

Technical Note for: “The Wisdom of the Crowd Playing the Price is Right”

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Rules of the Game

Four players are shown a prize, and then provide bids in a fixed and known sequence. The player whose bid is closest to the true price of the prize, without exceeding the price, is the winner. Each player has full knowledge of previous bids in the sequence. If all bids are above the true price, there is no winner. Bids are in \$1 increments, starting at \$1.

General Frame of Optimal Decision-Making

Assume the true price is a draw from some probability mass function $f_\theta(\cdot)$, with cumulative mass function $F_\theta(\cdot)$. We can now define a ‘win’ function $\omega_x(a, b, c, d, \theta)$ for the probability the x th player will win, given bids a, b, c, d for Players 1–4 respectively.

Player 4

Given existing bids a, b and c , the probability Player 4 will win if they made the bid d is just

$$\pi_4(d|a, b, c, \theta) = \omega_4(a, b, c, d, \theta) \tag{1}$$

We assume Player 4 makes their bid decision according to these probabilities, using a general exponentiated form of the Luce choice rule

$$p_4(d|a, b, c, \theta) = \frac{\pi_4(d|a, b, c, \theta)^\gamma}{\sum_{d'} \pi_4(d'|a, b, c, \theta)^\gamma} \tag{2}$$

where the parameter γ controls the continuum between deterministically choosing the bid most likely to result in winning at one extreme ($\gamma \rightarrow \infty$), and probability matching at the other ($\gamma = 1$). In the current paper, we restrict ourselves to the most standard setting $\gamma = 1$.

Player 3

Given existing bids a and b , the probability Player 3 will win if they made the bid c , assuming Player 4 subsequently ‘behaves optimally’ and bids according to Equation 2 above, is

$$\pi_3(c|a, b, \theta) = \sum_d p_4(d|a, b, c, \theta) \omega_3(a, b, c, d, \theta) \quad (3)$$

We again assume Player 3 makes their bid decision according to these probabilities, using the same determinism parameter γ , which gives

$$p_3(c|a, b, \theta) = \frac{\pi_3(c|a, b, \theta)^\gamma}{\sum_{c'} \pi_3(c'|a, b, \theta)^\gamma} \quad (4)$$

Player 2

Given an existing bid a the probability Player 2 will win if they made the bid b , assuming Players 3 and 4 subsequently behave optimally, is

$$\pi_2(b|a, \theta) = \sum_c p_3(c|a, b, \theta) \sum_d p_4(d|a, b, c, \theta) \omega_2(a, b, c, d, \theta) \quad (5)$$

This gives the bid decision probabilities

$$p_2(b|a, \theta) = \frac{\pi_2(b|a, \theta)^\gamma}{\sum_{b'} \pi_2(b'|a, \theta)^\gamma} \quad (6)$$

Player 1

Player 1 provides the first bid. If they bid a , their probability of winning, assuming subsequent optimal behavior is

$$\pi_1(a|\theta) = \sum_b p_2(b|a, \theta) \sum_c p_3(c|a, b, \theta) \sum_d p_4(d|a, b, c, \theta) \omega_1(a, b, c, d, \theta) \quad (7)$$

This gives the bid decision probabilities

$$p_1(a|\theta) = \frac{\pi_1(a|\theta)^\gamma}{\sum_{a'} \pi_1(a'|\theta)^\gamma} \quad (8)$$

We consider two models. In the first model, only Player 4 is optimal. In the second model, both Player 3 and Player 4 are optimal.

Model 1: Player 4 is optimal

We assume that the true price follows uniform distribution $U(l, u)$, and define a ‘win’ function $\omega_x(a, b, c, d, l, u)$ for the probability that the x th player will win, given bids a, b, c, d , for Players 1–4 respectively.

Player 4

Assuming the true price is a draw from the uniform distribution $U(l, u)$. Figure 1 illustrates the probability that Player 4 will win if they bid d , given the first three bids a, b, c , and the parameters l and u . Without loss of generality, we assume a, b and c are in ascending order within the lower bound l and upper bound u . If d is within (l, u) , then the corresponding winning probability is determined by how close d is to its next largest neighbor. If d is smaller than l , then the winning probability is determined by the distance from l and a . In this study, for the simplicity of notations and calculations, we assume that both l and u are integers, and we hope that this restriction won't affect the result.

