importance of endogenous predictor selection and macroeconomic dynamics were also shown by Marcet and Nicolini (2003), Tuinstra and Wagener (2007).

[8] In our simulations below, we initialize the model in an REE. In the 'neoclassical' limit, i.e. $\omega \to \infty$, the model will remain in an REE. Thus, this approach yields heterogeneity for $\omega < \infty$, so the existence of adaptive agents is a natural consequence of these measurement errors or random utility terms. In a sense, heterogeneity arises because of some uncertainty about the best forecasting model. In Branch and Evans (2006a), heterogeneity arises, in a stochastic univariate model, even as $\omega \to \infty$, and we would expect similar results if we extended that framework to a New Keynesian model.

of ω there may exist complex dynamics. The intuition for their finding is that the dynamic predictor selection coupled with the equilibrium price dynamics creates a tension between repelling and attracting dynamics. As we will explore in detail below, our analysis of the heterogeneous New Keynesian model with fixed predictor proportions indicates that the steady state may be dynamically unstable for low values of n and dynamically stable for large values of n; this creates precisely the sort of repelling/attracting forces that makes bounded, complex dynamics possible.

Since we are modeling the predictor selection as independent of the optimization problem, we assume that predictor success is measured in terms of mean-square error:

$$MSE^x_{jt} = MSE^x_{jt-1} + \mu\left((x_t - \hat{E}^j_{t-1}x_t)^2 - MSE^x_{jt-1}\right).$$

The predictor fitness metric is assumed to be

$$U^x_{jt} = -MSE^x_{jt} - C_j. \tag{6}$$

Below, in our numerical simulations we will assume that $\mu = 1$, so that agents react to last period's squared error only.[9] We make this assumption to minimize the number of bifurcation parameters.[10] Notice that we also assume a constant in the statistical metric function U_{jt}. This constant can be interpreted as the cost of using a particular predictor, or as Branch (2004) emphasizes, as a predisposition effect. Regardless of the interpretation we are not going to *a priori* impose a hierarchy on the C_j, and instead will treat them as bifurcation parameters. For simplicity, we set the cost of adaptive expectations equal to zero, so that C will always represent the relative cost of rationality. Because both adaptive and perfect foresight return the same forecast in a steady state, the value of ωC pins down the steady-state value of n, and thereby determines the local stability properties of the model.

We also assume C, ω are identical across forecasting variable. This may seem inconsistent with the assumption that θ_y, θ_π may differ. The approach here is flexible enough that we could expand the parameter space and consider the effects of altering the variable-specific C, ω as bifurcation parameters. We leave such an examination to future research. We do not impose that predictor proportions are identical, and the dynamics of predictor selection will be different for each n_x along a real-time path.

The predictor fitness metric may seem somewhat *ad-hoc* given the micro foundations of the model. We justify the form of (6) by appealing to the learning literature which models expectation formation as a distinct statistical problem. Thus, agents choose a forecasting model based on past success and then use that model to solve for

[9] The weight μ could be treated as a bifurcation parameter: see, for example, Brock, Dindo, and Hommes (2006).

[10] Branch and Evans (2006a) show that, in a cobweb model, similar results obtain provided μ is somewhat close to one. A larger value of μ can be justified if agents are concerned about structural change or uncertain about the right predictor to adopt.

their optimal plan in the anticipated utility sense. This is not difficult to justify since we constructed the agents' problem so that they are forecasting aggregate variables over which they exert no control.

2.2 The dynamic system

With dynamic predictor selection, the economy's law of motion becomes,

$$\begin{aligned}
y_t &= n_{yt-1}y_{t+1} + (1-n_{yt-1})\theta_y^2 y_{t-1} - \sigma^{-1}\left(i_t - \left(n_{\pi t-1}\pi_{t+1} + (1-n_{\pi t-1})\theta_\pi^2 \pi_{t-1}\right)\right) \\
\pi_t &= \lambda y_t + \beta\left(n_{\pi t-1}\pi_{t+1} + (1-n_{\pi t-1})\theta_\pi^2 \pi_{t-1}\right) \qquad (7) \\
i_t &= \alpha_y\left(n_{yt-1}y_{t+1} + (1-n_{yt-1})\theta_y^2 y_{t-1}\right) + \alpha_\pi\left(n_{\pi t-1}\pi_{t+1} + (1-n_{\pi t-1})\theta_\pi^2 \pi_{t-1}\right).
\end{aligned}$$

The laws of motion for $n_{yt}, n_{\pi t}$ are specified below.

The timing assumptions require special discussion. We follow the adaptive learning literature in assuming that current values of the endogenous state variables are not directly observable. This is usually assumed to avoid a simultaneity in least-squares parameter estimates and the endogenous variables. In this setting, the assumption preserves logical consistency for adaptive agents. Rational agents (who have one-step-ahead perfect foresight) know current values of all variables, but the adaptive agents do not. Under this natural assumption, predictor selection takes place at time $t-1$. The approach taken here assumes that agents have a menu of predictor choices, they look at their most recent past forecasting performance as of the end of the period $t-1$, and choose the predictor with which they forecast one-step-ahead x_t, π_t. These choices then aggregate into n_{t-1} upon which the current state variables depend.

As noted in the previous section, fully rational agents would be aware of the future evolution of predictor proportions. Obtaining this information and incorporating it into decision-making is a complicated problem and motivates the literature's assumption that agents treat the forecasting, or predictor selection, issue as a statistical problem distinct from their optimization. In the current setting, given n_{t-1}, agents behave to satisfy their current Euler equation and current optimal pricing equation, and ignore the time-varying nature of the predictor proportions[11]. A similar assumption motivates the Euler-equation approach of Evans, Honkapohja and Mitra (2003), Bullard and Mitra (2002), and to which Preston (2005a) is an alternative. We stay consistent with the Euler-equation approach with the difference here being that the time evolution of beliefs is via a pair of fixed predictors while in the adaptive learn-

[11]Importantly, by behaving in a manner that satisfies the Euler equation, but not the *ex ante* transversality condition, the perfect foresight agents' beliefs do not make choices so that the economy is necessarily on the stable saddle path associated to the model with fixed predictor proportions.

ing models the expectation operators are time varying. These assumptions justify working with the conditionally linear IS and AS equations (1)-(2).

To study the model's dynamics, impose the policy rule into the IS relation in (7) and simplify to get

$$Hx_t = F(n_{t-1})x_{t+1} + G(n_{t-1})x_{t-1}, \tag{8}$$

where $x = (y, \pi)'$, $n = (n_y, n_\pi)'$, and

$$\begin{aligned} F(n_t) &= \begin{pmatrix} n_{yt}(1-\sigma^{-1}\alpha_y) & \sigma^{-1}(1-\alpha_\pi)n_{\pi t} \\ 0 & \beta n_{\pi t} \end{pmatrix} \\ G(n_t) &= \begin{pmatrix} (1-n_{yt})(1-\sigma^{-1}\alpha_y)\theta_y^2 & \sigma^{-1}(n_{\pi t}-1)(\alpha_\pi-1)\theta_\pi^2 \\ 0 & \beta(1-n_{\pi t})\theta_\pi^2 \end{pmatrix} \\ H &= \begin{pmatrix} 1 & 0 \\ -\lambda & 1 \end{pmatrix}. \end{aligned} \tag{9}$$

Now let $z = (x', x'_{-1})'$. Since F is generically invertible, we may write

$$M(n_t, \xi) = \begin{pmatrix} F(n_t)^{-1}H & -F(n_t)^{-1}G(n_t) \\ I_2 & 0_2 \end{pmatrix},$$

where ξ is the vector of model parameters. Finally, set

$$f(z_t, n_t, \xi) = \begin{pmatrix} \exp(-\omega C)\left\{\exp(-\omega C) + \exp\left[-\omega\left(\left(e_1'M(n_t,\xi) - \theta_y^2 e_3'\right)z_t\right)^2\right]\right\}^{-1} \\ \exp(-\omega C)\left\{\exp(-\omega C) + \exp\left[-\omega\left(\left(e_2'M(n_t,\xi) - \theta_\pi^2 e_4'\right)z_t\right)^2\right]\right\}^{-1} \end{pmatrix}$$

where e_i is the i^{th} coordinate vector. Then the full dynamic system is

$$\begin{aligned} z_t &= M(n_{t-1}, \xi)z_{t-1} && (10) \\ n_t &= f(z_{t-1}, n_{t-1}, \xi). && (11) \end{aligned}$$

Given initial conditions z_{-1} and n_{-1}, the system (10), (11) determines the evolution of our economy.

