Business Cycle Amplification with Heterogeneous Expectations *

William A. Branch
University of California, Irvine

Bruce McGough
Oregon State University

November 12, 2009

Abstract

This paper studies the implications for business cycle dynamics of heterogeneous expectations in a stochastic growth model. The assumption of homogeneous, rational expectations is replaced with a heterogeneous expectations model where a fraction of agents hold rational expectations and the remaining fraction adopt parsimonious forecasting models that are, in equilibrium, optimal within a restricted class. Our approach nests the literature on rational expectations in business cycle models with a recent approach based on adaptive learning. We demonstrate that (i.) heterogeneous expectations can lead to substantial improvement in the internal propagation of equilibrium business cycle models, (ii.) the internal propagation depends on the degree of heterogeneity. A calibrated model with heterogeneity provides a closer fit to business cycle data than its representative agent, rational expectations counterpart.

JEL Classifications: E52; E32; D83; D84

Keywords: Heterogeneous expectations, Real Business Cycles, Restricted Perceptions, Propagation

1 Introduction

This paper studies business cycle dynamics in a framework that is similar to the stochastic neoclassical growth model, but which also incorporates heterogeneous ex-

*We are grateful to George Evans, Mordecai Kurz, and participants at the SITE Workshop on Diverse Beliefs for comments.

pectations. We assume a competitive economy populated by two different types of households, each selecting plans for consumption/savings and labor supply. The first group of agents are fully rational: they maximize lifetime utility taking as given prices and the behavior of the other group of households. These agents have rational expectations. The other group of agents hold *restricted perceptions* and employ a set of misspecified statistical models through which they form their expectations. With a statistical model in hand, the group of agents who are boundedly rational must decide on a household plan by satisfying a set of optimality conditions. A *restricted perceptions equilibrium* is a stochastic process whereby rational agents behave fully rationally and agents with restricted perceptions satisfy a least squares orthogonality condition that preserves many of the cross-equation restrictions that are a salient feature of rational expectations models, see (Sargent, 2008). The results of this paper demonstrate that a heterogeneous expectations business cycle model is capable of increasing the internal propagation of real business cycle models.

Equilibrium business cycle models feature intertemporal decision making by households as the primary means of generating empirically plausible output dynamics. It is well known however that these models have weak internal propogation mechanisms. In response to technology shocks, households optimally adjust their holdings of capital and labor, but the aggregate time series properties closely resemble the process for technology; that is, there is little additional impulse implied by the intertemporal decisions of households. For example, (Cogley and Nason, 1995) illustrate that in response to transitory technology shocks, the time series data for output exhibit a hump-shaped response, with output continuing to increase following the shock. Standard RBC models do not exhibit similar impulse responses.

The literature has proposed several alternative paths for enhancing the internal propogation in equilibrium models.[1] The approach in this paper begins with the observation that the implicit sophistication required by agents to form rational expectations may be too great for many economic agents; rational expectations requires that forecasters have the ability to compute expectations conditional on the true distribution of the endogenous variables. As an alternative, we assume that a fraction of the population has limited sophistication, or restricted perceptions, in forecasting. We do not abandon the hallmark cross equation restrictions of rational expectations models, however, and impose that all agents' forecasts are optimal within their (possibly) restricted class.

In real business cycle models, current period decisions for consumption/savings and labor supply depend on expectations of the future. The standard approach to modeling household behavior is to assume preferences, constraints, technologies

[1] See Section 6 for a discussion of related research.

are the primitives of the model and to assume expectations consistent with the dynamic programming problem; this is the rational expectations approach. In models of bounded rationality, the beliefs of the agents are primitive; and the behavioral rules are imposed to be consistent with those beliefs. For example, in (Evans and Honkapohja, 2006) agents are able to form one step-ahead forecasts and, to remain consistent, they also make consumption/savings decisions that satisfy their one period Euler equation. In (Preston, 2006), agents are able to form infinite step-ahead forecasts so that a consistent current consumption/savings decision satisfies the entire sequence of Euler equations along with the expected lifetime budget constraint.

A cognitive dissonance arises when agents have limited sophistication in forecasting but are able to solve for their infinite horizon optimal plan given those beliefs. Motivated by this observation, we take the forecasting model – model specification, parameters, and forecasting horizon – as a primitive and require that agents' optimizing behavior be consistent with those beliefs. Thus, if an agent has a forecasting horizon of N periods, then their current period optimizing decisions will depend on a planning horizon of the same length. We call this approach *bounded optimality*.

With a fraction of agents fully rational, and the remaining agents boundedly rational/optimal, we show that heterogeneity can enhance the internal propagation of equilibrium business cycle models. The key insight is that boundedly rational agents react more strongly to innovations in real wages and interest rates that result from technology shocks. Because they have limited sophistication, their misspecified statistical models are not able to completely forecast how temporary shocks impact future endogenous variables. As a result, boundedly rational/optimal agents will fail to smooth consumption as much as a fully rational agent at the time the shock is realized. The internal propagation of the business cycle model is altered precisely because of the distinct individual behavior of boundedly rational agents. We demonstrate that a calibrated version of the business cycle model with heterogeneous expectations is able to fit U.S. quarterly data significantly better than the model under rational expectations, i.e. the real business cycle model.

This paper proceeds as follows. Section 2 presents a stochastic growth model with heterogeneous expectations. Section 3 presents the main insights on business cycle propagation in a simple neoclassical model. Section 4 presents quantitative results in a calibrated real business cycle model.

2 The Model

We modify the benchmark Real Business Cycle (RBC) model to incorporate heterogenous expectations. To facilitate exposition, we begin with a brief review of the RBC model: for a detailed discussion see (King and Rebelo, 1999).

2.1 The RBC model

There is a continuum of identical households and a continuum of identical firms. Firms rent capital, hire labor, and produce consumption goods; households consume goods, and supply labor and capital. There are three competitive markets – consumption goods, capital, and labor – and prices are written with consumption as the numeraire.

The representative household's problem is given by

$$\max_{\{C_t, S_t, L_t, H_t\}} \quad E_0 \sum_{t \geq 0} \beta^t u(C_t, L_t) \tag{1}$$

$$\text{s.t.} \quad C_t + S_t = (1 + R_t) S_{t-1} + W_t H_t \tag{2}$$

where C_t is consumption, S_t is savings, R_t is the real net return on savings, W_t is the real wage, H_t is the quantity of labor supplied and L_t is leisure: Agents have a unit endowment of time each period, thus $L_t + H_t = 1$. The associated first order conditions are given by

$$u_c(C_t, L_t) = \beta E_t u_c(C_{t+1}, L_{t+1})(1 + R_{t+1}) \tag{3}$$

$$u_c(C_t, L_t) W_t = u_l(C_t, L_t). \tag{4}$$

Equation (3) is often called the Euler equation, and captures the inter-temporal consumption/savings decision faced by the household; equation (4) captures the contemporaneous, or intra-temporal, consumption/leisure trade-off.