We present the order of all bids and parameters using a notation illustrated in the following example: if d is in between of the minimum and the median of the previous three bids, we denote the order as $ladbcu$. Similarly, $labcd u$ means that d is the largest among all bids. We adopt an indicator function I_{order} which yields value 1 only if the corresponding order is true.

With the notations introduced above, the win function of Player 4, i.e. the probability Player 4 will win, given a, b, c, d, l and u , is just

$$\begin{aligned} \pi_4(d|a, b, c, l, u) &= \omega_4(a, b, c, d, l, u) \\ &= I_{dlabcu} \cdot \frac{a-l}{u-l} + I_{ldabcu} \cdot \frac{a-d}{u-l} + I_{ladbcu} \cdot \frac{b-d}{u-l} \\ &\quad + I_{labdcu} \cdot \frac{c-d}{u-l} + I_{labcd u} \cdot \frac{u-d}{u-l} + I_{labcd} \cdot 0. \end{aligned} \quad (9)$$

We assume Player 4 makes their bid decision according to these probabilities, using the Luce choice rule as shown in Equation 2. Fixing the parameter γ at 1, we have

$$p_4(d|a, b, c, l, u) = \frac{\pi_4(d|a, b, c, l, u)}{\sum_{d'} \pi_4(d'|a, b, c, l, u)} \quad (10)$$

When calculating the normalizing sum $\sum_{d'} \pi_4(d'|a, b, c, l, u)$, we consider a partition of the interval (l, u) , $\{(l, a), (a, b), (b, c), (c, u)\}$, because ω_4 has different expressions on these sub-intervals. Then we have

$$\begin{aligned} \sum_{d'=1}^u \pi_4(d'|a, b, c, l, u) &= \sum_{d'=1}^l \frac{a-l}{u-l} + \sum_{d'=l+1}^{a-1} \frac{a-d'}{u-l} + \sum_{d'=a+1}^{b-1} \frac{b-d'}{u-l} \\ &\quad + \sum_{d'=b+1}^{c-1} \frac{c-d'}{u-l} + \sum_{d'=c+1}^u \frac{u-d'}{u-l} \\ &= \frac{1}{u-l} \left((a-l)(a+l-1)/2 + (b-a)(b-a-1)/2 \right. \\ &\quad \left. + (c-b)(c-b-1)/2 + (u-c)(u-c-1)/2 \right) \end{aligned} \quad (11)$$

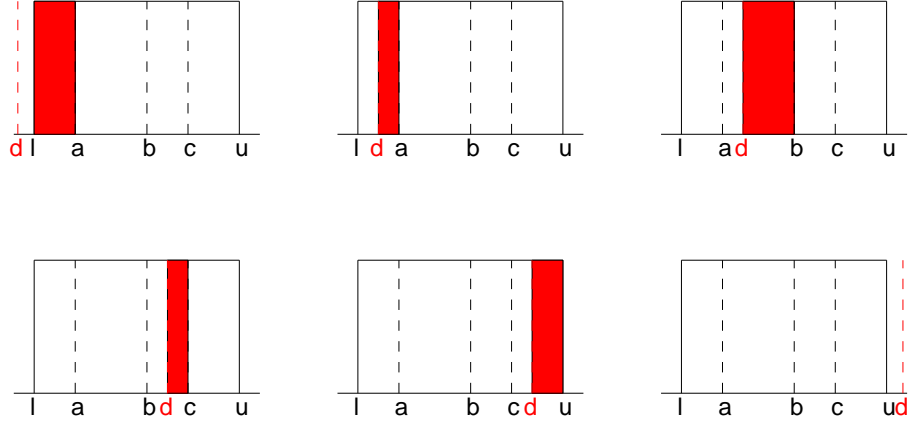


Figure 1. The winning probability for bid d given all bids and parameters.

By plugging Equation 9 and Equation 11 into Equation 10, we get $p_4(d|a, b, c, l, u)$.