3 Results

We now present results for the dynamical system (10)-(11). Because the state vector has six dimensions, analytic results are largely intractable.[12] In what follows, we provide a thorough numerical analysis.

[12]The state vector consists of the (4×1) vector z and the (2×1) vector n.

3.1 Local stability analysis

The system (10), (11) always has a steady state given by

$$z = 0 \qquad \text{and} \qquad \bar{n}_y = \bar{n}_\pi = \frac{\exp(-\omega C)}{\exp(-\omega C) + 1}. \tag{12}$$

We call this the "zero steady state," and note that the values of inflation and output correspond to the unique rational expectations equilibrium in case all agents are rational and the associated RE-model is determinate.

There also may exist steady states in which inflation and output are non-zero. When there exists a non-zero steady state, $\bar{y}, \bar{\pi}$, then the steady-state values for n_y, n_π are

$$\begin{aligned} n_y &= \frac{\exp(-\omega C)}{\exp(-\omega C) + \exp(-\omega(1+\theta_y)^2 \bar{y}^2)} \\ n_\pi &= \frac{\exp(-\omega C)}{\exp(-\omega C) + \exp(-\omega(1+\theta_\pi)^2 \bar{\pi}^2)}. \end{aligned}$$

Notice that in a non-zero steady state, adaptive agents make persistent forecasting errors, but because of the cost C to perfect foresight, they may still prefer the adaptive predictor.

The Jacobian of the dynamic system evaluated at the zero steady state is given by

$$J = \begin{pmatrix} M(\bar{n}, \xi) & 0 \\ f_z(0, \bar{n}, \xi) & 0 \end{pmatrix},$$

where $\bar{n}$ is given by (12). Thus the zero steady state's stability properties are determined by the properties of $M(\bar{n}, \xi)$. Furthermore, the time t dynamics of z_t are governed by the eigenvalues of $M(n_{t-1}, \xi)$; so to gain intuition about both local stability and other dynamic properties of the model, we turn to the numerical analysis of the matrix $M(n)$.

The properties of $M(\bar{n})$, and hence the stability of the dynamical system, depend in a critical way on the steady-state fraction of perfect foresight agents. The tension between the forward-looking expectations of rational agents and the backward-looking adaptive agents is at the crux of the results to be presented below, and $M(\bar{n})$ captures the mix of this tension. We can anticipate the results below by first assuming $\theta_y = \theta_\pi = 1$ and restricting attention to the two boundary cases:

1. The purely rational steady state, corresponding to $C \to -\infty$ and $\bar{n} = 1$, with equilibrium dynamics given by $x_t = F(1)^{-1} H x_{t-1}$;

2. The purely adaptive steady state,[13] corresponding to $C \to \infty$ and $\bar{n} = 0$, with equilibrium dynamics given by $x_t = H^{-1}G(0)x_{t-1}$.

Noting that $G(0) = F(1)$, we have the following result:

Proposition 1 *Assume $\theta_y = \theta_\pi = 1$. If the purely adaptive steady state is locally stable then the purely rational steady state is unstable. Conversely, if the purely rational steady state is stable, then the purely adaptive steady state is unstable.*

The key insight provided by this result is the evidenced tension between the forward and backward looking dynamics inherent to the heterogeneous expectations model: in the special case of naive expectations, i.e. $\theta = 1$, if the rational dynamics are attracting then the adaptive dynamics are repelling, and vice-versa. To gain further intuition about both local stability and other dynamic properties of the model, we turn to the numerical analysis of the matrix $M(n)$.

To conduct our numerical analysis, the model must be calibrated. Table 1 details the parameter constellations for the IS and AS relations: see Woodford (1999), Clarida, Gali and Gertler (2000), Evans and McGough (2005a) and McCallum and Nelson (1999). As in Branch and McGough (2009), all broad qualitative results are robust to the calibration employed.

Table 1: Calibrations

Author(s)	σ^{-1}	λ
W	1/.157	.024
CGG	4	.075
MN	.164	.3
EM	1/.157	.3

Completing our numerical specification requires choosing values for θ_y and θ_π and our work in Branch and McGough (2009) indicates that the size of these values relative to one impacts the dependence of the model's dynamic properties on predictor proportions. To account for the possible impact the magnitude of θ_x might have on the dynamics of our model, we consider both $\theta_x > 1$ and $\theta_x < 1$. For simplicity, we assume $\theta_\pi = \theta_y$.

To understand the local stability properties of the model, we first turn to an examination of the stable manifold, for given steady-state values of $\bar{n}$.[14] Denote by

[13]As noted above, $F(n)$ is invertible for $n \in (0, 1]$. As $n \to 0$, $\det(F) \to 0$ so that M becomes undefined. The equilibrium dynamics, in this case, are obtained via (8).

[14]Recall that at the zero steady state, $\bar{n}$ is determined by the parameters ω and C.

$W_s(n)$ the stable manifold of $M(n)$, for predictor proportions given by n; thus, $W_s(n)$ is the direct sum of the eigenspaces of the linear operator $M(n)$ whose corresponding eigenvalues have modulus less than one. Figure 1 plots the dimension of the stable manifold for various values of the policy parameters α_y, α_π and the predictor proportion n. Figure 1 sets $\theta = 1.1$ and adopts the Woodford calibration. Each panel provides the dimension of $W_s(n)$, for given n, across a subset of the policy space (α_π, α_y). Because the dimension of $W_s(n)$ gives the number of eigenvalues of $M(n)$ with modulus less than one, the zero steady state is stable if and only $\dim W_s(n) = 4$. The NW panel sets $n = 1$, and therefore corresponds to the model under rational expectations – this provides an interesting connection between the rational model's determinacy properties and the heterogeneous model's stability properties, and we will explore this connection in Subsection 3.3 below. Notice also that for $n = 1$, policy rules satisfying the Taylor principle generate instability.[15] Now consider the effect on the dimension of $W_s(n)$ as n decreases from unity: the sloped line anchored at the point $(\alpha_\pi, \alpha_y) = (1, 0)$ rotates clockwise thereby increasing the region of stability; however, as n gets increasingly small, the region of stability entirely disappears.

FIGURE 1 HERE

The rotation behavior evident in Figure 1 suggests that the cost parameter C is a natural bifurcation parameter. From (12), we see that $\partial\bar{n}/\partial C < 0$ and as C varies between $-\infty$ and ∞, $\bar{n}$ varies between one and zero: in a steady state, rational and adaptive predictors return the same forecast; thus we would expect the steady-state fraction of rational to decrease in C. Given a policy rule that satisfies the Taylor principle, provided there is enough weight in policy rule on the output gap, we can choose C so that the zero steady state is stable. For example, in the NE panel, there is a wedge of the upper left quadrant with $\alpha_\pi > 1$ and $\dim W_s(n) = 4$. By further increasing C, we lower the zero steady-state value $\bar{n}$, causing the anchored line to rotate clockwise so that the zero steady state destabilizes. When this happens a bifurcation occurs.

