Contest Functions: Theoretical Foundations and Issues in Estimation

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Abstract

Contest functions (alternatively, contest success functions) determine probabilities of winning and losing as a function of contestants' effort. They are used widely in many areas of economics that employ contest games, from tournaments and rent-seeking to conflict and sports. We first examine the theoretical foundations of contest functions and classify them into four types of derivation: stochastic, axiomatic, optimally-derived, and microfounded. The additive form (which includes the ratio or "Tullock" functional form) can be derived in all four different ways. We also exlore issues in the econometric estimation of contest functions, including concerns with data, endogeneity, and model comparison.

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1 Introduction

Contests are games in which each player exerts effort in order to increase his or her probability of winning a prize. There is a variety of areas of economics and other social sciences in which contests are applied. They include advertising by rival firms (Schmalensee, 1972, 1978), tournaments or influence-activities within organizations (Rosen, 1986; Tsoulouhas, Knoeber, and Agrawal, 2007; Müller and Wärneryd, 2001), patent and other technology races (Reinganum, 1989; Baye and Hoppe, 2003), lobbying and rent-seeking (Tullock, 1980; Nitzan, 1994), litigation (Hirshleifer and Osborne, 2001; Robson and Skaperdas, 2008), wars and other types of conflict (Hirshleifer, 1995, 2000; Garfinkel and Skaperdas, 2007; Levitin and Hausken, 2010), political campaigns (Baron, 1994; Skaperdas and Grofman, 1995), as well as sports (Szymanski, 2003). Konrad (2009) provides an excellent introduction to the basic theory and applications of contests.

How combinations of efforts by the players participating in a contest translate into probabilities of wins and losses is a critical component of a contest game. The functions that describe these probabilities as functions of efforts are often called *contest success* or simply *contest* functions.[1] In terms of their usage, they are analogous to production functions in production theory but they differ from production functions in two important ways. First, the outputs of contest functions are probabilities of wins and losses instead of deterministic outputs. Second, the inputs into contest functions, the efforts of the participating players, are *adversarially* combined so that a player's probability of winning is increasing in her or his effort but is decreasing in the efforts of all the adversaries.

The efforts themselves can be as varied as the particular social or economic environment to which the contest is meant to apply. In the case of tournaments and other intra-organizational competition the efforts are usually denominated in labor time expended. For advertising, lobbying, patent races, litigation, sports, wars, or political campaigns the cost of effort is typically represented by monetary expenditures but the effort itself can be the output of an ordinary production function that is a function of a large number of inputs (purchased with money). For advertising, the efforts can be advertising messages that are produced by means of different types of specialized labor (artistic staff, creative staff, film directors and crews, and so on) and all the capital and other inputs that go together with them. For lobbying, the efforts can be varied from the face-time of lobbyists with political decision-makers to grass-roots organizing, produced by means of different types of labor, capital, and material inputs. For sports, although the direct

[1] We use the simpler second term in this paper even though one of the authors was one of the first users of the former term (Skaperdas, 1996), for reasons that are not clear to him at this time. Probably, he followed Hirshleifer (1989) who used the term "conflict and rent-seeking success functions" in his exploration of different functional forms.

effort is that of the players on the team, how these efforts are combined as well as how the individual players and teams are nurtured, developed, and coached by managerial and coaching staff also clearly matter. This indicates that the ultimate "effort" of a sports team can also be best described by a production function that includes many inputs. For wars, the efforts of the adversaries can be thought of military capacities in the battlefield that are themselves outputs of different types of labor and arming (themselves produced with other inputs).

Contest functions are probabilistic choice functions that, to our knowledge, were first proposed by Luce (1959) in order to study individual choice. Later, and somewhat independently, econometricians developed such functions for the estimation of discrete choice variable (e.g., McFadden, 1974). Friedman (1958) is an early application of the popular "ratio" functional form to an advertising game.

In this paper we first review the different functional forms that have been employed in applications of contests and show how some of them can be derived using four different methods. First, *stochastic* derivations of contest functions start from the supposition that effort is a noisy contributor to some output and comparison of the different outputs of players determines the outcome of the contest. The probit and logit forms are the two most well-known and used forms that can be derived stochastically. Second, *axiomatic* derivations link combinations of properties (or, axioms) of contests to functional forms. The logit form can also be derived axiomatically as a special case of the more general additive form. Third, *optimal-design* derivations suppose that a designer with certain objectives about effort or other variables designs the contest, with the functional form being a result of such a design. Finally, *positive-microfoundations* derive contest functions by characterizing environments in which they naturally emerge as win probabilities of the contestants instead of being consciously chosen by a contest designer. We review incomplete information, search-based and Bayesian representations. By no means do all derivations relate to the different environments that contests have been applied and we will be indicating the areas of applications that each derivation is better suited for. We also we review some econometric issues in the econometric estimation of contest functions.