Model 2: Player 3 and 4 are optimal

Player 4

Without loss of generality, we assume a, b are in ascending order within the lower bound l and upper bound u . Please notice that in this case c does not have to be within (l, u) . Figure 2 illustrates the probability that Player 4 will win if they bid d , given all possible situations regarding other bids and parameters. These probabilities are shown in red.

Given existing bids a, b and c , the probability Player 4 will win if they made the bid d is just

$$\begin{aligned}
\pi_4(d|a, b, c, l, u) &= \omega_4(a, b, c, d, l, u) \\
&= I_{cdlabu} \cdot (a - l) + I_{dlcabu} \cdot (a - l) + I_{dlcabu} \cdot (c - l) + I_{dlacbu} \cdot (a - l) \\
&+ I_{dlabcu} \cdot (a - l) + I_{dlabuc} \cdot (a - l) + I_{cl dabu} \cdot (a - d) + I_{l cadbu} \cdot (a - d) \\
&+ I_{ldcabu} \cdot (c - d) + I_{ldacbu} \cdot (a - d) + I_{ldabcu} \cdot (a - d) + I_{ldabuc} \cdot (a - d) \\
&+ I_{cladb u} \cdot (b - d) + I_{l cadbu} \cdot (b - d) + I_{l acdbu} \cdot (b - d) + I_{l adcbu} \cdot (c - d) \\
&+ I_{ladbcu} \cdot (b - d) + I_{ladbuc} \cdot (b - d) + I_{clabdu} \cdot (u - d) + I_{lcabdu} \cdot (u - d) \\
&+ I_{lacbdu} \cdot (u - d) + I_{labcd u} \cdot (u - d) + I_{labdcu} \cdot (c - d) + I_{labduc} \cdot (u - d)
\end{aligned} \tag{12}$$

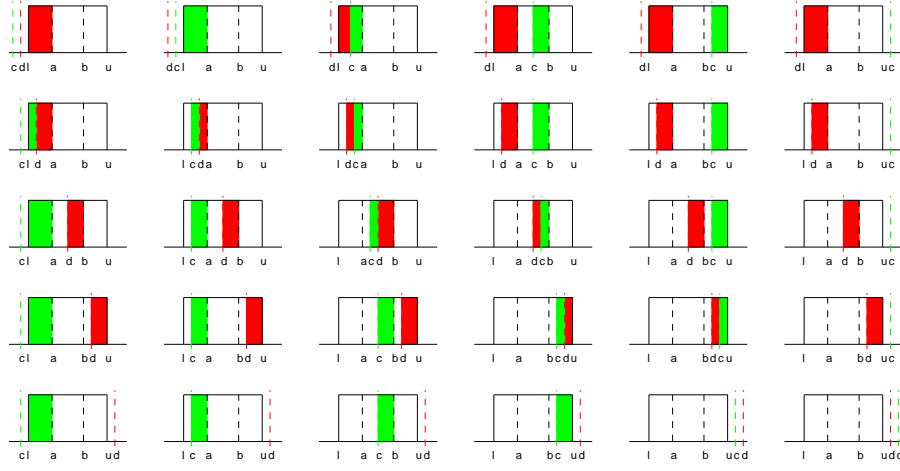


Figure 2. The winning probability for bid c (green, lighter) and d (red, darker) given all bids and parameters.

In this expression we omitted the denominator $u - l$ for simplicity, because it will be crossed off anyway in the normalization step.

Since Player 3 is also optimal, the rank of c matters, i.e. the quantity $\sum_{d'} \pi_4(d'|a, b, c, l, u)$ will have different expressions given different orders of a, b, c, l and u . There are five possible orders, namely $clabu, lcabu, lacbu, labcu, labuc$. We will calculate $\sum_{d'} \pi_4(d'|a, b, c, l, u)$ for each following the same logic in Equation 11.