The bottom two panels of Figure 1 indicate a change in the eigenvalue structure of M that is not captured by the rotational behavior of the sloped line anchored at the point $(\alpha_\pi, \alpha_y) = (1, 0)$. Consider the policy rule determined by $\alpha_\pi = 1.1$, $\alpha_y = .35$, as indicated by the circles in the bottom two panels. As n decreases, the dimension of the stable manifold changes from four to zero so that the associated steady state destabilizes. Close examination reveals that the bifurcation marking the destabilization of the steady state is characterized by the simultaneous passage of all four (complex) eigenvalues of M across the unit circle. The nature of this bifurcation can also be seen from Figure 4, which is drawn using the policy rule $\alpha_\pi = 1.1$, $\alpha_y = .35$. As noted above, increasing C corresponds to decreasing n. Now follow

[15] This instability under rationality is expected: see Subsection 3.3.

a horizontal line anchored at $\theta = 1.1$: as C increases past $\approx -.19$, the eigenvalues simultaneously cross the unit circle (see also footnote 17).

FIGURE 2 HERE

Figure 2 plots the $\dim W_s(n)$ under the same conditions except that $\theta = .9$. Here we find that the sloped line anchored at $(\alpha_\pi, \alpha_y) = (1, 0)$ rotates counterclockwise thereby decreasing the stability region. In this case, rules satisfying the Taylor principle will not yield stability, as there is not a part of the policy space where $\alpha_\pi > 1$ and $\dim W_s(n) = 4$. However, certain passive rules, i.e. $\alpha_\pi < 1$, lead to local stability of the zero steady state for large n, as is evident in the top two panels. As C increases and $\bar{n}$ falls, the region of stability will disappear, so that, again, C is a natural bifurcation parameter.

Figures 1 and 2 suggest that heterogeneity can increase the region of the parameter space that is stable when $\theta > 1$ or decrease the region for $\theta < 1$. That $\theta > 1$ can be stabilizing is intuitive. For moderately-sized values of $\bar{n}$, $\theta > 1$ works to offset the unstable forward dynamics from the rational agents. However, for $\bar{n}$ sufficiently small, e.g. $n = 0.5$, the extrapolative behavior of agents is explosive.

3.2 Simulations and bifurcations

The panels in Figures 1 and 2 suggest that, for appropriate policy parameters, small values of C will imply that the zero steady state is locally stable, while increases in C will lead to bifurcation. It may be possible to characterize the primary bifurcation by applying the center manifold reduction technique, though the daunting nature of this task compels us to proceed numerically.[16] For a given calibration, value of θ, and setting $\omega = 1$, we choose C so that the zero steady state is stable; then, we consider larger values of C and for each new value of C the model is simulated by choosing initial conditions at random near the zero steady state. The first 10,000 periods of transient dynamics are discarded, so that the remaining data will be near the invariant attractor. We then plot these simulations in a bifurcation diagram and in phase space.

3.2.1 Satisfying the Taylor principle

Figure 3 provides a bifurcation diagram for the Woodford calibration, $\theta = 1.1$, and we set $\alpha_\pi = 1.1$ and $\alpha_y = .35$, so that the Taylor principle is satisfied. As expected, low costs imply that the zero steady state is stable, but when C is approximately

[16]For more information on bifurcation theory, see Kuznetsov (1998), Guckenheimer and Holmes (1983), and Palis and Takens (1993).

$C = -.19$. a bifurcation occurs. For larger costs, the model may exhibit complicated behavior, as the bifurcation diagram seems to exhibit periodic and aperiodic dynamics.

FIGURE 3 HERE

Because we have not conducted the center manifold analysis, a precise statement of the primary bifurcation can not be assessed. However, we can learn quite a bit about the primary bifurcation by looking at the eigenvalues of the Jacobian evaluated at the steady state. Figure 4 plots the moduli of the eigenvalues in (C, θ) space. Each line in the plot represents a boundary where the eigenvalues switch from being real or complex and/or contracting or expanding. One can interpret the primary bifurcation witnessed in Figure 3 by moving horizontally from left to right when $\theta = 1.1$. For sufficiently small values of C, i.e. $\bar{n}$ is high but bounded away from one, the steady state is locally stable. For larger values of C the contracting eigenvalues are all complex. However, when $C \approx -.19$ both pairs of complex eigenvalues simultaneously crosse the unit circle.[17]

FIGURE 4 HERE

Figure 4 plots values for $\theta > 1$, as there is no stable steady state when the Taylor principle is satisfied and $\theta < 1$. Notice also in Figure 4 that when $C = 0$, so that $\bar{n} = 0.5$ the steady-state is always locally unstable.[18]

The primary bifurcation for the calibrated model occurs for values of $C \approx -.19$.[19] Complicated dynamics do not depend on negative values of C. The main role played by C is controlling the steady-state value $\bar{n}$, and thus determining the stability properties of the model as seen in Figure 1. Figure 5 demonstrates that positive costs also can lead to periodic and (possibly) aperiodic complex dynamics. Figure 5 provides a series of attractor plots for the same calibration as Figure 3. In Figure 5, $C = .45$ and $C = .4625$ yield a 14-cycle. For larger values of $C > 0$, more complex behavior arises.

FIGURE 5 HERE

[17]That all four eigenvalues cross the unit circle at the same time suggests a double Neimark-Sacker bifurcation; however, the codimension of a generic double Neimark-Sacker bifurcation is two, and the codimension of our bifurcation is one.

[18]In alternative calibrations, it is possible that the steady state corresponding to $C = 0$ will be locally stable.

[19]The figure also indicates that as C decreases past ≈ -2.5 the steady state destabilizes. In this case one of the real eigenvalues passes through $+1$. Numerical simulations suggest that the dynamics become explosive after the bifurcation. This suggests that a subcritical pitchfork bifurcation occurs, with 3 steady states (one stable and two unstable) before and 1 (unstable) steady state after the subcritical pitchfork bifurcation.

The intuition for the existence of complicated dynamics for larger values of C is easily expressed by re-examining Figure 1. If the steady state is unstable and the model is initialized near it, then the time-paths of output and inflation begin to diverge. This divergence reduces the performance of the adaptive predictor so that agents switch to the rational predictor, thus increasing the value of n_t. As n_t increases, the sloped line anchored at $(1, 0)$ rotates counter-clockwise so that the eigenvalues of $M(n_t)$ reduce in size until they are all smaller than one in modulus, that is, $\dim W_s(n_t) = 4$. This contracts the economy's time-path of output and inflation sending the system back toward the steady state. As the system nears the steady state, the adaptive predictor's performance improves and agents begin switching to it, thus reducing n, causing clockwise rotation of the anchored line, and a corresponding increase in the size of the eigenvalues of $M(n_t)$ until $\dim W_s(n_t) < 4$, at which point the economy begins to diverge from the zero steady state, and the process repeats indefinitely. Similarly complex behavior arises for different costs, ω values, calibrations, and for the alternative policy rules and different values of θ; however, $\theta > 1$ is required for a policy rule which follows the Taylor principle to generate complex dynamics.