In the next section, we review the different classes of functional forms that have appeared in the literature and discuss some of their properties. In section 3 we explore the four different types of derivations of contest functions, in section 4 we examine some issues in estimation, and we conclude in section 5.

2 Probit, Logit and other functional forms

Our purpose in this section is to introduce and discuss the properties of different functional forms of contest technologies before exploring their theoretical foundations in the next section.

Consider two adversaries or contestants, labeled 1 and 2. Denote their choice of efforts as e_1 and e_2. We suppose that efforts are themselves outputs of production functions of different inputs as discussed in the introduction. These production functions can be the same for the two adversaries or they can be different. Associated with them are cost functions $c^1(e_1)$ and $c^2(e_2)$. Since we are solely concerned with how pairs of efforts translate into probabilities of wins and losses and not how efforts might be chosen, we will keep these cost and production functions in the background. For any given combination of efforts, each rival has a probability of winning and a probability of losing. Denote the probability of party $i = 1$ winning as $p_1(e_1, e_2)$ and the probability of party $i = 2$ winning as $p_2(e_1, e_2)$.

For the p_i's to be probabilities, they need to take values between zero and one, and add up to one: $p_2(e_1, e_2) = 1 - p_1(e_1, e_2) \geq 0$. Moreover, we can expect an increase in one party's effort to increase its winning probability and reduce the winning probability of its opponent; that is, we should have $p_1(e_1, e_2)$ be strictly increasing in e_1 (when $p_1(e_1, e_2) < 1$) and strictly decreasing in e_2 (when $p_1(e_1, e_2) > 0$).

A class of functions that has been widely examined takes the following additive form:

$$p_1(e_1, e_2) = \begin{cases} \frac{f(e_1)}{f(e_1)+f(e_2)} & \text{if } \sum_{i=1}^{2} f(e_i) > 0; \\ \frac{1}{2} & \text{otherwise,} \end{cases} \tag{1}$$

where $f(\cdot)$ is a non-negative, strictly increasing function. This class has been employed in a number of fields, including in the economics of advertising (Schmalensee, 1972, 1978), sports economics (Szymanski, 2003), rent-seeking (Tullock, 1980; Nitzan, 1994), as well as contests in general (Konrad, 2009).

One unique and appealing feature of the class of contest functions in (1) is that it naturally extends to contests involving more than two parties. Thus, if there were n parties to the contest, denoting the effort of rival i by e_i, and the vector of efforts by all other agents $j \neq i$ by e_{-i}, the winning probability of i would be as follows:

$$p_i\left(e_i, e_{-i}\right) = \begin{cases} \frac{f(e_i)}{\sum_{j=1}^{n} f(e_j)} & \text{if } \sum_{j=1}^{n} f(e_j) > 0; \\ \frac{1}{n} & \text{otherwise.} \end{cases} \tag{2}$$

The most commonly used functional form is the one in which $f(e_i) = e_i^{\mu}$,[2] where $\mu > 0$ (and often, for technical reasons of existence of pure-strategy

[2] A variation on this form is $f(e_i) = ae_i^{\mu} + b$ where $a, b > 0$. Amegashie (2006) examined the properties of this form.

Nash equilibrium, $\mu \leq 1$), so that

$$p_1(e_1, e_2) = \frac{e_1^\mu}{e_1^\mu + e_2^\mu} = \frac{(\frac{e_1}{e_2})^\mu}{(\frac{e_1}{e_2})^\mu + 1}. \tag{3}$$

This functional form, sometimes referred to as the "power" form or as the "ratio" form, is that which was employed by Tullock (1980) and the ensuing voluminous literature on rent-seeking. This is also the workhorse functional form used in the economics of conflict. As Hirshleifer (1989) has noted, the probability of winning in this case depends on the *ratio* of efforts, $\frac{e_1}{e_2}$, of the two parties.

A suitable modification of (1) can accommodate asymmetric effects of contestant efforts on the win probabilities as shown by the following functional form, where $f_i(\cdot)$ is a non-negative, strictly increasing function:

$$p_i(e_1, e_2) = \frac{f_i(e_i)}{f_1(e_1) + f_2(e_2)}. \tag{4}$$

Assuming $f_i(e_i) = a_i f(e_i)$, a particularly convenient version of (4) is given by:

$$p_1(e_1, e_2) = \frac{a_1 f(e_1)}{a_1 f(e_1) + a_2 f(e_2)}, \tag{5}$$

where $a_i > 0$ for every $i = 1, 2$, with $a_i > a_j$ for $i \neq j$ implying that contestant i has an edge over contestant j in transforming its effort into a win probability. Variants of this functional form have been applied to litigation (Hirshleifer and Osborne, 2001; Robson and Skaperdas, 2008) and political campaigns (Baron, 1994) among other settings.