$$\begin{aligned}
\sum_{d'=1}^u \pi_4(d'|a, b, c, l, u) &= \frac{1}{2} \cdot [\\
&I_{clabu} \cdot (2(a-l)(l-c) + (a-l)(a-l-1) + (b-a)(b-a-1) + (u-b)(u-b-1)) \\
&+ I_{lcabu} \cdot ((c-l)(c+l-1) + (a-c)(a-c-1) + (b-a)(b-a-1) + (u-b)(u-b-1)) \\
&+ I_{lacbu} \cdot ((a-l)(a+l-1) + (c-a)(c-a-1) + (b-c)(b-c-1) + (u-b)(u-b-1)) \\
&+ I_{labcu} \cdot ((a-l)(a+l-1) + (b-a)(b-a-1) + (c-b)(c-b-1) + (u-c)(u-c-1)) \\
&+ I_{labuc} \cdot ((a-l)(a+l-1) + (b-a)(b-a-1) + (u-b)(u-b-1))] \\
\end{aligned} \tag{13}$$

Plugging Equation 12 and Equation 13 into Equation 10, we obtain $p_4(d|a, b, c, l, u)$.

Player 3

Given existing bids a and b , the probability Player 3 will win if they made the bid c , assuming Player 4 subsequently ‘behaves optimally’ and bids according to

Equation 10 above, is

$$\begin{aligned}
\pi_3(c|a, b, l, u) &= \sum_d p_4(d|a, b, c, l, u) \omega_3(a, b, c, d, l, u) \\
&= \sum_d \frac{\omega_4(a, b, c, d, l, u)}{\sum_{d'} \omega_4(a, b, c, d', l, u)} \omega_3(a, b, c, d, l, u) \\
&= \frac{\sum_d \omega_4(a, b, c, d, l, u) \omega_3(a, b, c, d, l, u)}{\sum_d \omega_4(a, b, c, d, l, u)}
\end{aligned} \tag{14}$$

Similar to ω_4 , ω_3 can be written down according to Figure 2. For each possible order of a, b, c, l and u , we calculate $\sum_d \omega_4$, $\omega_3 \omega_4$ and $\sum_d \omega_3 \omega_4$, in order to get π_3 , as shown below:

1° *clabu*.

$$\begin{aligned}
\omega_3 \omega_4 &= I_{dclabu} \cdot 0 + I_{cdlabu} \cdot 0 + I_{cl dabu} \cdot (d-l)(a-d) \\
&\quad + I_{cladb u} \cdot (b-d)(a-l) + I_{clabdu} \cdot (a-l)(u-d) + I_{clabud} \cdot 0
\end{aligned}$$

$$\pi_3(c|a, b, l, u) = \frac{\sum_d \omega_3 \omega_4}{\sum_d \omega_4} = \frac{\lambda_1}{\lambda_2 c + \lambda_3} \tag{15}$$

where

$$\begin{aligned}
\lambda_1 &= (a^2 + l^2)(a-l-1)/2 - a(a-1)(2a-1)/6 + l(l+1)(2l+1)/6 \\
&\quad + (a-l)(b-a)(b-a-1)/2 + (a-l)(u-b)(u-b-1)/2 \\
\lambda_2 &= l-a \\
\lambda_3 &= (a-l)(a+l-1)/2 + (b-a)(b-a-1)/2 + (u-b)(u-b-1)/2
\end{aligned}$$

2° *lcabu*.

$$\begin{aligned}
\omega_3 \omega_4 &= I_{dlcabu} \cdot (c-l)(a-c) + I_{ldcabu} \cdot (c-d)(a-c) + I_{lcdabu} \cdot (a-d)(d-c) \\
&\quad + I_{l cadbu} \cdot (b-d)(a-c) + I_{lcabdu} \cdot (u-d)(a-c) + I_{lcabud} \cdot 0
\end{aligned}$$

$$\pi_3(c|a, b, l, u) = \frac{\sum_d \omega_3 \omega_4}{\sum_d \omega_4} = \frac{\rho_{11}c^3 + \rho_{12}c^2 + \rho_{13}c + \rho_{14}}{c^2 + \rho_{15}c + \rho_{16}} \tag{16}$$

where

$$\begin{aligned}
\rho_{11} &= -2/3 \\
\rho_{12} &= a + 1/2 \\
\rho_{13} &= 1/6 - (a + l)(a - l + 1)/2 - (b - a)(b - a - 1)/2 \\
&\quad - (u - b)(u - b - 1)/2 \\
\rho_{14} &= a(a - l)(a + l - 1)/2 - a(a - 1)(2a - 1)/6 \\
&\quad + a(b - a)(b - a - 1)/2 + a(u - b)(u - b - 1)/2 \\
\rho_{15} &= -a \\
\rho_{16} &= -(l - a)(l + a - 1)/2
\end{aligned}$$