Firms have access to a constant returns to scale technology, $Y_t = Z_t f(K_t, H_t)$, where K is aggregate capital stock. The variable Z_t is a serially correlated productivity shock satisfying $Z_t = V_t Z_{t-1}^{\rho}$, where $|\rho| < 1$ and V_t is i.i.d. with small, compact, positive support. In maximizing profits, firms hire labor and capital from competitive factor markets and face no inter-temporal trade-offs, implying that relative prices satisfy

$$W_t = Z_t f_h(K_t, H_t) \tag{5}$$

$$R_t = Z_t f_k(K_t, H_t) - \delta, \tag{6}$$

where δ is the rate of depreciation.

The model is closed by the market clearing condition $S_t = K_{t+1}$, which, together with the pricing relations (5) and (6), yields the capital accumulation equation

$$K_{t+1} = Z_t f(K_t, H_t) + (1-\delta)K_t - C_t. \tag{7}$$

Definition. *An equilibrium of the real business cycle model is a collection of processes* $\{C_t, K_t, H_t, L_t, R_t, W_t, S_t\}$ *satisfying (2)–(7), the representative household's transversality condition,* $S_t = K_{t+1}$ *and* $L_t + H_t = 1$.

For the remainder of the paper, we adopt the functional forms in (King and Rebelo, 1999):

$$f(K,H) = K^{\alpha}H^{1-\alpha} \text{ and } u(C,L) = \log C + \Theta\frac{L^{1+\eta}}{1+\eta}. \tag{8}$$

With this specification (and more generally) it can be shown that the RBC model has a unique equilibrium. Analysis of this equilibrium is often accomplished by log-linearizing the system (2)–(7) about the unique deterministic steady state. The resulting reduced form system of linear expectational difference equations may then be solved using, for example, the techniques of (Blanchard and Kahn, 1980); the solution to the linearized system is a covariance-stationary vector autoregression that may be used to compute impulse response functions and aggregate co-movements.

2.2 The Heterogeneous Business Cycle Model

We modify the benchmark RBC model by assuming heterogeneous households. Households differ in the way they form expectations and in the way they behave given their forecasts: we take the forecasting model as the primitive, and assume optimizing behavior consistent with the boundedly rational forecasting model. We first describe the behavior of the boundedly rational agents before defining the restricted perceptions equilibrium for the heterogeneous expectations business cycle (HBC) model.

2.3 Bounded optimality and Bounded Rationality

In most dynamic stochastic general equilibrium models, rational agents are assumed to make decisions by forming contingency plans that solve their stochastic dynamic programming problem. Forming contingency plans requires significant sophistication on the part of agents, both as forecasters and as decision makers. To model the behavior of agents with limited sophistication, we adapt the approach commonly used

in the learning literature: agents form expectations at finite horizons using forecasting models; then agents make current decisions based on these expectations.

There have two main approaches to modeling household behavior when agents are boundedly rational and learning. The first, "Euler equation learning," assumes households forecast one period ahead and make current decisions to satisfy their one period Euler equation. As an alternative, (Preston, 2006) argues that since a rational household solves an infinite horizon optimization problem, boundedly rational agents should have infinite horizon forecasts used to solve that problem. We generalize Euler equation learning by taking the beliefs, in the form of a forecasting model and a forecasting horizon, as the primitive and look for boundedly optimal decision rules that are *consistent* with these beliefs. We call this approach *bounded optimality.*

We begin by describing agents' behavioral decision rules which are consistent with a *given* forecasting horizon; then, in the next subsection, we describe the specifics of the forecasting models.

2.3.1 Bounded Optimality

Given its forecasting model and planning horizon, we base the households' behavioral primitives on the usual household problem, as given by (1). Instead of assuming our agents solve the dynamic programming problem, we identify agent behavior by taking the log-linearized versions of some of the associated optimality restrictions as primitive. First, though, some notational issues. Denote by lowercase letters without subscripts the steady-state values of variables, and define lowercase letters with subscripts as being written in proportional deviation from steady state. Finally, a superscript will indicate a type-specific variable, whereas no super-script indicates an aggregate quantity. For example, k is the steady-state value of aggregate capital, k_t is the proportional deviation of time t capital from the steady-state value, and c_t^τ is the proportional deviation of the time t consumption of an agent of type τ from its steady-state value.[2]

[2]Boundedly optimal behavior and boundedly rational forecasts will be defined in such a way as to guarantee that all type-specific variables have the same steady-state values, that is, the steady-state value of consumption etc. for agents of type τ_1 and τ_2 will be the same.

The log-linearized versions of equations (2), (3), and (4) are given by

$$s_t^\tau + \frac{c}{k}c_t^\tau - w\frac{h}{k}h_t^\tau = \beta^{-1}s_{t-1}^\tau + w\frac{h}{k}w_t + rr_t \tag{9}$$

$$c_t^\tau = E_t^\tau c_{t+N^\tau}^\tau - \beta r\sum_{i=1}^{N^\tau} E_t^\tau r_{t+i}, \tag{10}$$

$$c_t^\tau + \eta\frac{h}{l}h_t^\tau = w_t. \tag{11}$$

Here E_t^τ is agent τ's expectations operator, which we specify in the next section. Equation (10) is obtained by iterating the log-linearized Euler equation forward N^τ steps. Under rational expectations, or infinite horizon learning, (10) will be satisfied for all N^τ. Instead, we identify a household type τ as one having a forecasting/planning horizon of N^τ periods. Then our behavioral model assumes that a boundedly optimal plan satisfies their flow budget constraint (9), their N^τ period Euler equation (10) , and their intra-temporal condition (11). Intuitively, the condition (10) requires that a boundedly optimal agent with a N^τ period horizon select a consumption level in time t so as to equate the marginal utility at t with the expected discounted marginal utility at $t + N^\tau$.

Solving these equations for agent τ's contemporaneous choices, we get

$$c_t^\tau = E_t^\tau c_{t+N^\tau}^\tau - \beta r\sum_{i=1}^{N^\tau} E_t^\tau r_{t+i} \tag{12}$$

$$h_t^\tau = \frac{l}{\eta h}w_t - \frac{l}{\eta h}E_t^\tau c_{t+N^\tau}^\tau + \frac{\beta lr}{\eta h}\sum_{i=1}^{N^\tau} E_t^\tau r_{t+i} \tag{13}$$

$$s_t^\tau = \beta^{-1}s_{t-1}^\tau + rr_t + \frac{\eta b + l}{\eta k}w_t - \chi E_t^\tau c_{t+N^\tau}^\tau + \beta r\chi\sum_{i=1}^{N^\tau} E_t^\tau r_{t+i}, \tag{14}$$

where $\chi = \frac{wl+c\eta}{\eta k}$. Equations (12) – (14) identify how agent τ makes decisions. Importantly, these equations show how agent τ's time t decisions are determined by his savings, s_{t-1}^τ, time t prices, and his time t forecasts of the future.