Notice that in all of the above contest functions, one of the contestants always wins the prize. A variation allows one to accommodate the possibility of a "draw", where with a positive probability neither contestant wins the prize as shown by the functional form below:

$$p_1(e_1, e_2) = \frac{f_1(e_1)}{s + f_1(e_1) + f_2(e_2)}, \tag{6}$$

where $f_1(\cdot)$ and $f_2(\cdot)$ are non-negative strictly increasing functions and $s > 0$ is a constant term. As discussed subsequently, this functional form with $s = 1$ has been axiomatized by Blavatskyy (2010). A stochastic derivation is provided by Jia (2009). It has also been investigated as an optimal contest design by Dasgupta and Nti (1998).

Another well-known functional form is the following "logit" specification, in which $f(e_i) = \exp(\mu e_i)$, where $\mu > 0$, so that,

$$p_1(e_1, e_2) = \frac{\exp(\mu e_1)}{\exp(\mu e_1) + \exp(\mu e_2)} = \frac{1}{1 + \exp(\mu(e_2 - e_1))}. \tag{7}$$

Again as Hirshleifer (1989) has noted and as is evident from the expression following the second equal sign in (7), by this specification, the probability of winning depends on the *difference* in efforts between the two parties. This is also the case with the *probit* functional form (see, e.g., Train, 2003; Albert and Chib, 1993), where the win probabilities take the form:

$$p_1(e_1, e_2) = \Phi(e_1 - e_2), \tag{8}$$

In the above expression, Φ is the cumulative distribution function of the standard normal distribution. As discussed subsequently, (8) has stochastic foundations.

Another type of contest functions where the probability of winning depends on the difference in efforts is the so-called "Difference form" as shown below:

$$p_1(e_1, e_2) = \alpha + h_1(e_1) - h_2(e_2), \tag{9}$$

where $\alpha \in (0, 1)$ and the functions $h_1(e_1)$ and $h_2(e_2)$ are suitably constrained so that $p_1(e_1, e_2) \in [0, 1]$. Contest games under specific cases of this class of functions have been explored by Baik (1998) and Che and Gale (2000). Skaperdas and Vaidya (2009) have derived this class in a Bayesian framework as an outcome of an audience's (for example, a judge) inference from "evidence" produced by two contestants, with $h_1(e_1)$ and $h_2(e_2)$ being probabilities. Corchón and Dahm (2010) also derive a particular class of the difference form (similar to that examined by Che and Gale (2000)) in an axiomatic setting in which the contest success function is thought of as a share instead of as a probability.

All the above contest functions are imperfectly discriminating in the sense that with all of them, the prize at stake is awarded probabilistically to one of the contestants with higher effort leading to a higher probability of winning the prize. In the all-pay auction contest, the prize allocation is extremely sensitive to the efforts put in so that the contestant with the highest effort wins the prize with certainty as illustrated by a 2-player all-pay contest function below:

$$p_1(e_1, e_2) = \begin{cases} 1 & \text{if } e_1 > e_2; \\ \frac{1}{2} & \text{if } e_1 = e_2; \\ 0 & \text{if } e_1 < e_2. \end{cases} \tag{10}$$

It has been established that the all-pay auction contest (10) can be understood as a limiting form of (3) as $\mu \to \infty$. Applications of all-pay auctions include ownership structure and non-price competition among firms.[3] As we shall see in the next section, the general additive form as well as the other

[3] Konrad (2000) examines location choice among firms when they engage in non-price competition to attract consumers via an all-pay auction in the second stage. Konrad (2006) examines how ownership structure such as minority holdings affect bidding behavior and profits of firms competing for a prize via an all-pay auction.

contest functions listed above have various stochastic, axiomatic and other theoretical foundations.

3 Theoretical foundations of contest functions

We will consider four types of derivations. Stochastic foundations are based on assumptions about how performance in a contest – in the sense of probabilities of winning and losing – might be a noisy function of efforts. Axiomatic foundations are derived from general properties (or, axioms) that a contest function might be expected to have and the implications of combinations of such properties would be expected to have for functional forms. We also explore the circumstances under which some of the above contest functions can be justified as optimal choices from a contest designer's perspective. Given the wide variety of applications of some contest functions (such as the additive form) in areas such as lobbying, political campaigns and research tournaments, we also survey the literature that provides Bayesian inferential and other "positive" micro-underpinnings that can justify the usage of these functions in the relevant contexts.