This intuition also correctly predicts the existence of explosive dynamics for some parameter values and initial conditions. Consider the lower left region of the SW panel in Figure 1 corresponding to a passive Taylor rule with low weight placed on output variation. Assume that C and w are chosen so that $\bar{n} = .7$. Because $\dim W_s(\bar{n}) = 3$, the economy begins to diverge from the zero steady state. This divergence makes the rational predictor more attractive, and n_t rises. In this case, however, the rise in n_t only lowers the dimension of $W_s(n_t)$, so that the divergence of the economy continues.

3.2.2 Ignoring the Taylor principle

We now turn to the case where the Taylor principle is not satisfied so that $\alpha_\pi < 1$. When monetary policy is passive, the dynamics can be quite different than under an aggressive response to inflation. In this section, we find that there may be multiple stable attractors. These attractors may take the form of multiple stable steady states, or, a unique steady state from which the ensuing bifurcations may produce multiple stable attractors.

Consider setting $\theta = .9$, $\alpha_\pi = .75$ and $\alpha_y = .5$, and again adopt the Woodford calibration. Numerical analysis indicates that for C sufficiently small there is a unique stable steady state at zero. However, as C increases past ≈ -4.5, the zero steady state destabilizes as one of the real eigenvalues crosses $+1$. A bifurcation diagram (not shown) indicates the emergence of two new steady states which correspond to non-zero values of output and inflation, thus suggesting a pitchfork bifurcation.[20] As costs

[20] At these non-zero steady states, the $\bar{z}$ is an eigenvector of $M(\bar{n})$ corresponding to a unit eigen-

further increase, each of these steady states destabilize through a bifurcation process leading to multiple stable attractors. Figure 6, which is constructed analogously to Figure 4, indicates the behavior of the relevant eigenvalues. For small (negative) values of C, and $\theta < 1$, the steady-state is locally stable with 4 contracting real eigenvalues. However, at $C \approx -4.5$ one real eigenvalue equals $+1$, indicating a change in the structural stability. Notice, as well from Figure 6 that if $\theta > 1$, and the Taylor principle is ignored, then there is bifurcation analogous to the one found in the previous subsection. Hence, failure to abide by the Taylor principle, may lead to complex dynamics across the range of adaptive coefficients θ.

FIGURE 6 HERE

Qualitatively different behavior may arise under alternative calibrations, as is nicely illustrated in the bifurcation diagram under the Evans-McGough calibration, which corresponds to a model with strong elasticities in the IS and AS relations. Figure 7 sets $\theta = .5$, $\alpha_\pi = .3$ and $\alpha_y = .5$. For small values of C the zero steady state is stable, as expected. However, Figure 7 illustrates that as C increases the zero steady state destabilizes and a stable two cycle emerges. Interestingly, for C near $-.05$ the system again bifurcates, but this time two distinct stable 2-cycles emerge, suggesting a pitchfork bifurcation of the dynamic map's second iterate. Here the "+" and "$\circ$" indicate dynamics resulting from different initial conditions.

FIGURE 7 HERE

Figure 8 gives a more complete picture of the bifurcations under the alternative calibration and $\alpha_\pi < 1$. Figure 8 plots the bifurcation diagram for the upper stable attractor, which was de-marked by "$\circ$" in Figure 7. A similar picture (not shown) emerges for the lower attractor. Figure 8 illustrates that as costs rise further a period doubling cascade emerges and results in complex dynamics.

FIGURE 8 HERE

3.2.3 Discussion of Results

The results in the above subsections indicate that complex dynamics can arise under various settings for monetary policy and whether adaptive agents are extrapolative or dampening. It is useful to briefly review these results.

If adaptive agents are extrapolative, i.e. $\theta > 1$, then Figure 1 demonstrates that increasing the steady-state fraction of adaptive agents can enhance the stability of the

value.

steady state. With dynamic predictor selection, the economy switches between stable and unstable dynamics, and this can occur regardless of whether policy satisfies the Taylor Principle. With $\theta > 1$, altering the cost of the rational predictor can bifurcate the stable steady state and lead to stable attractors with some exhibiting complex dynamics.

On the other hand, if adaptive are dampening, i.e. $\theta < 1$, then Figure 2 shows that increasing the steady-state fraction of adaptive agents can lead to instability of the steady state. It follows that the nature of the dynamics depends on the monetary policy coefficients. If $\theta < 1$ and monetary policy satisfies the Taylor Principle, then there does not exist a stable steady state with heterogeneous expectations. The dampening of adaptive expectations in this case is not sufficient to offset the (unstable) forward dynamics arising from the perfect foresight agents. However, if monetary policy does not satisfy the Taylor Principle, then depending on the cost to the rational predictor, then can exist multiple steady states and multiple stable attractors. The results from this paper suggest that the interaction between the forward-looking behavior of rational agents and the backward-looking behavior of adaptive agents can destabilize the economy, even when monetary policy is set to satisfy the Taylor principle.

3.3 Determinacy, stability, and monetary policy

Consider again the model

$$Hx_t = Fx_{t+1} + Gx_{t-1}, \tag{13}$$

where $x = (y, \pi)'$ and F, G, and H are given by (9) and depend on n. Recall also that

$$M(n, \xi) = \begin{pmatrix} F^{-1}H & -F^{-1}G \\ I_2 & 0_2 \end{pmatrix}.$$

If n is taken to be fixed and set $n = 1$, then (13) corresponds to the usual New Keynesian model under rational expectations (in this case, $G = 0$). This model is said to be determinate if there is a unique non-explosive perfect foresight path and indeterminate otherwise (because $G = 0$ the RE-model can not be explosive). Whether the model is determinate can be assessed using the usual Blanchard-Kahn technique; and, in this case, determinacy depends on the eigenvalues of M: when $n = 1$, M necessarily has two zero eigenvalues; if the remaining two are outside the unit circle then the model is determinate.

To make the link between determinacy, stability and monetary policy, consider again the Woodford calibration and $\theta = 1.1$. Consider the NW panel of Figure 1 corresponding to $n = 1$; here, determinacy corresponds to $\dim(W_s) = 2$, indeterminacy to $\dim(W_s) > 2$, and explosiveness to $\dim(W_s) < 2$. We see that when the Taylor principle is satisfied and when there is low weight placed on the output coefficient in the policy rule, the associated RE model is determinate: there is a unique non-explosive perfect foresight solution. We also see from Figure 1 that for low values of C and thus high vaules of $\bar{n}$, the zero steady state of our model is unstable when the Taylor Principle is satisfied.[21] Indeed, this dynamic instability exactly reflects the determinacy of the RE model: they are both due to the large eigenvalues of $M(1)$. In a model with fully rational agents and n fixed at one, the agents must make their initial choices so that the economy lies on W_s, the stable manifold of $M(1)$: otherwise their ex-ante (and ex-post) transversality condition would be violated. In case $\dim(W_s) = 2$ there is precisely one way for agents to make their choices and the equilibrium path is unique.[22] In our model, n is not fixed, and as n evolves so do the eigenvalues, and thus the stable manifold, of $M(n)$. As we have seen, as n evolves, $\dim(W_s)$ may move between two and four, alternately repelling and attracting inflation and output. This evolution may cause the time path to remain bounded and converge to an invariant set, regardless of how the initial conditions are locally chosen.

The determinacy region reflects the standard policy advice imparted by the NK model: by choosing policy within this region, the model is guaranteed to have a unique equilibrium. This uniqueness is thought to be of benefit because it precludes the existence of non-fundamental "sunspot" equilibria, many of which exhibit welfare reducing volatility. Furthermore, as shown by Bullard and Mitra (2002), policy choices within this region yield equilibria that are stable under learning, so that policy makers can be confident that agents can learn to coordinate on the unique equilibrium.