The form of equation (12) warrants further comment. By incorporating the N-step Euler equation into agent τ's behavioral primitives, household decisions are based not only on forecasts of tomorrow, but also of events further in the future. We view the N-step Euler equation as a natural behavioral primitive: agent τ forecasts future consumption and trade-offs, and chooses consumption today so that marginal benefit equals expected marginal cost. While this behavior is natural, it does not characterize fully optimal behavior, even given agent τ's subjective beliefs (as captured by the

expectations operator E^τ): the agent's planning horizon is finite and does not account for the transversality condition ex-ante. We say that an agent making decisions based on (12) – (14) is boundedly optimal, with an N^τ-period planning horizon.[3]

2.3.2 Bounded Rationality

The beliefs of a given agent are fully specified by the functional form of the forecasting model together with a vector of perceived parameters and a forecasting horizon. Rational expectations require an unrealistic degree of sophistication by forecasters. They must be able to compute forecasts at all future horizons that coincide with conditional expectations taken with respect to the true distribution of the endogenous state variables that are also functions of these forecasts. Instead, we follow (Branch and Evans, 2006) and assume that those agents who do not have the sophistication to form rational expectations, adopt parsimonious forecast models. In particular, we assume their forecasting models are univariate, and thereby underparameterized.

We assume that boundedly rational agents of type τ form their forecasts from the following perceived laws of motion,

$$c_t^\tau = \psi_c^\tau s_{t-1}^\tau + \varepsilon_{c,t} \tag{15}$$

$$s_t^\tau = \psi_s^\tau s_{t-1}^\tau + \varepsilon_{s,t} \tag{16}$$

$$r_t = \psi_r^\tau r_{t-1} + \varepsilon_{r,t}, \tag{17}$$

where $\varepsilon_{j,t}$ is a perceived white noise error. Expectations are formed by iterating the perceived laws of motion forward N^τ periods. These forecasting models capture several desirable features of reasonable forecasting behavior by agents with limited sophistication. First, as univariate models they are parsimonious. Second, they impose the plausible assumption that aggregate prices are observable but capital holdings of other types of agents are not observable. In a representative agent model, a rational agent would forecast future consumption as depending on both future savings levels and future prices. These models are in the spirit of such a forecasting model but subject to the parsimony restriction. Third, these perceived laws of motion

[3]To study equilibrium stability learning in a New Keynesian model, (Evans and Honkapohja, 2006) employed similar behavioral assumptions, but used one period planning horizons and correctly specified forecasting models. They called their implementation "Euler equation learning." (Evans and McGough, 2009) instead assume that agents make decisions based on forecasts of shadow prices. They show that under shadow price learning, agents with correctly specified forecasting models may learning to make optimal decisions. Finally, (Preston, 2006) takes as behavioral primitives both the Euler equations at all iterations, and the lifetime budget constraint: he calls this implementation "infinite horizon learning," and he finds that it may result in stability conditions which differ from those predicted by Euler equation learning.

resemble linear regression models and so the implicit assumption is that even though these agents have limited sophistication, they still forecast like a good econometrician. The next subsection details how the regression parameters, ψ_c, ψ_s, ψ_r are pinned down within a restricted perceptions equilibrium.

Parsimony in forecasting is often favored because of model uncertainty, computational constraints, degree of freedom limitations, etc. In this simple model, it may seem forced that agents would favor parsimony, after all the correct forecasting model only involves one additional variable which agents are already forecasting anyway. However, this assumption is a reasonable approximation to real economies where adopting a correctly specified model requires too much sophistication, computing power, degrees of freedom, etc., and so some agents make decisions based on forecasting models that are under-specified. The purpose of this paper is to show that a model with heterogeneity in expectations, and as a result heterogeneity in decision making, has important implications for business cycle dynamics.

The timing of when certain variables are observable to the agent requires brief comment. It is natural to assume that time t prices are observed by agents when forecasts are made, therefore we assume that $E_t^\tau r_t = r_t$. It is less obvious whether time t values of choice variables are observed by agents when forecasts are made. The learning literature, e.g. (Evans and Honkapohja, 2001), views both options as reasonable. Below, when we turn to numerical analysis, we will consider both timing conventions. Under the lagged-timing convention, $E_t^\tau s_t^\tau = \psi_s^\tau s_{t-1}^\tau$, and under the contemporaneous-timing convention, $E_t^\tau s_t^\tau = s_t^\tau$. Thus, under lagged timing, $E_t^\tau c_{t+N^\tau}^\tau = \psi_c^\tau (\psi_s^\tau)^{N^\tau} s_{t-1}^\tau$, and under contemporaneous timing, $E_t^\tau c_{t+N^\tau}^\tau = \psi_c^\tau (\psi_s^\tau)^{N^\tau - 1} s_t^\tau$. In either case

$$\sum_{i=1}^{N^\tau} E_t^\tau r_{t+i} = \frac{\psi_r^\tau}{1-\psi_r^\tau}\left(1-(\psi_r^\tau)^{N^\tau}\right) r_t. \tag{18}$$

Note that agent τ's forecasts – and hence beliefs – are completely characterized by the perceived parameters $\psi^\tau = (\psi_c^\tau, \psi_s^\tau, \psi_r^\tau)$. These forecasts may be combined with the behavioral equations (12) – (14) to write the contemporaneous decisions of agent τ as functions of prices and his lagged savings.

2.4 Equilibrium

We now describe equilibria in the heterogeneous expectations business cycle (HBC) model. There are $M+1$ agent types: there is a proportion θ of rational agents, and a proportion $(1-\theta)\phi^\tau$ of boundedly optimal, boundedly rational agents of type τ, where $\sum_{\tau=1}^{M} \phi_\tau = 1$. For notational simplicity, time-subscripted variables associated to rational agents have no superscript.

Definition. *Given beliefs and forecasting horizons* $\{\psi^\tau, N^\tau\}_{\tau=1}^M$*, proportions* θ *and* $\{\phi^\tau\}_{\tau=1}^M$*, an equilibrium of the heterogeneous expectations business cycle model is a collection*

$$\{\{s_t^\tau, c_t^\tau, n_t^\tau\}_{\tau=1}^M, s_t, c_t, k_t, r_t, w_t\}$$

satisfying

$$\begin{aligned}
c_t &= E_t c_{t+1} - \beta r E_t r_{t+1} \\
s_t &= \beta^{-1} s_{t-1} + w\frac{n}{k} w_t + r r_t - \frac{c}{k} c_t + w\frac{n}{k} n_t \\
\begin{pmatrix} c_t^\tau \\ n_t^\tau \\ s_t^\tau \end{pmatrix} &= A^\tau(N^\tau, \psi^\tau) \begin{pmatrix} s_{t-1}^\tau \\ w_t \\ r_t \end{pmatrix}, \qquad \text{for } \tau = 1, \ldots M \\
k_t &= \theta s_{t-1} + (1-\theta) \sum_{\tau=1}^M \phi^\tau s_{t-1}^\tau \\
r_t &= (1-\alpha)\left(\frac{r+\delta}{r}\right)(n_t - k_t) + \frac{r+\delta}{r} z_t \\
w_t &= \alpha(k_t - n_t) + z_t \\
z_t &= \rho z_{t-1} + v_t,
\end{aligned} \tag{19}$$

where (19) is obtained by imposing agents' boundedly rational forecasts into (12) – (14).

To guarantee that rational agents are satisfying their transversality condition, and to remain true to the linearization of the model, we focus on bounded solutions to the above system, which may be computed in the usual way. The definition of an equilibrium to the HBC model illustrates that a heterogeneous expectations equilibrium can be found by solving the associated rational model. In (Branch and McGough, 2004) it was shown that the number and nature of heterogeneous expectations equilibria may be very different from a model with homogeneous, rational expectations. Below, we restrict attention to determinate HBC models.