3.1 Stochastic derivations

The stochastic derivation relies on the basic premise that the outcome of a particular contest can be thought of as a noisy function of the two rivals' efforts. In particular, we can posit that each rival's "performance", denoted by Y_i, is a function of his effort and noise so that $Y_i = h(e_i, \theta_i)$ where θ_i represents a random variable and $h(\cdot,\cdot)$ is a function of the two variables. Then, the probability of side 1 winning can be represented by the probability that its performance is higher than that of its adversary so that:

$$p_1(e_1, e_2) = \Pr[Y_1 > Y_2] = \Pr[h(e_1, \theta_1) > h(e_2, \theta 2)]$$

From this stochastic perspective, each side's probability of winning depends not only on the efforts of both sides, but also on the functional form of $h(\cdot,\cdot)$ and the distribution of the θ_is'.[4]

The most commonly used form of $h(\cdot,\cdot)$ is the linear form so that $h(e_i, \theta_i) = e_i + \theta_i$. In that case, when the θ_i's are independently identically distributed according to the normal distribution, the resultant probabilities of winning and losing for the two sides are described by (8), the *probit* form (see, e.g., Train, 2003; Albert and Chib, 1993), where Φ is the cumulative

[4] As discussed later, Corchón and Dahm (2010) provide a similar derivation of various contest functions, except that in their setting, the performance of each contestant is actually the utility of the prize awarder from giving the prize to that contestant and randomness arises out of incomplete information on the part of contestants about a parameter governing the utility function of the awarder for which every contestant has a common prior.

distribution function of the standard normal distribution. Most likely because there is no analytical functional form representing the probit, it has not been used as much in the contest literature as for (3) or its more general form in (1).

Still when $h(e_i, \theta_i) = e_i + \theta_i$ but the θ_i's are independently identically distributed according to the extreme value distribution, the n-player version of the logit form in (7) is obtained (McFadden, 1974):

$$p_i(e_i, e_{-i}) = \frac{\exp(\mu e_i)}{\sum_{j=1}^{n} \exp(\mu e_j)}. \tag{11}$$

The cumulative distribution function of the (type I) extreme value distribution is

$$G_{\theta_i} = \exp(-\exp(-z)),$$

which has been known as the double exponential distribution (Yellott, 1977; Luce, 1977) or the log-Weibull distribution. The above discussion suggests that the stochastic derivation is particularly amenable to contest functions where win probabilities depend on differences in efforts.

However, motivated by the derivations of the probit and logit forms, Jia (2008b) provides a stochastic interpretation of the ratio form (3) by assuming the performance function $h(\cdot, \cdot)$ has the multiplicative form $h(e_i, \theta_i) = e_i\theta_i$. This result also extends to the n-party contest model:

$$p_i(e_i, e_{-i}) = \frac{e_i^{\mu}}{\sum_{j=1}^{n} e_j^{\mu}}. \tag{12}$$

Jia (2008b) shows that, for $n > 2$, the contest model has the generalized ratio form (12) if and only if the independent random shocks $\{\theta_i\}_{i=1}^{n}$ have a specific distribution, which is known as the *Inverse Exponential Distribution*.

Specifically, a random variable belongs to the *Inverse Exponential Distribution with Parameters α and μ, $\alpha, \mu > 0$* [$IEXP(\alpha, \mu)$ for short] if and only if its probability density function (p.d.f.) has the form:

$$g(z) = \alpha\mu z^{-(\mu+1)} \exp(-\alpha z^{-\mu}) I_{[z>0]}, \tag{13}$$

where I is the indicator function which is equal to 1 when $z > 0$ and 0 otherwise. Accordingly, the cumulative distribution function (c.d.f.) of $IEXP(\alpha, \mu)$ is

$$G(z) = \int_0^z h(s)ds = \exp(-\alpha z^{-\mu}).$$

For an $IEXP(\alpha, \mu)$ distributed random variable, one can verify that neither its expectation nor variance exist, and its mode is located at $(\frac{\mu\alpha}{\mu+1})^{1/\mu}$. When α increases, its p.d.f. becomes flatter, and more and more mass is being pushed to the right. The parameter μ plays an opposite role. When μ decreases, the p.d.f. becomes flatter.

Jia (2008b) interprets the parameter μ in (11) and (12) as the "noise" of the contest. Common to all contestants, the parameter μ captures the marginal increase in the probability of winning caused by a higher effort. Contests with low μ can be regarded as poorly discriminating or "noisy" contests. When μ converges to zero, the contest outcome converges to a random lottery with no dependence upon the efforts of the adversaries. Conflicts with high μ can be regarded as highly discriminating; as μ approaches infinity, the contest outcome is determined by an all-pay auction of the type in (10).

A lucid interpretation of the power μ, and an alternative derivation of (12) from (13), has been given by Fu and Lu (2008), which draws an equivalence between contest games and research tournaments.

The stochastic approach developed in Jia (2008b) can be easily extended to the general additive contest model (2). As such the logit form (11) is isomorphic to the ratio or power form (12) up to a logarithmic transformation. In addition, asymmetric functional form such can be rationalized by relaxing the assumption that all the random variables θ_is' are identically distributed.