Of interest to us is whether policy chosen in this advised region may also, in the presence of heterogeneity in expectations, yield complex, and possibly welfare reducing dynamics. In particular, we consider the following experiment: we choose a policy in the determinacy region, that is, a policy that would be recommended under the assumption of full rationality. We then investigate whether there exist values of C for which the corresponding equilibrium time paths exhibit complex dynamics. Consider again the Woodford calibration, with $\theta = 1.1$, $\alpha_\pi = 1.1$ and $\alpha_y = .32$. This policy specification yields determinacy when all agents are rational, and also yields stability under learning. We find that, in our model, complex dynamics emerge. An example attractor is given in Figure 9.

FIGURE 9 HERE

In the determinate RE-model there is a unique bounded path, which will arise if rational agents' initial beliefs place the economy on the stable manifold. However, as

[21]The eigenvalues of $M(n)$ are continuous in n which, itself, is decreasing in C.

[22]In case $n = 1$, M has two zero eigenvalues – the associated RE model is purely forward looking – and so $\dim(W_s) \geq 2$.

indicated by Figure 9, the determinacy of the rational model does not prevent complex bounded behavior if, in fact, time-varying heterogeneous expectations prevail. Thus, a key finding of this research is that with a natural evolution of n, a model that is determinate under rational expectations may yield complex dynamics.

3.4 Further discussion of policy implications

The monetary policy literature, though wide and diverse, typically settles on the same recommendation: set policy so that the REE is determinate. At the end of the day, this is the main message communicated to policy makers. At the heart of this recommendation is the property that a determinate steady state reduces the volatility of inflation and output. Our results indicate that a determinate REE may lead to inefficient outcomes if there exists heterogeneous expectations: these inefficient outcomes may either take the form of multiple equilibria – as found by Branch and McGough (2009) – or bounded complex dynamics.[23]

To illustrate this point in the starkest terms, we parameterized the model so that it is determinate under rational expectations, and we granted agents the choice of whether to be rational or adaptive. This is exactly the scenario discussed extensively in the monetary policy literature and forms much of the basis for policy advice in "ideal" conditions. Our results indicate that choosing a policy rule in this fashion may result in bounded, possibly chaotic equilibria with inefficiently high inflation and output volatility. Most strikingly, our arguments do not require coordination of expectations on sunspot equilibria. We instead allow for an empirically realistic degree of expectational heterogeneity. The dynamically evolving heterogeneity has been documented in survey data by Mankiw, Reis, and Wolfers (2003) and Carroll (2003). We illustrate that if policy attempts to achieve a determinate REE in a New Keynesian model and these heterogeneous expectations dynamics are present, the policy maker may unwittingly destabilize the economy. This suggests that policy should be designed to account for potentially destabilizing heterogeneity in a way that simple linear interest rate rules can not accomplish.

One may wonder how sensitive our results are to our specification of adaptive expectations, the predictor choice dynamic, and the model parameterization. We adopted an adaptive predictor with the same form as the MSV REE because it is the least *ad hoc* specification of adaptive expectations. We could instead specify adaptive beliefs in the Cagan sense as a geometric average of past observations. We believe that our qualitative results are robust to this specification because the key for generating our findings is that adaptive and rational predictors return distinct forecasts out of

[23] It has also been emphasized that policy rules should be chosen so that the associated unique equilibrium is stable under learning: see for example, Bullard and Mitra (2002), Honkapohja and Mitra (2005) and Evans and McGough (2005a). We note that the rule used to generate the plot in Figure 9 does produce an equilibrium that is stable under learning.

steady state; this property alters the stability properties of the steady state.

Similarly, the results are not sensitive to the predictor choice mechanism. In Branch and McGough (2005), we illustrated that a replicator dynamic will yield similar dynamic behavior, in a cobweb model, as the MNL of Brock and Hommes (1997). Finally, we have endeavored to verify the existence of complicated dynamics across a broad spectrum of calibrations. The key is that for some for some $\bar{n}$ we have $\dim(W_s(\bar{n})) = 4$. Branch and McGough (2009) document an extensive region of the parameter space with this property. An open empirical question is whether the values of ω and C are of reasonable magnitudes. Since ω parameterizes the MSE, and C is measured in MSE units, there is no natural interpretation of these values in terms of utility or consumption units. An interesting extension would be to embed the predictor choice into the agent's recursive optimization problem.

4 Conclusion

This paper examines the impact of endogenous expectations heterogeneity on a model's dynamic properties. Our central finding is that an otherwise linear model may exhibit bounded complex dynamics if agents are allowed to select between competing costly predictors (e.g. rational versus adaptive). These dynamics arise through the dual attracting and repelling nature of the steady-state values of output and inflation – the nature of which depends on the proportions of rational and adaptive agents. If the steady state is attracting for higher proportions of rational agents and repelling for lower proportions, then the natural tension between predictor cost and forecast accuracy mirrors the implied tension of attracting and repelling dynamics. When the economy is far from the steady state, the accuracy benefits of the rational predictor outweighs its costs, and the proportion of rational agents rises, causing the steady state to become attracting and thereby drawing the economy toward it. As the economy approaches the steady state, the relative effectiveness of the rational predictor falls, so that agents begin switching to the cheaper adaptive predictor. This switching causes the steady state to repel the economy and the process repeats itself.

The complex dynamics produced by our model are not outcomes limited to unusual calibrations or *a priori* poor policy choices: complex behavior appears to be an almost ubiquitous feature of a time-varying heterogeneous expectations New Keynesian model. Even policy designed to induce determinacy and stability under learning when levied against a rational version of the model may be insufficient to guard against the mentioned bad outcomes. We find that specifications of policy rules satisfying the Taylor principle, and which yield determinacy under rationality, may result in bounded complex dynamics, and this possibility obtains even if all agents are initially rational.

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Figure 1: Stability of $M(n)$, for various fixed values of n, $\theta = 1.1$, and alternative settings for the monetary policy parameters (α_π, α_y) under the Woodford calibration.