Although boundedly rational/boundedly optimal agents hold misspecified forecasting models, we require that they forecast in a statistically optimal manner, i.e. we require that the forecast model parameters are optimal linear projections. It follows that ψ^τ satisfy the following least squares orthogonality conditions

$$Es_{t-1}^\tau \left(c_t^\tau - \psi_c^\tau s_{t-1}^\tau\right) = 0 \tag{20}$$

$$Es_{t-1}^\tau \left(s_t^\tau - \psi_s^\tau s_{t-1}^\tau\right) = 0 \tag{21}$$

$$Er_{t-1} \left(r_t - \psi_r^\tau r_{t-1}\right) = 0 \tag{22}$$

For each orthogonality condition, the expectation is taken with respect to the unconditional equilibrium distribution. Notice that in case these orthogonality conditions are satisfied, all boundedly rational agents have the same beliefs, which we now simply label as ψ, though these agents may still differ with respect to planning horizon. Least squares orthogonality conditions appear frequently in the macroeconomics literature. (Sargent, 2008; Evans and Honkapohja, 2001) show that learning models often converge to parameters that satisfy orthogonality conditions like (20)-(22). A key feature of beliefs that satisfy orthogonality conditions like (20)-(22) are that within the context of their forecasting model, agents are unable to detect their misspecification. We are now ready to define our main equilibrium concept.

Definition. *Given the forecasting horizons* $\{N^\tau\}_{\tau=1}^M$*, proportions* θ *and* $\{\phi^\tau\}_{\tau=1}^M$*, a restricted perceptions equilibrium is a collection* $\{\{s_t^\tau, c_t^\tau, n_t^\tau\}_{\tau=1}^M, s_t, c_t, k_t, r_t, w_t\}$ *such that*

1. *given* ψ*,* $\{\{s_t^\tau, c_t^\tau, n_t^\tau\}_{\tau=1}^M, s_t, c_t, k_t, r_t, w_t\}$ *is an equilibrium of the HBC model;*
2. ψ *satisfies the least squares orthogonality conditions (20)-(22).*

3 Inelastic Labor Suppy: Comparative Dynamics

The HBC framework is flexible enough to incorporate many forms of heterogeneity. This section provides critical insights into the way in which heterogeneity alters business cycle dynamics by restricting attention to the special case of inelastic labor supply, i.e. the Ramsey model. The next section presents quantitative results for the HBC with elastic labor supply.

Because the complicated nature of the model makes analytic results intractable, this section presents numerical analysis. There are two agent types: a proportion θ who are rational, and a proportional $1-\theta$ who are boundedly rational with an N-period planning horizon. The functional forms for utility and production are given by (8). For results presented here, we employ the following parameter values (consistent with annual data): $\beta = .95$, $\alpha = 1/3$, $\delta = .1$ and $\rho = .9$. In this section, we set $\Theta = 0$ and focus on the model with inelastic labor supply; this allows us to explore more intuitively the individual and aggregate implications of bounded rationality and optimality.

3.1 Boundedly Rational Agent Behavior

We begin by comparing the individual behavior of rational agents to boundedly rational agents. This subsection assumes the economy is in a representative agent environment, with $\theta = 1$, and a zero mass of boundedly rational agents. In this subsection, there is no heterogeneity and so, for the moment, beliefs are fixed. We now illustrate how forecasting/planning horizons alter individual behavior.

Figure 1: Individual Behavior, Various Horizons, Lagged Timing. Impulse responses to a 1% technology shock, with $\theta = 1$ and a zero-mass of boundedly rational agents. The dashed lines are impulse responses for rational agents, the solid lines correspond to the zero-mass boundedly rational agent. The arrow indicates the direction in which N increases.

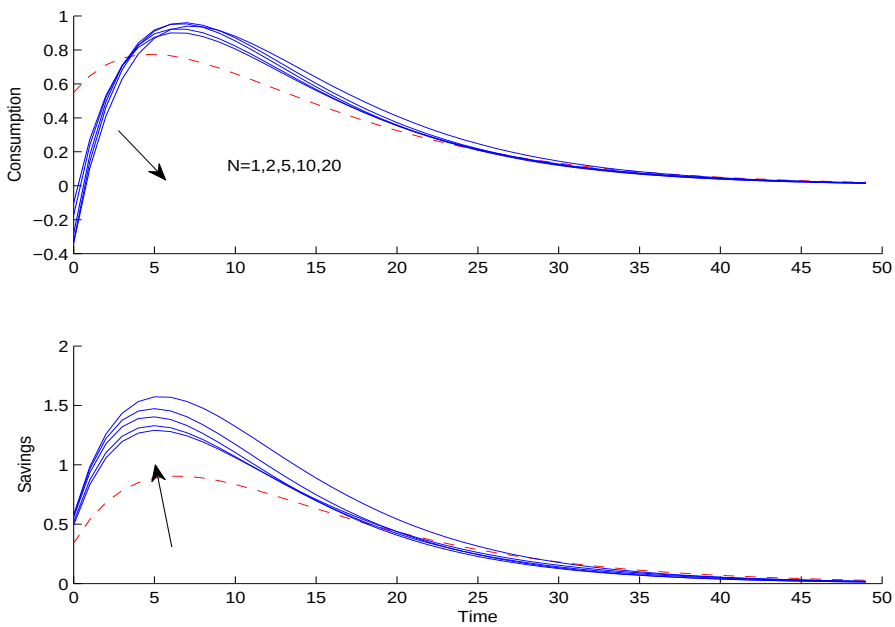


Figure 1 plots the individual agent type's consumption and savings responses to a technology shock. To construct this figure, we performed the following experiment: we begin by computing the impulse response functions from the model following a 1% shock to productivity when $\theta = 1$, i.e. the fully rational case. The impulse responses provide the consumption and savings behavior of the rational agents as well as a simulated series of prices. Given these data, we compute, via the behavioral equations, the corresponding impulse responses for the zero-mass boundedly rational

agent. In the figure, the dashed line indicates the behavior of the rational agents, and the solid lines indicate the behavior of the boundedly rational agents at various planning horizons $N = 1, 2, 5, 10, 20$. The arrow indicates the way in which the impulse responses morph as the planning horizon increases. The exogenously set beliefs are given by $\psi_c = 0.7$, $\psi_r = 0.7$, and $\psi_s = 0.99$, which are approximately the values that would arise in a restricted perceptions equilibrium when $N = 1$.