Jia (2009) extends the stochastic approach further to allow for the possibility of a draw (or, stalemate) in contest, which corresponds to the situation that no party can force a win. This is accomplished by introducing a "threshold" c into the performance comparing process. The intuition is simple. A draw can arise if every performance comparison is decided by estimates of the difference in the adversaries' performances with error and is a draw if this difference is smaller than a "threshold" value $c > 0$. Indeed, in most contests, the outcomes are not determined by each party's performance, but by measures of their performances, which is the process of estimating the magnitude of all parties' performances against some unit of measurement. Adopting the assumptions that (1) adversaries' performances are determined by their efforts and some random variables θ_is', and (2) the random variables are independently and identically distributed with an *inverse exponential distribution*, Jia (2009) derives the following functional forms:

$$p_i(e_i, e_{-i}) = \frac{f(e_i)}{f(e_i) + c\sum_{j\neq i} f(e_j)}, \quad c > 1, \tag{14}$$

and

$$p_i(e_i, e_{-i}) = \frac{f(e_i)}{(n-1)c + \sum_{j=1}^{n} f(e_j)}, \quad c > 0. \tag{15}$$

Notice that (15) is essentially the n-player version of (6) with $f_i(\cdot) = f(\cdot)$, and $s = (n-1)c$. Again, by relaxing the i.i.d. assumption to require only independence, one can easily obtain more general asymmetric forms.

3.2 Axiomatic foundations

Luce (1959) first axiomatized probabilistic choice functions such as those in

(2) relation to utility theory, while Skaperdas (1996) provided an axiomatization in relation to contests and conflict. Key to both axiomatizations is an *Independence of Irrelevant Alternatives* property. This property requires that the outcome of a contest between any two parties depend only on efforts of these two parties and not on the efforts of any third parties to the contest.

The particular "ratio" form in (12) has the property of *homogeneity of degree zero* in efforts, or $p_i(te_i, te_{-i}) = p_i(e_i, e_{-i})$ for all $t > 0$. This is an analytically convenient property and likely accounts for the popularity of this functional form in applications.

The "logit" form (11) can be derived under the property that each adversary's probability of winning is invariant to the addition of a constant D to the effort of each adversary (i.e., $p_i(e_i + D, e_{-i} + D) = p_i(e_i, e_{-i})$ for all D such that $e_j + D > 0$ for all j).[5] Though the logit form also has analytical advantages, it has not been used as much as the power form shown in (3) because for a number of well-specified models, no pure-strategy Nash equilibrium exists.

Thus, both the "ratio" functional form in (3) and the "logit" form in (7) can be derived axiomatically as well as stochastically.

The class in (1) and the specific forms in (3) and (7) have the property of symmetry or anonymity, in the sense that if the efforts of two adversaries were switched, their probabilities of winning would switch as well. Consequently when two adversaries have the same efforts, they have equal probabilities of winning and losing. There are circumstances, however, in which one party might be favored over another even though they might have the same levels of effort. This could be due to the awarder having a bias in favor of one opponent over the other or one of the contestants may have a natural advantage (such as the side arguing for the "truth" in a litigation trial as in Hirshleifer and Osborne (2001)). A simple way to extend (1) to take account of such asymmetries is via (5). Note that when the adversaries have the same efforts, $e_1 = e_2$, 1's probability of winning equals $\frac{a_1}{a_1+a_2}$ and 2's probability of winning is $\frac{a_2}{a_1+a_2}$. Consequently, when $a_1 > a_2$, 1 has the advantage, whereas when $a_1 < a_2$, 2 has the advantage. Clark and Riis (1998) have axiomatized this asymmetric form for the case of the ratio form (i.e., where $f(e) = e^{\mu}$). Rai and Sarin (2009) have provided more general axiomatizations of this that also allow for the function $f(\cdot)$ to be one of many inputs and not just of effort (which we have assumed, in general, to be of other inputs as well). Finally, Münster (2009) provided a reinterpretation and extension of the axioms in Skaperdas (1996) and Clark and Riis (1998) by allowing contestants to be members of groups.

For functional forms that allow for the possibility of a draw, Blavatskyy

[5] Hirshleifer (1989, 1995, 2000) provide many insightful discussions of contest technologies and comparisons of the functional forms in (3) and (7).

(2010) axiomatized (6) with $s = 1$. However, restricting s to 1 does not have any special significance because if we were to multiply the numerator and denominator of (6) with $s = 1$ by any positive number we would get an equivalent functional form.[6] One way of thinking about (6) is to consider a third party, say "Nature," that has a constant effort, e', which is defined by $f(e') = 1$ (where $f(\cdot)$ is non-negative and increasing). When Nature "wins," a draw occurs.[7] Blavatskyy (2010) has extended (6) to more than 2 adversaries but not in the straightforward way that (2) extends (1).

3.3 Optimal-design derivations

A central feature of any imperfectly discriminating contest function is the stochastic awarding of the prize to one of the contestants as opposed to an all-pay auction contest where the contestant with the highest effort wins the prize with certainty. In some cases, as in some forms of rent-seeking, the prize awarder might also be able to design the contest. A natural question that arises is why should the contest designer prefer to allocate the prize stochastically and what form of the contest function might be optimal in this regard? The literature that studies this question of optimal contest design suggests that the answer depends on the objective of the contest designer, the choice set of contest functions at his disposal and the extent of heterogeneity among contestants.