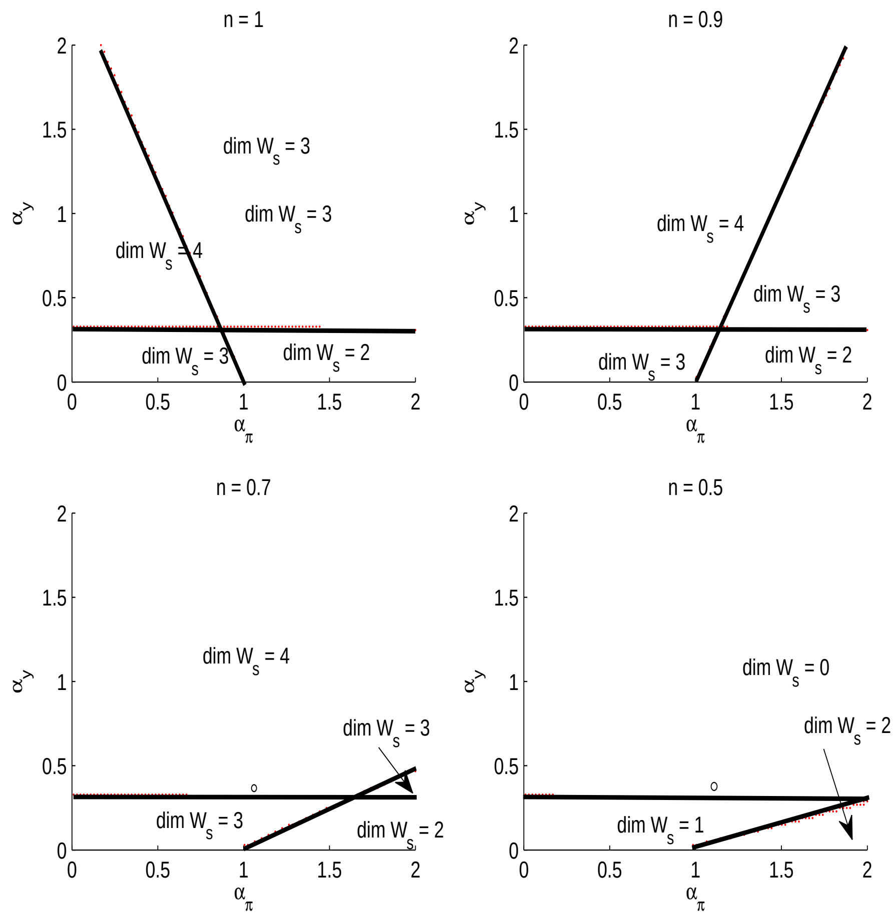

Figure 2: Stability of $M(n)$, for various fixed values of n, $\theta = .9$, and alternative settings for the monetary policy parameters (α_π, α_y) under the Woodford calibration.

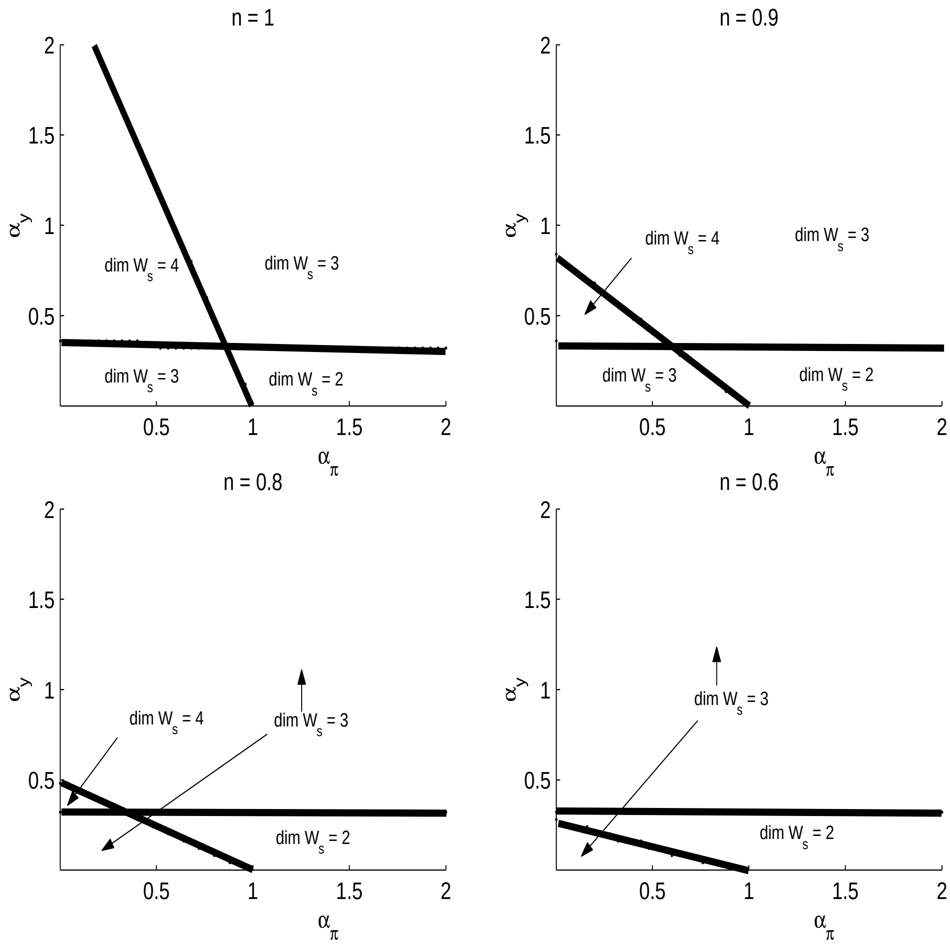

Figure 3: Bifurcation diagram: satisfying the Taylor Principle. Woodford calibration, $\theta = 1.1, \alpha_\pi = 1.1, \alpha_y = .35$. For $C \approx -0.19$ a bifurcation occurs.

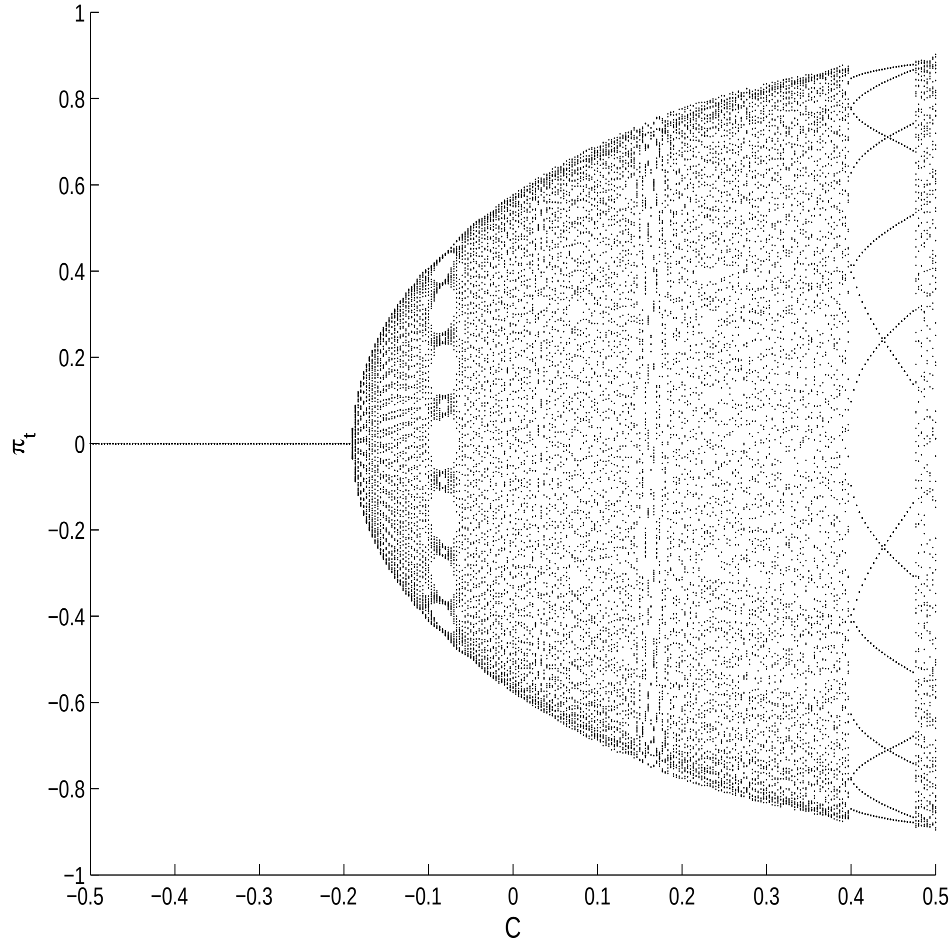

Figure 4: Local stability of the steady state: satisfying the Taylor Principle. Given (C, θ), figure calculates eigenvalues of the Jacobian in the Woodford calibration, $\alpha_\pi = 1.1, \alpha_y = .35$.

