Risk Aversion, Risk Premia, and the Labor Margin with Generalized Recursive Preferences

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Coefficient of Relative Risk Aversion

Suppose a household has preferences:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t),$$

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta l_t$$

What is the household’s coefficient of relative risk aversion?
Coefficient of Relative Risk Aversion

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What is the household’s coefficient of relative risk aversion?

Answer: 0
Suppose a household has preferences:

\[
E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, l_t), \]

\[
u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1 - \gamma} - \eta \frac{l_t^{1+\chi}}{1 + \chi}
\]

What is the household’s coefficient of relative risk aversion?

Answer: \[
\frac{1}{\gamma + \chi}
\]
Empirical Relevance of the Labor Margin

Imbens, Rubin, and Sacerdote (2001):
- Individuals who win a lottery prize reduce labor supply by $.11 for every $1 won (note: spouse may also reduce labor supply)

Coile and Levine (2009):
- Older individuals are 7% less likely to retire in a given year after a 30% fall in stock market

Coronado and Perozek (2003):
- Individuals who held more stocks in late 1990s retired 7 months earlier

Large literature estimating wealth effects on labor supply (e.g., Pencavel 1986)
Household with Generalized Recursive Preferences

Household chooses state-contingent \( \{(c^t, l^t)\} \) to maximize

\[
V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta \left( E_t V(a_{t+1}; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}
\]
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\]

Note: Generalized recursive preferences are often written as:

\[
U(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} \left[ \tilde{u}(c_t, l_t)^{\rho} + \beta \left( E_t U(a_{t+1}; \theta_{t+1})^{\tilde{\alpha}} \right)^{\rho/\tilde{\alpha}} \right]^{1/\rho}
\]
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U(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} \left[ \tilde{u}(c_t, l_t)^{\rho} + \beta \left( E_t U(a_{t+1}; \theta_{t+1})^{\tilde{\alpha}} \right)^{\rho/\tilde{\alpha}} \right]^{1/\rho}
\]

It’s easy to map back and forth from \( U \) to \( V \); moreover,

- \( V \) is more closely related to standard dynamic programming results, regularity conditions, and FOCs
- \( V \) makes derivations, formulas in the paper simpler
- additively separable \( u \) is easier to consider in \( V \)
Household with Generalized Recursive Preferences

Household chooses state-contingent \( \{ (c_t, l_t) \} \) to maximize

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V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta \left( E_t V(a_{t+1}; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)}
\]

subject to flow budget constraint

\[
a_{t+1} = (1 + r_t) a_t + w_t l_t + d_t - c_t
\]

and No-Ponzi condition.
Household with Generalized Recursive Preferences

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\{w_t, r_t, d_t\} are exogenous processes, governed by \theta_t.
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\( \{w_t, r_t, d_t\} \) are exogenous processes, governed by \( \theta_t \).

State variables of the household’s problem are \( (a_t; \theta_t) \).

Let:

\[
c^* = c^*(a_t; \theta_t),
\]

\[
l^* = l^*(a_t; \theta_t).
\]
Technical Conditions

**Assumption 1.** The function \( u(c_t, l_t) \) is increasing in its first argument, decreasing in its second, twice-differentiable, and strictly concave.

**Assumption 2.** Either \( u: \Omega \to [0, \infty) \) or \( u: \Omega \to (-\infty, 0] \).

**Assumption 3.** A solution \( V:X \to \mathbb{R} \) to the household’s generalized Bellman equation exists and is unique, continuous, and concave.

**Assumption 4.** For any \((a_t; \theta_t) \in X\), the household’s optimal choice \((c^*_t, l^*_t)\) exists, is unique, and lies in the interior of \( \Gamma(a_t; \theta_t) \).

**Assumption 5.** For any \((a_t; \theta_t) \) in the interior of \( X \), the second derivative of \( V \) with respect to its first argument, \( V_{11}(a_t; \theta_t) \), exists.
Assumptions about the Economic Environment

**Assumption 6.** The household is infinitesimal.

**Assumption 7.** The household is representative.

**Assumption 8.** The model has a nonstochastic steady state, $x_t = x_{t+k}$ for $k = 1, 2, \ldots$, and $x \in \{c, l, a, w, r, d, \theta\}$.

**Assumption 8'.** The model has a balanced growth path that can be renormalized to a nonstochastic steady state after a suitable change of variables.
Arrow-Pratt in a Static One-Good Model

Compare:

\[ E \, u(c + \sigma \varepsilon) \quad \text{vs.} \quad u(c - \mu) \]
Arrow-Pratt in a Static One-Good Model

Compare:

\[ E u(c + \sigma \varepsilon) \quad \text{vs.} \quad u(c - \mu) \]

Arrow-Pratt coefficient of absolute risk aversion:

\[ \lim_{\sigma \to 0} \frac{2\mu(\sigma)}{\sigma^2} \]
Consider a one-shot gamble in period $t$:

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t + \sigma \varepsilon_{t+1},$$

vs.