Epstein and Nitzan (2006) study contest design in a lobbying context with two contestants (denoted by 1 and 2 respectively) where the "prize" at stake is a discrete policy choice made by the government between status quo or policy change. The government needs to choose between randomizing its policy choice via the ratio functional form (3) with $\mu \leq 2$ or going for an all-pay auction as per (10) or outrightly choosing a policy in favor of one contestant. This choice critically depends on the government's objective function $U(.)$ which they assume to be of the form:

$$U(\cdot) = \alpha(E(w_1) + E(w_2)) + (1 - \alpha)(e_1 + e_2).$$

Notice that in general $U(\cdot)$ not only depends on aggregate effort $(e_1 + e_2)$ but also on the welfare of the contestants where $E(w_1) = \mathrm{Pr}_1(\cdot)v_1 - e_1$ represents the expected net payoff to interest group 1 that values its preferred policy stance at v_1 and its realization depends on the policy choice method used by the government (as embodied in probability of selection of its favored policy Pr_1). Analogously, $E(w_2) = (1 - \mathrm{Pr}_1(\cdot))v_2 - e_2$ represents the expected net payoff of interest group 2 from adoption of its favored policy which it

[6]Just define $f_i(e_i) \equiv \frac{1}{(n-1)c} f(e_i)$. Also note that (6) is essentially the same as the 2-player asymmetric version of (15) derived stochastically by Jia (2009).

[7]Dasgupta and Nti (1998) show how a linear symmetric version of (6) with $f_i(\cdot) = ae_i$, $a > 0$, can be an optimal choice of contest technology for the prize awarder when the "draw" results in him retaining the prize.

values at v_2.We assume that $v_1 = bv_2$ where $b \geq 1$ so that the interest group 1 has a higher stake. It is implicitly assumed that either interest group does not derive any benefits if their favored policy is not chosen.

From $U(\cdot)$, it is straightforward to see that if $\alpha = 1$ so that the government only cared about interest groups' welfare and not their efforts, then it would never randomize and always choose against the status quo in favor of interest group 1 as $b \geq 1$. More generally, the government would prefer to set up a contest and *not* choose outrightly in favor of 1 only if: $\alpha(E(w_1) + E(w_2)) + (1 - \alpha)(e_1 + e_2) > \alpha b v_2$, which gives the following condition:

$$\frac{(1-2\alpha)}{\alpha}\frac{(e_1+e_2)}{(1-\Pr_1)} > (b-1)v_2.$$

Using the above condition, they show that the all-pay auction is preferred over both (3) with $\mu \leq 1$ and an outright decision in favor of 1 when $\alpha < \frac{1}{3}$. Hence in this setting the usage of the standard Tullock lottery contest ((3) with $\mu = 1$) can be justified only when an all-pay auction is not feasible. However, they show that (3) with $2 > \mu > 1$ can be the policy maker's optimal choice and preferred over the all-pay auction even when $\alpha < \frac{1}{3}$, if for a given μ, the stake asymmetry b is sufficiently large or for a given b the return to lobbying μ is sufficiently high. Hence, the paper provides a partial justification for usage of (3) with $2 > \mu > 1$ in the lobbying context when the relative weight α, stake asymmetry b and return to lobbying μ are in a certain range. However, it also casts a doubt over the validity of widespread use of the standard Tullock lottery contest from an optimal choice perspective.

In contrast to Epstein and Nitzan (2006), Dasgupta and Nti (1998) examine a case where the prize awarder's utility does not depend on the utilities of the contestants per se but rather on their total effort. Further, they allow for the prize awarder to derive value v_0 from the prize if it remains unallocated. Assuming $n \geq 2$ symmetric contestants, in their setting, the awarder's choice problem involves selecting among additive concave contest functions as given by an n-player symmetric version of (6) where $f_i(\cdot) = f(\cdot)$ with $f(\cdot)$ restricted to being a twice differentiable, increasing and concave function, $f(0) = 0$ and $s \geq 0$. Notice that the term s allows the awarder to retain the prize with a positive probability. In this set up, they show that for any given, n, v, s, the second stage symmetric contestants' equilibrium induced by any specific concave $f(\cdot)$ can be exactly equivalently induced by a linear specification $\widetilde{f}(\cdot) = ae_i + b$ where $a > 0$, $b \geq 0$. Hence in determining the optimal contest function in the first stage, it is sufficient to focus on the linear specification with a, b being the choice variables. Next, they show that for any given s, the optimal contest function always takes the homogenous linear form with $f(\cdot) = ae_i$. They fully characterize the optimal choice of both a and s for the case of a risk neutral contest designer. Their results are intuitive. They show that when v_0 is very small so that the contest designer cares very little about the prize and mostly about total effort, he chooses $s = 0$ and $a = 1$ so that the optimal contest function is (12) with $\mu = 1$ (which is the standard Tullock lottery contest). When the contest designer cares about the prize sufficiently, the optimal s is positive but non-unique and the size of a is influenced by his valuation of the prize. A higher valuation of the prize induces the designer to reduce a, as although doing so induces lower aggregate effort, it also boosts his probability of retaining the prize.