3° $lacbu$.

$$\begin{aligned}
\omega_3\omega_4 &= I_{dlacbu} \cdot (a - l)(b - c) + I_{ldacbu} \cdot (a - d)(b - c) + I_{ladcbu} \cdot (c - d)(b - c) \\
&\quad + I_{lacdbu} \cdot (b - d)(d - c) + I_{labcdu} \cdot (u - d)(b - c) + I_{lcabud} \cdot 0
\end{aligned}$$

$$\pi_3(c|a, b, l, u) = \frac{\rho_{21}c^3 + \rho_{22}c^2 + \rho_{23}c + \rho_{24}}{c^2 + \rho_{25}c + \rho_{26}} \quad (17)$$

where

$$\begin{aligned}
\rho_{21} &= -2/3 \\
\rho_{22} &= a + b + 1/2 \\
\rho_{23} &= 1/6 + l(l - a) - (a + b)(a + b + 1)/2 \\
&\quad - (a - l)(a - l - 1)/2 - (u - b)(u - b - 1)/2 \\
\rho_{24} &= b(u - b)(u - b - 1)/2 + b(a - l)(a - l - 1)/2 + lab \\
&\quad - b(b - 1)(2b - 1)/6 + b(a^2 + a + b^2 - b)/2 - bl^2 \\
\rho_{25} &= -(a + b) \\
\rho_{26} &= (a^2 + a + b^2 - b)/2 + (a - l)(a + l - 1)/2 + (u - b)(u - b - 1)/2
\end{aligned}$$

4° $labcu$.

$$\begin{aligned}
\omega_3\omega_4 &= I_{dlabcu} \cdot (a - l)(u - c) + I_{ldabcu} \cdot (a - d)(u - c) + I_{ladbcu} \cdot (b - d)(u - c) \\
&\quad + I_{labdcu} \cdot (c - d)(u - c) + I_{labcdu} \cdot (u - d)(d - c) + I_{lcabud} \cdot 0
\end{aligned}$$

$$\pi_3(c|a, b, l, u) = \frac{\rho_{31}c^3 + \rho_{32}c^2 + \rho_{33}c + \rho_{34}}{c^2 + \rho_{35}c + \rho_{36}} \quad (18)$$

where

$$\begin{aligned}
\rho_{31} &= -2/3 \\
\rho_{32} &= u + b + 1/2 \\
\rho_{33} &= 1/6 + l(l - a) - (u + b)(u + b + 1)/2 \\
&\quad - (a - l)(a - l - 1)/2 - (b - a)(b - a - 1)/2 \\
\rho_{34} &= u(a - l)(a - l - 1)/2 + u(b - a)(b - a - 1)/2 \\
&\quad - u(u + 1)(2u + 1)/6 + lau - ul^2 + u(b^2 + b + u^2 - u)/2 \\
\rho_{35} &= -(u + b) \\
\rho_{36} &= (u^2 - u + b^2 + b)/2 + (a - l)(a + l - 1)/2 + (b - a)(b - a - 1)/2
\end{aligned}$$

When the order is $labuc$, $\pi_3(c|a, b, l, u)$ is 0. Putting things together, we have

$$\begin{aligned}
\pi_3(c|a, b, l, u) &= I_{clabu} \cdot \text{Equation 15} \\
&\quad + I_{lcabu} \cdot \text{Equation 16} \\
&\quad + I_{lacb u} \cdot \text{Equation 17} \\
&\quad + I_{labcu} \cdot \text{Equation 18}
\end{aligned} \tag{19}$$

We again assume Player 3 makes their bid decision according to these probabilities, using the same determinism parameter $\gamma = 1$, which gives

$$p_3(c|a, b, l, u) = \frac{\pi_3(c|a, b, l, u)}{\sum_{c'} \pi_3(c'|a, b, l, u)} \tag{20}$$

The normalizing denominator can be found through two different approaches. The first approach is to use computer brute force. For any given a, b, l and u , let c take value from 1 to u , compute the corresponding $\pi_3(c|a, b, l, u)$, and then the sum. This approach does not require further analytical calculations, but is computationally expensive. We implemented the model using this approach in C++.