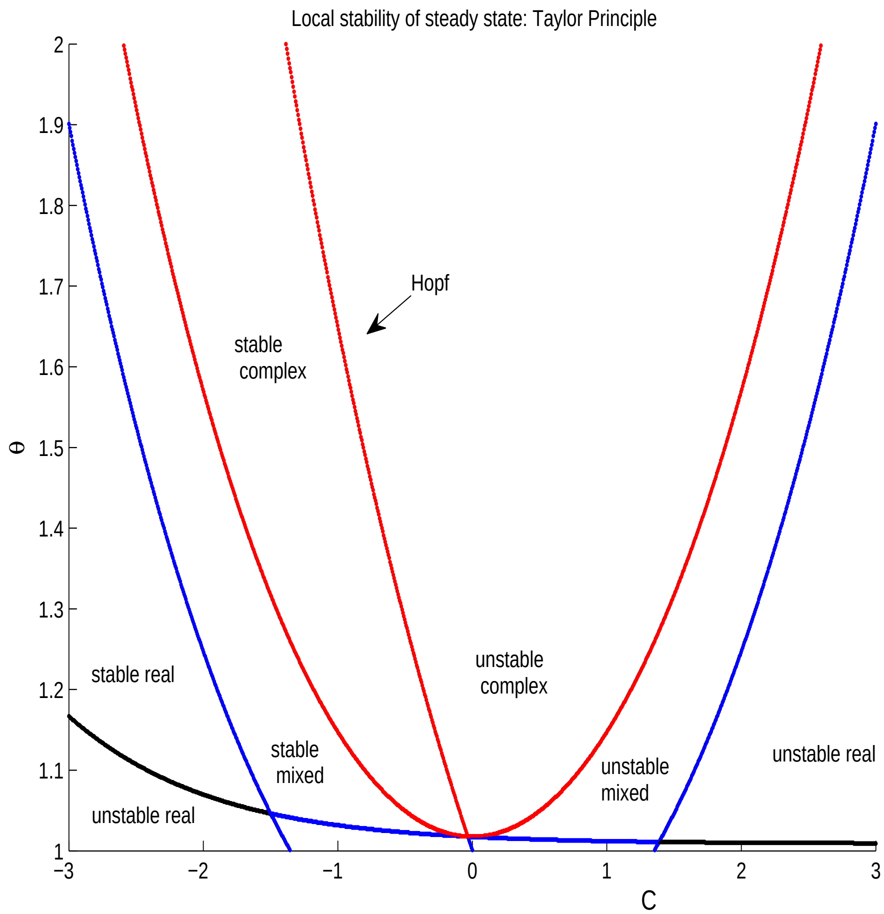

Figure 5: Attractors illustrating periodic and complex dynamics: satisfying the Taylor Principle. Each figure was generated by simulating the model following a 10,000 length transient period. Woodford calibration, $\theta = 1.1, \alpha_\pi = 1.1, \alpha_y = .35$.

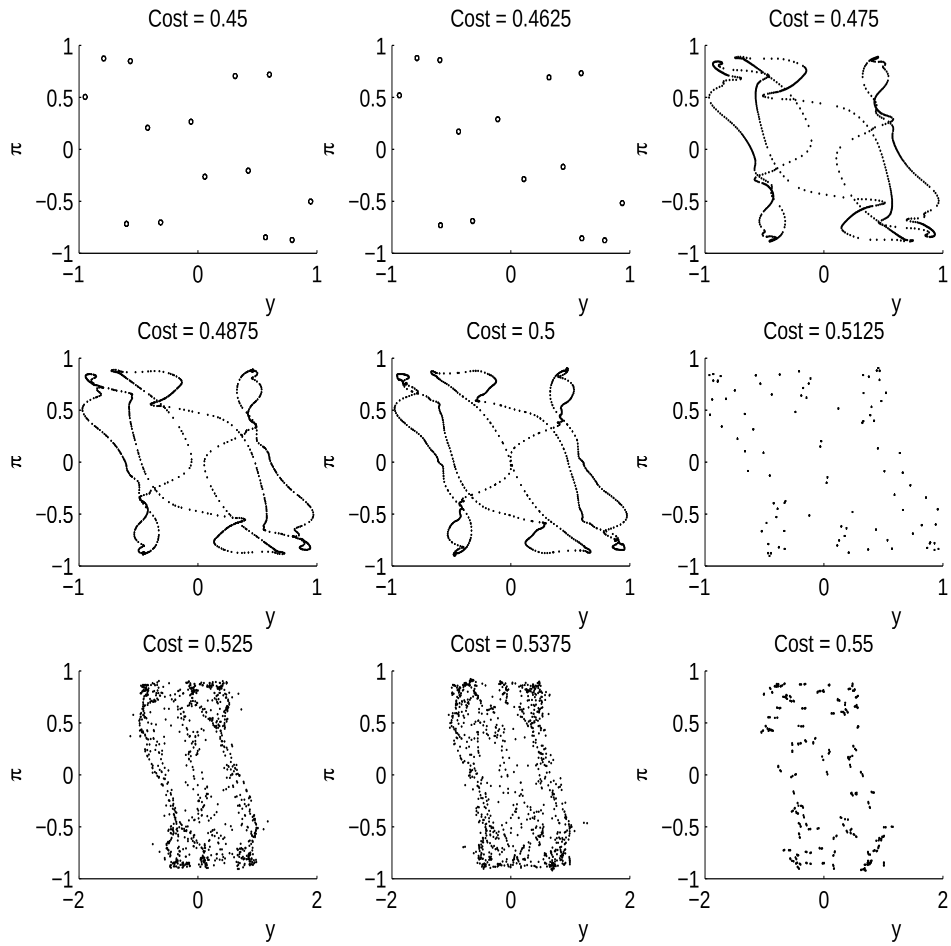

Figure 6: Local stability of the steady state: ignoring the Taylor Principle. Given (C, θ), figure calculates eigenvalues of the Jacobian in the Woodford calibration, $\alpha_\pi = .75, \alpha_y = 0.5$.