Nti (2004) further simplifies the contest designer's objective function to assume that he only cares about maximizing aggregate effort. With this assumption, the paper examines 2 player contests where players may differ in their valuations of the prize ($v_1 \geq v_2$). Player asymmetry makes the problem of eliciting effort complicated as a player with the lower valuation has a lower incentive to put in effort which in turn allows the higher stake player to economize on its effort too. Hence asymmetric player valuations might induce the contest designer to introduce a corresponding asymmetry in the contest functions to mitigate the negative effect on total effort. Indeed, when the contest designer's domain is restricted to the class of concave asymmetric functions as in (4), the paper shows that the linear asymmetric function as per (5) with $f(e_i) = e_i$ is optimal with $a_2 > a_1$ where contestant 2 with a lower stake is assigned a higher weight in the contest to boost his incentive to put in effort. The paper also shows, that if the contest designer's domain was unrestricted, his optimal contest design would be an all-pay auction with a reservation price equal to the higher valuation, v_1 so that:

$$p_1(e_1, e_2) = \begin{cases} 1 & \text{if } e_1 \geq v_1 \text{ and } e_1 > e_2; \\ \frac{1}{2} & \text{if } e_1 = e_2 \geq v_1; \\ 0 & \text{if } e_1 < v_1. \end{cases}$$

This finding further supports Epstein and Nitzan (2006) claim of the all-pay auction tending to be the superior mechanism for eliciting aggregate effort in a two player framework.

Franke, Kanzow, Leininger, and Väth (2009) also assume that the objective of the contest designer is to solely maximize aggregate effort ($\sum_i e_i$) and extend Nti (2004) to allow for $n \geq 2$ potentially heterogenous contestants. They restrict attention to the n-player linear version of the additive asymmetric contest function (5)) with $f(e_i) = e_i$. They determine for any given number of players and their valuation profiles, the optimal subset of active players and their corresponding contest weights a_i that would maximize the equilibrium aggregate effort. An interesting insight is that when players have asymmetric valuations, the contest designer may want to restrict the number of active participants and keep out players with very low valuations. However, they show that the optimal set of active players for $n \geq 3$ case always involves at least 3 players. This suggests that starting from a 2 player contest, the contest designer can always benefit by allowing at least one more player. However, in general, admitting an extra contestant need not always be for the better. Further they show as in Nti (2004) that among active players, the optimal weights a_i are biased in favor of the weaker contestants to reduce the extent of heterogeneity among the participants in order to boost the total equilibrium effort.

Throughout this discussion, we have assumed that the contest function itself is a choice variable and is optimally chosen by the prize awarder given his choice set. Next, we examine the branch of the literature which characterizes environments in which various contest functions naturally emerge as win probabilities of the contestants instead of being consciously chosen by a contest designer.

3.4 "Positive" microfoundations

3.4.1 Incomplete information and search based foundations

Corchón and Dahm (2010) provide an alternative positive justification of the randomness implicit in an imperfectly discriminating contest function. In their framework, the decision maker always decides deterministically in favor of a contestant i as long as $U_i(B_i(\theta), e_i) > U_j(B_j(\theta), e_j)$ for $i \neq j$ where $U_i(\cdot)$ represents his utility from supporting contestant i, that is increasing not only in e_i but also in a state-dependent component $B_i(\theta)$ which depends on the realized value of a parameter θ distributed uniformly over the interval $[0, 1]$. They assume that the realized value of θ is only known to the decision maker and not to the competing contestants. From the contestants' perspective, it is this uncertainty over θ that creates a randomness in the allocation of the prize. This justification is similar in spirit to the foundations provided for probabilistic voting models in the political science literature where competing political candidates are uncertain about voter preferences and therefore about how their chosen policy positions would affect voters' choice. In a two contestant setting, they show that any contest function, $p_i = p_i(e_1, e_2)$ that satisfies intuitive regularity conditions such as $\partial p_i/\partial e_i > 0, \partial p_i/\partial e_j < 0$ for all $j \neq i$ and some reasonable boundary conditions including $p_i \to 1$ when $e_i \to \infty$ and $p_i \to 0$ when $e_i \to 0$ is always rationalizable by a suitable pair of payoff functions U_i, as long as $B_i(\theta)$ satisfies a single crossing condition.[8] Hence they show that the general asymmetric additive form as in (4) can be rationalized using a multiplicative pair of utility functions for the decision maker $U_1(B_1(\theta), e_1) = (1-\theta)f_1(e_1)$ and $U_2(B_2(\theta), e_2) = \theta f_2(e_2)$. The various versions of difference form contest function as in (9) can be rationalized via additive forms of utility functions such as $U_i(B_i(\theta), e_i) = B_i(\theta) + a_i e_i + A_i$, where a_i and A_i are constants. Unfortunately, they find that with more than two contestants, in general, this