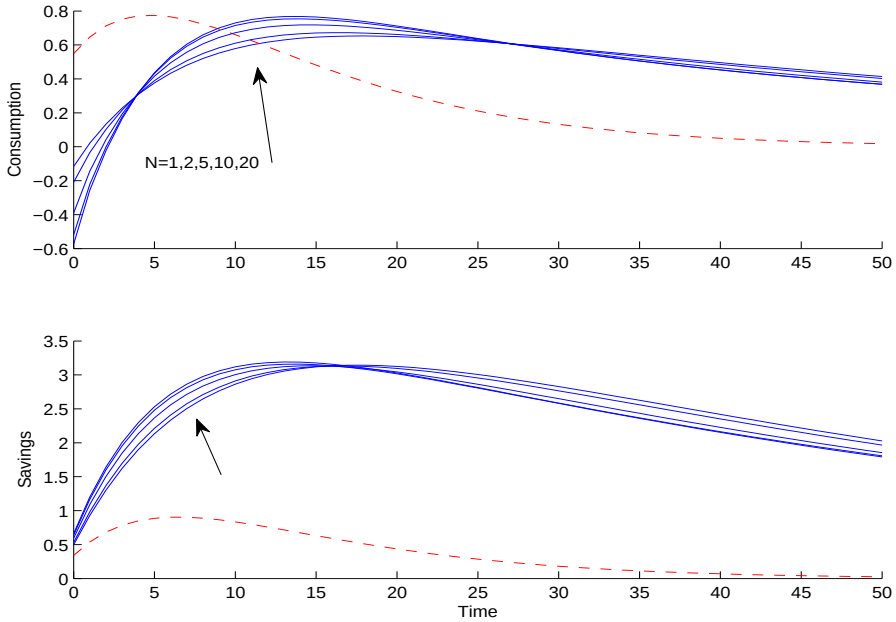
A number of interesting features are evident from Figure 1. First, notice that in response to the productivity shock, rational agents increase both savings and consumption, whereas boundedly rational agents shift toward savings and reduce consumption. These divergent responses to technology shocks are due to the backward-looking nature of the boundedly rational agents: the productivity shock increases the expected interest rate which raises the relative price of current consumption. This puts downward pressure on current consumption for both types of agents. However, rational agents, by correctly forecasting that future consumption will rise, lift current period consumption to promote smoothing. Because of their backward-looking beliefs, boundedly rational agents fail to predict the rise in future consumption and their current consumption falls. Increasing the planning horizon exacerbates this result by magnifying the interest rate effect.

The fall in consumption of the boundedly rational agents is very short-lived. Because savings increases rapidly – indeed faster for boundedly rational agents than for rational agents – forecasts of future consumption rise quickly as well. The higher consumption forecasts promote rapidly increasing consumption for the boundedly rational agents: they too want to smooth consumption across periods. In fact, we see that both the savings and consumption paths of the boundedly rational agents peak higher than the rational agents, and remain higher for at least twenty periods. As before, these effects are magnified by increasing the planning horizon. This result gives credence to our intuition that the presence of heterogeneity may amplify the propagation mechanism inherent in these models.

As noted, the initial fall in of the boundedly rational agent's consumption is due to the backward-looking nature of his forecasting model; it is further exacerbated in the lagged timing specification by the inability of the agent to condition expected future consumption on current savings, which, in turn, depends positively on current wage levels. The contemporaneous timing specification mitigates this issue, so that the boundedly rational agent's current consumption responds negatively to the interest rate shock, but positively to the wage shock. Figure 2 provides the impulse response functions for the same rational behavior, but now with boundedly rational agents abiding the contemporaneous timing protocal. The propagation magnification is greatly increased relative to the assumption of lagged timing. This increase may

be traced to agents' optimal restricted perceptions. To produce Figure 2, we used $\psi_c = 0.2$, $\psi_r = 0.8$, and $\psi_s = 0.999$, which approximately correspond to optimal restricted perceptions given a one period planning horizon. The high value ψ_s indicates that the agent believes savings will remain high for some time; however, the low value of ψ_c reduces the corresponding response of expected future consumption. This reduction lowers the agent's perceived need to smooth consumption, so savings levels build high and stay high for many periods.

Figure 2: Individual Behavior, Various Horizons, Contemporaneous Timing. Impulse responses to a 1% technology shock, with $\theta = 1$ and a zero-mass of boundedly rational agents. The dashed lines are impulse responses for rational agents, the solid lines correspond to the zero-mass boundedly rational agent. The arrow indicates the direction in which N increases.



Figures 1-2 seem to suggest that consumption is countercyclical. The intuition discussed above illustrates that this arises with inelastic labor supply because the backward-looking agents fail to forecast the increase in future consumption and so their optimal response to technology shocks is to increase savings and decrease consumption. When labor supply is elastic, there is an additional wealth effect that arises when workers work more in response to positive technology shocks, and this effect will lead to procyclical consumption. Though the contemporaneous correlation

with output will be much weaker for the reasons mentioned in this subsection.

3.2 Equilibrium Behavior

Now we turn to analyzing the model's restricted perceptions equilibria for various planning horizons and proportions of rational agents. We begin by allowing the planning horizon to vary, and by setting $\theta = 0.22$.[4] Figure 3 plots the equilibrium output impulse responses to a unit shock to productivity.

Figure 3: Restricted Perceptions Equilibria, Various Horizons, Lagged Timing. Impulse response to a 1% technology shock. Dashed line is the RBC model, solid lines are the HBC model with various horizons N. The arrow indicates the direction of increasing N.

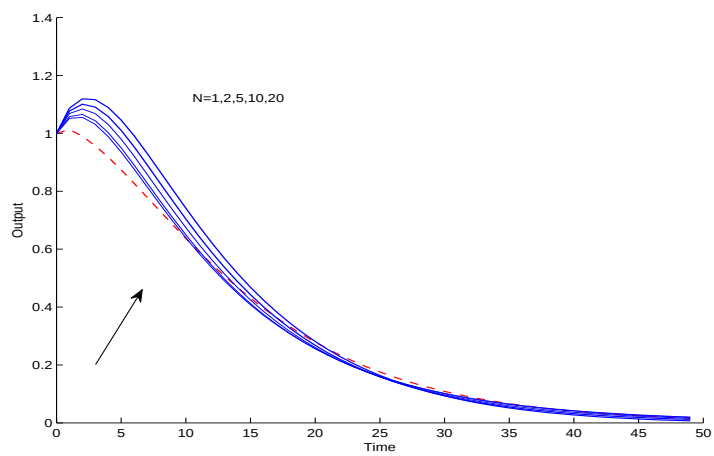
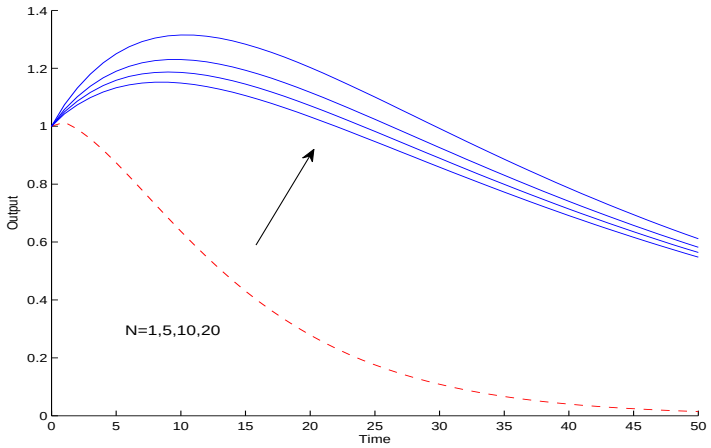


Figure 3 illustrates that the presence of boundedly rational agents magnifies the model's propagation mechanism: a unit shock to productivity results in an output time-path that, under heterogeneous beliefs, has a higher peak and is more hump shaped than it is under rational expectations; further, for long planning horizons output under heterogeneous beliefs remains higher for up to 22 periods. This presence of propagation magnification can be traced to the increased savings of the boundedly rational agents. Recall from Figure 1 that, facing a unit productivity shock, boundedly rational agents save more than their rational counterparts, and maintain higher savings for some time. This increase in savings results in an increase in aggregate

[4]Throughout, we calibrate $\theta = 0.22$, the fraction of the US population living in the midwestern, or freshwater, states.