$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu.$$
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$$a_{t+1} = (1 + r_t)a_t + w_t l_t + d_t - c_t - \mu.$$ 

**Definition 1.** *The household’s coefficient of absolute risk aversion at $(a_t; \theta_t)$ is given by $R^a(a_t; \theta_t) = \lim_{\sigma \to 0} 2 \mu(\sigma)/\sigma^2$.***
Proposition 1. The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\), denoted \(R^a(a_t; \theta_t)\), satisfies

\[
-E_t \left[ V(a^*_{t+1}; \theta_{t+1})^{-\alpha} V_{11}(a^*_{t+1}; \theta_{t+1}) - \alpha V(a^*_{t+1}; \theta_{t+1})^{-\alpha-1} V_1(a^*_{t+1}; \theta_{t+1})^2 \right]
\]

\[
E_t \ V(a^*_{t+1}; \theta_{t+1})^{-\alpha} V_1(a^*_{t+1}; \theta_{t+1})
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\]

\[
E_t \frac{V(a^*_{t+1}; \theta_{t+1})^{-\alpha} V_1(a^*_{t+1}; \theta_{t+1})}{E_t V(a^*_{t+1}; \theta_{t+1})^{-\alpha} V_1(a^*_{t+1}; \theta_{t+1})}
\]

Evaluated at the nonstochastic steady state, this simplifies to:

\[
R^a(a; \theta) = - \frac{V_{11}(a; \theta)}{V_1(a; \theta)} + \alpha \frac{V_1(a; \theta)}{V(a; \theta)}.
\]
**Proposition 1.** The household’s coefficient of absolute risk aversion at \((a_t; \theta_t)\), denoted \(R^a(a_t; \theta_t)\), satisfies

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R^a(a; \theta) = \frac{- V_{11}(a; \theta)}{V_1(a; \theta)} + \alpha \frac{V_1(a; \theta)}{V(a; \theta)}.
\]

Solve for $V_1$ and $V_{11}$

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c^*_t, l^*_t).$$

\[ (*) \]
Solve for $V_1$ and $V_{11}$

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c^*_t, l^*_t). \quad (*)$$

Differentiate $(*)$ to get:

$$V_{11}(a_t; \theta_t) = (1 + r_t) \left[ u_{11}(c^*_t, l^*_t) \frac{\partial c^*_t}{\partial a_t} + u_{12}(c^*_t, l^*_t) \frac{\partial l^*_t}{\partial a_t} \right].$$
Solve for $V_1$ and $V_{11}$

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c_t^*, l_t^*).$$

Differentiate ($*$) to get:

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Intratemporal optimality:

$$\frac{\partial l_t^*}{\partial a_t} = -\lambda \frac{\partial c_t^*}{\partial a_t}, \quad \lambda = \frac{wu_{11} + u_{12}}{u_{22} + wu_{12}}.$$

Euler equation and BC:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1 + w\lambda}.$$
Solve for $V_1$ and $V_{11}$

Benveniste-Scheinkman:

$$V_1(a_t; \theta_t) = (1 + r_t) u_1(c_t^*, l_t^*).$$

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Euler equation and BC:

$$\frac{\partial c_t^*}{\partial a_t} = \frac{r}{1 + w\lambda}.$$

**Proposition 3.** The household’s coefficient of absolute risk aversion in Proposition 1, evaluated at steady state, satisfies:

$$R^a(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{r}{1 + w\lambda} + \alpha \frac{r u_1}{u}.$$
Relative vs. Absolute Risk Aversion

Relative risk aversion depends on household wealth.

Household wealth includes:
- financial assets $a_t$
- present value of nonlabor income, $d_t$
- present value of labor income, $w_t l_t$
- maybe present value of leisure, $w_t (\bar{l} - l_t)$?
Relative vs. Absolute Risk Aversion

Relative risk aversion depends on household wealth.

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- present value of labor income, $w_t l_t$
- maybe present value of leisure, $w_t(\bar{l} - l_t)$?

Leisure can be hard to define, e.g.,

$$u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi}$$
Two Coefficients of Relative Risk Aversion

Definition 2. The consumption-wealth coefficient of relative risk aversion, $R^c(a_t; \theta_t) \equiv A_t^c R^a(a_t; \theta_t)$, where $A_t^c$ denotes the present discounted value of household consumption.

At steady state:

$$R^c(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c}{1 + w\lambda} + \alpha \frac{c u_1}{u}.$$ 

Definition 3. The consumption-and-leisure-wealth coefficient of relative risk aversion, $R^{cl}(a_t; \theta_t) \equiv A_t^{cl} R^a(a_t; \theta_t)$, where $A_t^{cl}$ denotes the present discounted value of consumption and leisure.