The second approach is to use human brute force to find an analytical solution for $\sum_{c'} \pi_3(c'|a, b, l, u)$. Since the range of the bids are large, assuming c being continuous over its range will be a good approximation, i.e. we want to approximate $\sum_{c'} \pi_3(c'|a, b, l, u)$ by

$$\int_1^l (\text{Equation 15}) dx + \int_l^a (\text{Equation 16}) dx + \int_a^b (\text{Equation 17}) dx + \int_b^u (\text{Equation 18}) dx \tag{21}$$

Notice that the kernels are in the form of $\frac{\rho_1 x^3 + \rho_2 x^2 + \rho_3 x + \rho_4}{x^2 + \rho_5 x + \rho_7}$ or $\frac{\lambda_1}{\lambda_2 x + \lambda_3}$. Closed-form solutions for integrals of this type are available.

$$\int_{q_1}^{q_2} \frac{\lambda_1}{\lambda_2 x + \lambda_3} dx = \frac{\lambda_1}{\lambda_2} \ln \frac{|\lambda_2 q_2 + \lambda_3|}{|\lambda_2 q_1 + \lambda_3|} \tag{22}$$

and

$$\int_{q_1}^{q_2} \frac{\rho_1 x^3 + \rho_2 x^2 + \rho_3 x + \rho_4}{x^2 + \rho_5 x + \rho_6} dx = \frac{\rho_1}{2}(q_2^2 - q_1^2) + (\rho_2 - \rho_1 \rho_5)(q_2 - q_1) + A \quad (23)$$

where

$$A = \begin{cases} \frac{m}{2} \ln \frac{|q_2^2 + \rho_5 q_2 + \rho_6|}{|q_1^2 + \rho_5 q_1 + \rho_6|} + \frac{2n - \rho_5 m}{\sqrt{4\rho_6 - \rho_5^2}} \left(\arctan \frac{2q_2 + \rho_5}{\sqrt{4\rho_6 - \rho_5^2}} - \arctan \frac{2q_1 + \rho_5}{\sqrt{4\rho_6 - \rho_5^2}} \right) & \text{if } \rho_5^2 < 4\rho_6 \\ \frac{m}{2} \ln \frac{|q_2^2 + \rho_5 q_2 + \rho_6|}{|q_1^2 + \rho_5 q_1 + \rho_6|} - \frac{2n - \rho_5 m}{\sqrt{\rho_5^2 - 4\rho_6}} \left(\operatorname{artanh} \frac{2q_2 + \rho_5}{\sqrt{\rho_5^2 - 4\rho_6}} - \operatorname{artanh} \frac{2q_1 + \rho_5}{\sqrt{\rho_5^2 - 4\rho_6}} \right) & \text{if } \rho_5^2 > 4\rho_6 \\ \frac{m}{2} \ln \frac{|q_2^2 + \rho_5 q_2 + \rho_6|}{|q_1^2 + \rho_5 q_1 + \rho_6|} - \frac{2n - \rho_5 m}{2q_2 + \rho_5} + \frac{2n - \rho_5 m}{2q_1 + \rho_5} & \text{if } \rho_5^2 = 4\rho_6 \end{cases}$$

and

$$\begin{aligned} m &= \rho_3 - \rho_1 \rho_6 - \rho_5 \rho_2 + \rho_1 \rho_5^2 \\ n &= \rho_4 - \rho_2 \rho_6 + \rho_1 \rho_5 \rho_6 \end{aligned}$$

We already know the coefficients through Equations 15 to 18. We tested the second approach by comparing the results to what we got from numeric computations, and they are very close.