Figure 4: Restricted Perceptions Equilibria, Various Horizons, Contemporaneous Timing. Impulse response to a 1% technology shock. Dashed line is the RBC model, solid lines are the HBC model with various horizons N. The arrow indicates the direction of increasing N.

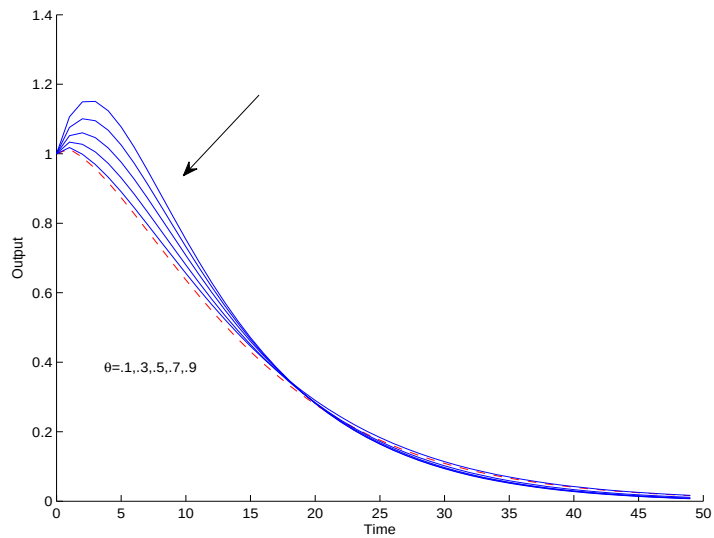


capital stock which increases firms' production levels. However, the increase in capital stock also lowers the marginal product of capital, and therefore the real interest rate, which encourages increased consumption. This increase mitigates somewhat the impact of the boundedly rational agents on equilibrium output, which is why the impulse response functions seen in Figure 3 are less exaggerated than might be anticipated given the individual behavior witnessed in Figure 1.

The impulse responses corresponding to a unit shock under the contemporaneous timing protocal are given in Figure 4. In this case, the heterogeneous model's output response to a unit shock is greatly magnified relative to the rational model. This, again, is due to the interaction between the restricted optimal beliefs coefficients ψ_c and ψ_s.

Varying the proportion of rational agents also leads to interesting comparative dynamics. Consider the impulse response functions presented in Figure 5: here we take the planning horizon of boundedly rational agents to be $N = 20$. The arrow indicates the way in which the impulse response functions morph as the proportion of rational agents increases. A *decrease* in the proportion of rational agents magnifies the model's propagation mechanism: as θ becomes smaller, the model's dynamics are increasingly governed by the behavior of the boundedly rational agents, and their backward-looking forecasting models direct them to save more than rational agents.

Figure 5: Restricted Perceptions Equilibria, Various Proportions of Rational Agents, Lagged Timing. Impulse response to a 1% technology shock. Dashed line is the RBC model, solid lines are the HBC model with various fractions of rational agents θ. The arrow indicates the direction of increasing θ.



In case contemporaneous timing is assumed, the propagation amplification is exaggerated as above, and decreasing the proportion of rational agents serves to magnify the amplification as anticipated.

4 Elastic Labor Supply: Quantitative Results

The previous section demonstrated that a heterogeneous expectations version of the neoclassical model, with inelastic labor supply, is capable of increasing the propagation of technology shocks. This section shows that a heterogeneous expectations version of the real business cycle model provides improved quantitative results over the benchmark RBC model. We again assume two types of agents, a fraction θ of which are rational, and now relax the inelastic labor supply assumption, i.e. $\Theta > 0$. We calibrate the model with standard parameter configurations adopted by the RBC literature, and compare the fit of the HBC model to the RBC model for various second moments of interest.

There are a number of dimensions that the benchmark RBC model fails to match

empirical business cycle properties. The previous section focused on the hump-shaped output impulse responses documented by (Cogley and Nason, 1995). The RBC models have also been criticized for relying on large technology shocks in order to match volatilities observed in data: see, for example, (Summers, 1986; King and Rebelo, 1999). A key feature of RBC models is the labor/leisure trade-off made in response to temporary movements in real wages. The standard model relies on an unrealistically high degree of labor elasticity to generate empirically realistic business cycles. Even with these assumptions, RBC models underpredict volatility in output, consumption, and hours. We show that a heterogeneous expectations business cycle model can overcome these limitations and provide improved quantative results.

We now calibrate the model in Section 2 according to the parameter values chosen by (King and Rebelo, 1999). These values facilitate model comparisons to U.S. quarterly data. Table 1 details the assumed parameter values. We begin by computing business cycle moments for the HBC model with these standard parameter values. We subsequently show that for less elastic labor supply and smaller shocks, the HBC continues to fit the data well. The values $\eta = 1$ and $\theta = 3.48$ imply that the Frisch labor supply elasticity is approximately 4, a value that is critizicized as unrealistically high. We will also consider alternative values for η, θ so that the labor supply elasticity is equal to one. The parameters governing the stochastic process for productivity are standard and are derived from the Solow residual. The size of the shocks have been criticized by (Summers, 1986; Kurz and Motolese, 2005). We follow Kurz and Motolese in also considering the properties of the model with much smaller shocks by setting σ_v to a smaller value.

Table 1: Calibration

β	α	δ	ρ	η	Θ	σ_v
0.984	1/3	.025	0.979	1	3.48	.0072

This section compares the fit of the RBC model, the HBC model, and quarterly U.S. data from 1947.1-2009.2. To generate moments from the models, we simulate the (log-linearized) model for 248 periods and calculate key moments and correlations. We then repeat the simulation 20,000 times and average across simulations.[5] To remain consistent with the RBC literature, we filter the data – actual and simulated – using the Hodrick-Prescott filter. We report results on the unconditional volatilities of aggregate output, aggregate consumption, and aggregate hours. To illustrate the

[5] We found identical quantitative results when we simulated for 5000 periods and averaged across 5000 simulations.

internal propagation of the model we calculate the contemporaneous correlations of output with hours and productivity shocks and calculate the autocorrelation function. Table 2 presents the results for the baseline case of contemporaneous timing.

Table 2: Business Cycle Moments: contemporaneous savings version

	Data	RBC	HBC	HBC small shocks	HBC lower elasticity
σ_y	1.699	1.39	1.85	1.68	1.78
σ_c	1.272	0.62	0.70	1.08	1.21
σ_n	1.909	0.68	1.42	2.05	1.27
corr(y,n)	0.86	0.97	0.92	0.95	0.89
corr(y,z)	0.78	1.00	0.98	0.88	0.98

The first column presents moments for U.S. quarterly data.[6] The second column presents results for the RBC model, which arises by setting $\theta = 1$, that are quantitatively identical to King and Plosser. These results illustrate a number of the common shortcomings of the benchmark RBC model. Output volatility, consumption volatility, and hours volatility in the model are lower than in the data. Moreover, the contemporaneous correlation between output and hours or productivity is close to one, while these correlations are much lower in data. Studies based on structural VAR identification of shocks, place the fraction of output explained by technology shocks even lower. These results lead many to conclude that the RBC model has weak internal propagation.