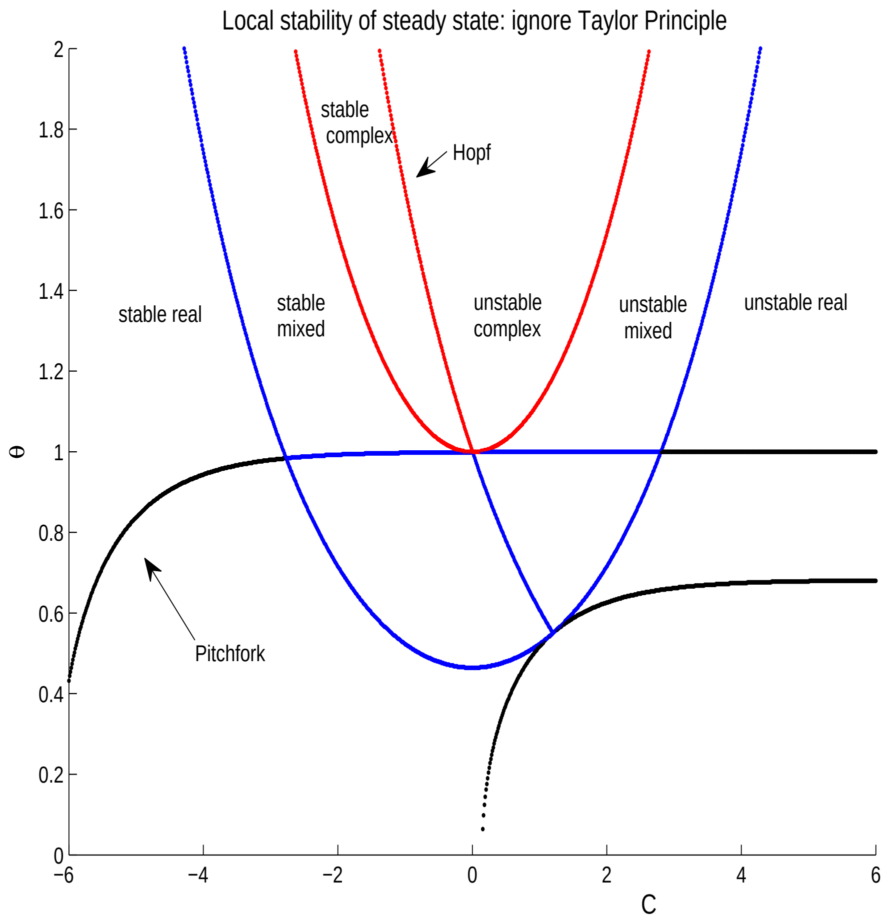

Figure 7: Stable 2-cycle bifurcates into two co-existing stable 2-cycles: ignoring the Taylor Principle. For $-.2 < C < -.005$ there is a unique stable two-cycle, which for $C > -.005$ bifurcates into two co-existing two cycles. Evans-McGough calibration, $\theta = .5, \alpha_\pi = .3, \alpha_y = .5$.

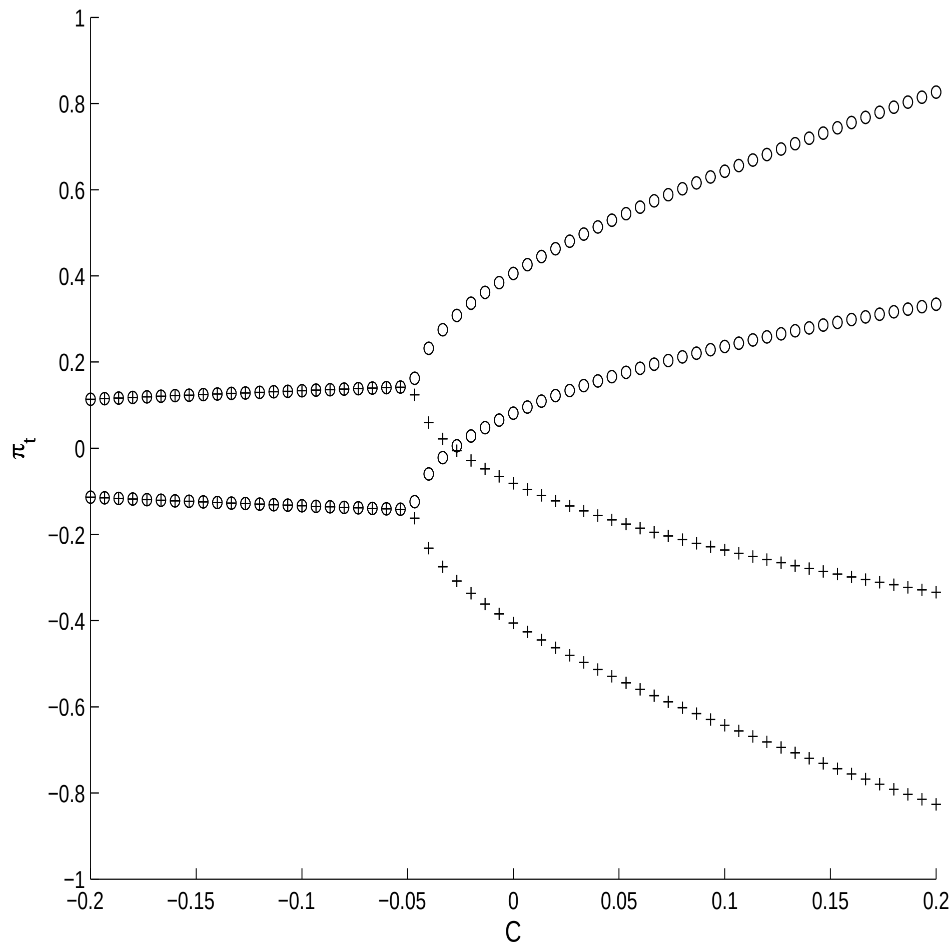

Figure 8: Bifurcation diagram for upper stable attractor: ignoring the Taylor Principle. Along the upper stable attractor, a cascade of period-doubling bifurcations. Evans-McGough calibration, $\theta = .5, \alpha_\pi = .3, \alpha_y = .5$.

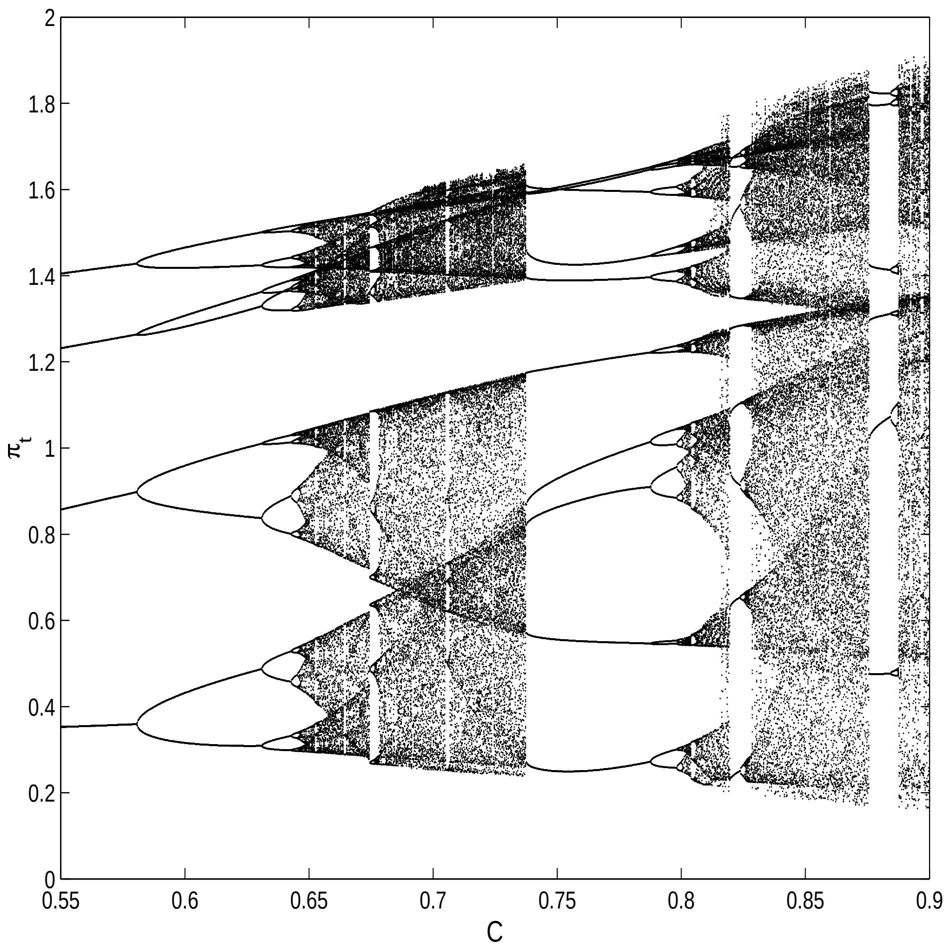

Figure 9: Stable attractor: determinate under rationality. Woodford calibration, $\theta = 1.1, \alpha_\pi = 1.1, \alpha_y = 0.32$.