At steady state:

$$R^{cl}(a; \theta) = \frac{-u_{11} + \lambda u_{12}}{u_1} \frac{c + w(\bar{l} - l)}{1 + w\lambda} + \alpha \frac{(c + w(\bar{l} - l)) u_1}{u}.$$
Expected excess return on asset $i$:

$$\psi_t^i \equiv E_t r_{t+1}^i - r_{t+1}^f$$

$$= -\text{Cov}_t(m_{t+1}, r_{t+1}^i)$$
Expected excess return on asset $i$:

$$
\psi_t^i \equiv E_t r_{t+1}^i - r_{t+1}^f \\
= -\text{Cov}_t(m_{t+1}, r_{t+1}^i)
$$

**Proposition 7.** To first order around the nonstochastic steady state,

$$
dm_{t+1} = -R^a(a; \theta) d\hat{A}_{t+1} + d\Phi_{t+1}
$$

To second order around the nonstochastic steady state,

$$
\psi_t^i = R^a(a; \theta) \text{Cov}_t(dr_{t+1}^i, d\hat{A}_{t+1}) - \text{Cov}_t(dr_{t+1}^i, d\Phi_{t+1})
$$
Economy is a very simple, standard RBC model:

- Competitive firms
- Cobb-Douglas production, \( y_t = Z_t k_t^{1-\zeta} l_t^\zeta \)
- AR(1) technology, \( \log Z_{t+1} = \rho_z \log Z_t + \varepsilon_t \)
- Capital accumulation, \( k_{t+1} = (1 - \delta) k_t + y_t - c_t \)
- Equity is a consumption claim
- Equity premium is expected excess return,

\[
\psi_t = \frac{E_t(C_{t+1} + p_{t+1})}{p_t} - (1 + r^f_t)
\]
Numerical Example: Preferences

Period utility

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi} \]

Generalized recursive preferences

\[ V(a_t; \theta_t) = \max_{(c_t, l_t) \in \Gamma(a_t; \theta_t)} u(c_t, l_t) + \beta \left( E_t V(a_{t+1}; \theta_{t+1})^{1-\alpha} \right)^{1/(1-\alpha)} \]
Numerical Example: Preferences

Period utility

\[ u(c_t, l_t) = \frac{c_t^{1-\gamma}}{1-\gamma} - \eta \frac{l_t^{1+\chi}}{1+\chi} \]

Generalized recursive preferences

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Note:

- IES = 1/\gamma
- If labor fixed, relative risk aversion is \( R^{fl} = \gamma + \alpha(1-\gamma) \)
- Epstein-Zin, Weil define \( \tilde{\alpha} = \gamma + \alpha(1-\gamma) \)
- If labor flexible, relative risk aversion is \( R^c \), depends on \( \chi, \gamma, \alpha \)
Additively Separable Period Utility

- Fixed-labor risk aversion measure, $R^l$ (left axis)
- Equity premium (right axis)
- Coefficient of relative risk aversion $R^c$ (left axis)
Additively Separable Period Utility

![Graph showing the relationship between the coefficient of relative risk aversion and equity premium.](image)

- **Fixed-labor risk aversion measure, \( R_l \) (left axis)**
- **Coefficient of relative risk aversion \( \gamma \)**
- **Equity premium (right axis)**

The graph illustrates the additively separable period utility function, where the coefficient of relative risk aversion and equity premium are plotted against the parameter \( \gamma \). The fixed-labor risk aversion measure is also shown for reference.
Second Numerical Example

Same RBC model as before, with Cobb-Douglas period utility

\[ u(c_t, l_t) = \frac{(c_t^\chi(1-l_t)^{1-\chi})^{1-\gamma}}{1-\gamma} \]

and random-walk technology, \( \rho_z = 1 \).

Note:

- IES = 1/\( \gamma \)
- If labor fixed, risk aversion is \( R^{fl} = (1 - \chi(1 - \gamma)) + \alpha(1 - \gamma) \)
- For composite good, risk aversion is \( R^{cl} = \gamma + \alpha(1 - \gamma) \)
- Epstein-Zin-Weil consider \( \chi = 1 \), define \( \tilde{\alpha} = \gamma + \alpha(1 - \gamma) \)
- Risk aversion \( R^c \) recognizes labor is flexible, excludes value of leisure from household wealth, \( R^c = \chi \gamma + \chi \alpha(1 - \gamma) \)
Cobb-Douglas Period Utility

Fixed-labor risk aversion measure, $R^f$ (left axis)

Coefficient of relative risk aversion $R^c$ (left axis)

Equity premium (right axis)

Coefficient of relative risk aversion $R^c$ (left axis)
Cobb-Douglas Period Utility

Coefficient of relative risk aversion $R^e$ (left axis)

Equity premium (right axis)

Coefficient of relative risk aversion $R^f$ (left axis)

Fixed-labor risk aversion measure, $R^f$ (left axis)
Conclusions

1. A flexible labor margin affects risk aversion
2. Risk premia are related to risk aversion
3. Fixed-labor and composite-good measures of risk aversion perform poorly
4. For multiplier preferences, risk aversion is very sensitive to scaling by \((1 - \beta)\)
5. Simple, closed-form expressions for risk aversion with:
   - flexible labor margin
   - generalized recursive preferences
   - external or internal habits
   - validity away from steady state
   - correspondence to risk premia in the model
6. Ongoing work: frictional labor markets