The RBC literature has proposed a number of alternative formulations, particularly in the labor market, to address some of the shortcomings of the benchmark RBC model. This paper proposes expectations heterogeneity as a mechanism to enhance the propagation of shocks. Column 3 of Table 2 illustrates the results for the HBC model under the standard calibration. These results assume that $\theta = 0.22$, $N = 5$, and agents observe contemporaneous savings when forming expectations.[7] Column

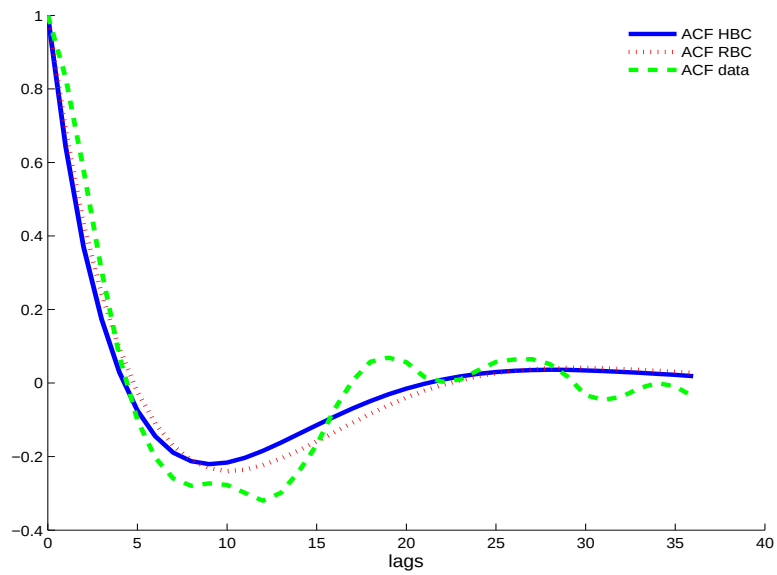
[6]These data come from the St. Louis Federal Reserve Bank's FRED database. Output is measured as detrended log real GDP, consumption is log real personal consumption expenditures, and hours are in the nonfarm business sector.

[7]Throughout, we set $\theta = 0.22$ and choose $N = 5$ so that the results of the HBC model match the data. The results of the previous section show that lower values of θ and higher values of n increase the internal propagation. We calibrate θ to the fraction of the U.S. population in the midwestern states. We treat N as a free parameter governing the behavior of the adaptive agents. An extension of our approach would be to combine heterogeneous expectations with a model where agents choose their planning horizon as well. By assuming a cost to being rational, and having a longer planning horizon, it is possible to pin these parameters down in equilibrium. We leave this to future research.

3 demonstrates that the HBC model increases the internal propagation of the RBC model. There is substantially more volatility in hours, with 108% more volatility than in the RBC model, which translates into 33% more output volatility than the representative agent model. The additional propagation provided by beliefs is seen in the lower correlation between output and hours, with the HBC providing a 6% reduction in this correlation.

The improved fit of the HBC model over the RBC model is not at the expense of its autocorrelation properties. Figure 6 plots the autocorrelation functions for detrended output in the data, the RBC model, and the HBC model. These autocorrelation functions were computed from the same data generating columns 2 and 3 of Table 2. This figure shows that the RBC model and HBC model do a good job capturing the positive autocorrelation at short lags, and the negative autocorrelation at medium-range lags. Because the HBC model has very similar autocorrelation properties but substantially improved second moments, Table 2 and Figure 6 demonstrate that the HBC model delivers a significantly better fit than the RBC model.

Figure 6: Autocorrelation functions for data, RBC, and HBC.



The fit in column 3 of Table 2 adopts the standard RBC calibration and, in particular, relies on large productivity shocks. Column 4 demonstrates that the model can improve the fit over an RBC model with lower productivity shocks. The HBC results in column 4 assume $\sigma_v = 0.003$, following the value set by (Kurz and Motolese, 2005), and sets $N = 36$ (a planning horizon of 9 years). The results in column 3 show

a substantial improvement. The HBC model delivers output volatility that matches the data, and has volatility in hours and consumption that are substantially closer to the data than in the RBC model. The smaller shocks, and longer planning horizon, lead to a much weaker connection between output and technology shocks, which in our view is a significant success of the HBC model. The key intuition for the improved fit of the HBC model relative to the RBC model is the increased propagation provided by the boundedly rational agents. A positive technology shock that increases the real wage and the real interest rate will induce substantial increased savings and hours by the boundedly rational agents. They forecast higher future real interest rates, leading them to increase labor and increase their savings accumulation. The longer the forecast horizon the stronger this response. Column 5 demonstrates that this intuition continues to hold even for lower labor supply elasticities. The HBC results in column 5 occur when η, θ are set so that the Frisch elasticity equals one, $\sigma_v = 0.0072$, and $N = 20$. As in the previous formulations, the HBC model with a more reasonable labor supply elasticity is capable of improving the fit of business cycle models to data.

Table 3: Business Cycle Moments: lagged savings version

	Data	RBC	HBC	HBC small shocks	HBC lower elasticity
σ_y	1.699	1.39	2.05	1.59	1.72
σ_c	1.272	0.62	0.87	1.02	1.14
σ_n	1.909	0.68	1.77	1.94	1.20
corr(y,n)	0.86	0.97	0.92	0.95	0.89
corr(y,z)	0.78	1.00	0.97	0.87	0.98

As a robustness check, Table 3 conducts the same model comparisons but where agents do not observe contemporaneous savings when forming forecasts of future savings. This assumption is not as natural in the present setting, since savings are an agent's own choice variable. However, Table 4 demonstrates that the results in Table 4 are not sensitive to this assumption.

5 Discussion of Related Research

This paper demonstrates that heterogeneity in expectations can improve the internal propagation of equilibrium business cycle models. The literature has proposed several alternative paths. One promising avenue has been to incorporate "news" shocks

into RBC models. For example, (Beaudry and Portier, 2006) illustrate that when households build news about future technology shocks into their current plans, then an equilibrium business cycle model is capable of generating empirically realistic business cycle fluctuations. A second approach has been to constrain household behavior. (Deaton, 1991) assumes some households face liquidity constraints. In (Krusell and Smith, 1996) households pay a utility cost to calculating the fully optimal consumption plan, or for no cost they can solve for an optimal linear consumption plan. Interestingly, there can exist an equilibrium with households split between being fully rational and constrained rational.

An approach more closely related to the present study has been to assume that agents have limited information and adopt econometric models to forecast future prices. (Williams, 2004) develops a standard RBC model where agents know the form of the (linear) law of motion under rational expectations, but they must learn in real time the model's parameters. Williams finds that adaptive learning does not enhance the internal propagation of the RBC model. Closely related, (Huang et al., 2009), in a model very close to (Williams, 2004), demonstrate that with a one period planning/forecasting horizon and an optimally misspecified model, the intertemporal substitution effects can be strengthened and improve the propagation of RBC models. Importantly, these two papers assume households form forecasts based on "Euler equation learning," e.g. (Marcet and Sargent, 1989; Bullard and Mitra, 2002), which requires agents to only form forecasts one period ahead in order to satisfy their Euler equation. On the other hand, (Eusepi and Preston, 2008) consider a closely related model where agents, conditional on their current parameter estimates, satisfy their Euler equation and their lifetime budget constraint, requiring them to forecast infinitely far into the future. Eusepi and Preston demonstrate that, under "infinite horizon learning," the median impulse response across simulated economies exhibit hump-shaped responses to transitory technology shocks.

Our paper also fits into a broader literature on diverse beliefs and optimal misspecification. (Branch and Evans, 2006, 2009) show that intrinsic heterogeneity can arise when agents are restricted to underparameterized statistical models but the distribution is determined endogenously in a misspecification equilibrium. (Sargent, 2008) argues forcefully that misspecification can be a pervasive factor in forecasting since with finite data it may be difficult for econometricians to uncover their misspecification. This argument justifies orthogonality conditions similar to those in the restricted perceptions equilibria.

In other settings, diverse beliefs can play a prominent role. (Kurz and Motolese, 2009) and (Guo and Wu, 2009) demonstrate the strong empirical implications of diverse beliefs for asset pricing and asset returns. These papers adopt the Rational

Belief Equilibrium (RBE) approach of (Kurz, 1994) that emphasizes that beliefs are themselves a state variable. The RBE requires that the empirical distribution of the economy aligns with the subjective beliefs of the agents. In De Grauwe (2009); Wieland (2009), diverse beliefs are studied in New Keynesian models. Finally, the effect of heterogeneity on expectational coordination is addressed by (Guesnerie and Jara-Moroni, 2009).

6 Conclusion

This paper incorporates heterogeneous expectations into an equilibrium business cycle model. We assume a competitive economy populated by two different types of households, each selecting plans for consumption/savings and labor supply. The first group of agents are fully rational and form their expectations rationally while the other group of agents employ a set of misspecified statistical models through which they form their expectations. With a statistical model in hand, the group of agents who are boundedly rational must decide on a household plan by satisfying a set of optimality conditions. A fraction of agents hold restricted perceptions because they lack the forecasting sophistication required by rational expectations. In an approach we call *bounded optimality*, we require that households have the same sophistication in decision making as in forecasting, thereby avoiding a cognitive dissonance. The results of this paper demonstrate that a heterogeneous expectations business cycle model is capable of increasing the internal propagation of real business cycle models.

The approach of this paper is most closely related to (Kurz and Motolese, 2005), who develop a business cycle model with diverse beliefs, variable capacity utilization, and monetary and technology shocks. They show that their model can match key business cycle moments even with smaller shocks, e.g. $\sigma_v = .003$. Their approach to heterogeneity is complementary to ours. They adopt the rational belief approach that requires agents' subjective beliefs to be consistent with the empirical distribution generated by those beliefs. Their beliefs, like ours, are optimally misspecified. The primary difference between their study and ours, is they study the interaction between diverse, possibly misspecified, beliefs, while the present paper studies the interaction between rational and boundedly rational beliefs.

The quantitative results presented in this paper suggest heterogeneous expectations and bounded rationality/optimality as a promising avenue for future research on business cycles. There is an extensive literature that considers alternative assumptions for labor markets, preferences, shocks, capacity utilization, and so on, that have been useful for improving the fit of RBC models. The results in this paper suggest

that also including heterogeneous expectations into these models may also improve their fit and at the same time brings more realism to the model by departing from the representative agent structure.

References

Beaudry, P. and F. Portier (2006). When can changes in expectations cause business cycle fluctuations in neo-classical settings. *Journal of Econimic Theory 135*, 458–477.

Blanchard, O. and C. Kahn (1980). The solution of linear difference models under rational expectations. *Econometrica 48*, 1305–1311.

Branch, W. A. and G. W. Evans (2006). Intrinsic heterogeneity in expectation formation. *Journal of Economic Theory 127*, 264–295.

Branch, W. A. and G. W. Evans (2009). Monetary policy and heterogeneous expectations. working paper.

Branch, W. A. and B. McGough (2004). Multiple equilibria in heterogeneous expectations models. *Contributions to Macroeconomics 4*(1), 1–16.

Bullard, J. and K. Mitra (2002). Learning about monetary policy rules. *Journal of Monetary Economics 49*, 1105–1129.

Cogley, T. and J. Nason (1995). Output dynamics in real business cycle models. *American Economic Review 85*, 492–511.

De Grauwe, P. (2009). Animal spirits and monetary policy.

Deaton, A. (1991). Saving and liquidity constraints. *Econometrica 59*(5), 1221–1248.

Eusepi, S. and B. Preston (2008). Expectations, learning and business cycle fluctuations. mimeo.

Evans, G. W. and S. Honkapohja (2001). *Learning and Expectations in Macroeconomics*. Princeton, New Jersey: Princeton University Press.

Evans, G. W. and S. Honkapohja (2006). Monetary policy, expectations and commitment. *Scandinavian Journal of Economics 108*, 15–38.

Evans, G. W. and B. McGough (2009). Learning to optimize. mimeo.

Guesnerie, R. and P. Jara-Moroni (2009). Expectational coordination in a class economies with strategic substitutabilities: the complete information case". working paper.

Guo, W.-C. and H.-M. Wu (2009). Financial leverage and market volatility with rational beliefs.

Huang, K., Z. Liu, and T. Zha (2009). Learning, adaptive expectations and technology shocks. *Economic Journal,* forthcoming.

King, R. G. and S. T. Rebelo (1999). *Resuscitating Real Business Cycles.* Amsterdam: Elsevier.

Krusell, P. and A. A. Smith, Jr. (1996). Rules of thumb in macroeconomic equilibrium: A quantitative analysis. *Journal of Economic Dynamics and Control 20*, 527–558.

Kurz, M. (1994). On the structure and diversity of rational beliefs. *Economic Theory 4*, 1–24.

Kurz, M. and M. Motolese (2005). The role of expectations in economic fluctuations and the efficacy of monetary policy. *Journal of Economic Dynamics and Control 29*, 2017–2065.

Kurz, M. and M. Motolese (2009). Diverse beliefs and time variability of risk premia. mimeo.

Marcet, A. and T. J. Sargent (1989). Convergence of least-squares learning mechanisms in self-referential linear stochastic models. *Journal of Economic Theory 48*, 337–368.

Preston, B. (2006). Adaptive learning, forecast-based instrument rules and monetary policy. *Journal of Monetary Economics 53*, 507–535.

Sargent, T. J. (2008). Evolution and intelligent design. *American Economic Review 98*, 5–37.

Summers, L. H. (1986). Some skeptical observations on real business cycle theory. *Federal Reserve Bank of Minneapolis Quarterly Review 10*, 23–27.

Taylor, J. and M. Woodford (Eds.) (1999). *Handbook of Macroeconomics, Volume 1.* Amsterdam: Elsevier.

Wieland, V. (2009). The diversity of forecasts from macroeconomic models of the u.s. economy.

Williams, N. (2004). Adaptive learning and business cycles. working paper.