[8]This condition requires that θ' that solves $U_i(B_i(\theta'), e_i) = U_j(B_j(\theta'), e_j)$ for $i \neq j$, is well defined and unique.

approach cannot be used to rationalize the n-player versions of the above contest functions. However, through an innovative application of the Salop (1979) circular city model, they show how the difference form (9) can be rationalized for multiple contestants when the prize awarder's utility function takes the specific form $U_i(\theta, e_i) = u - k|l_i - \theta| + e_i^\alpha$ with u, k, and $\alpha \leq 1$ being constants, l_i representing each contestant's fixed position on a unit circle and θ representing the decision maker preferred position on this circle which is unknown to the contestants with them having a common uniform prior over it.

An interesting commonality between the optimal contest design literature surveyed above and Corchón and Dahm (2010) is that in both these frameworks, it is assumed that the efforts put in by the contestants directly contribute to the utility of the decision maker. This is easiest to appreciate when the efforts can be interpreted as bribes or transfers which is plausible in certain rent-seeking contexts. However, such contest functions have also been used in various other settings such as trials, research tournaments and advertising where the contestant efforts need to be interpreted differently. In an important work, Baye and Hoppe (2003) extend the work of Fullerton and McAfee (1999) to show that a Tullock contest game can be naturally interpreted as a reduced form version of an innovation tournament or patent race where the effort put in by the contestants are expenditures on research trials and the randomness in the contest can be understood as a natural outcome of the stochastic nature of innovation outcomes. They establish such equivalence across innovation tournaments and patent races.

Let us consider their innovation tournament model first. In such a game, $n \geq 2$ firms compete to establish their monopoly over a new product via discovering the most valued design. In the first stage, a firm i simultaneously determines how many parallel trials or experiments e_i, $i = 1, 2, \ldots, n$, it wants to invest to search for the most valuable design or idea, where each trial costs c and it represents an independent draw from a distribution of values over the interval $[0, 1]$ with an associated distribution function F. At the end of this stage, the outcomes of all parallel trials run by firms are realized to them and each firm puts forward its best idea z_i to the patent allocating bureaucrat who then allocates the patent deterministically to the firm with the highest valued idea while the others gain nothing. The paper establishes that such an innovation game is equivalent to an n-player Tullock contest game with the form in (12) where $\mu = 1$ except that the effort choices $e_i \in \{0, 1, 2, \ldots\}$ are discrete and the prize is given by:

$$v(e_1, e_2, \ldots, e_n) = \frac{1}{c}\left[1 - \int_0^1 [F(z)]^{\sum_j e_j} dz\right],$$

which is increasing in individual effort (e_i), rivals' efforts ($\sum_{j \neq i} e_j$), and total efforts ($\sum_j e_j$), but at a decreasing rate (via exponent of the the distribution function) rather than being fixed. However, they show that via a suitable transformation of the distribution function, even the fixed prize case can be shown to be strategically identical to an innovation tournament albeit with discrete effort choice.

The paper also examines patent races where instead of the most highly valued design becoming the winner, a prize of fixed value v goes to a firm that produces the design first. In this setting, e_i (representing research expenditures incurred by firm i) influences the "hazard rate" $f(e_i) \in [0,1]$ which gives firm i's conditional instantaneous win probability between time t and $t+dt$ given that no innovation has occurred at or before t. They show that player i's payoff in such a patent race can be expressed as

$$\pi_i(e_1, e_2, \ldots, e_n) = v \frac{f(e_i)}{\sum_{j=1}^{n} f(e_j) + r} - e_i.$$

In the above expression, r represents the discount rate. Given that $f(\cdot)$ is defined as an increasing and concave function over the interval $[0,1]$, they show that a patent race game indeed becomes strategically equivalent to a contest game with the form in (12) as r approaches 0 when $f(e_i) = e_i^{\mu}$, $\mu \leq 1$, and the prize $v \leq n^2/n-1$. Notice that while the restriction of discrete effort choice does not apply here, the equivalence does impose restrictions on the sensitivity of the contest function to efforts and the size of the prize.

Such strategic equivalence has also been established between contest games and various types of market interactions among firms. Szidarovszky and Okuguchi (1997) establish strategic equivalence between the n-player version of the additive asymmetric contest game involving (4) and Cournot competition among firms under certain specific demand and cost conditions. Similarly, Menezes and Quiggin (2010) interpret the Tullock contest game involving (12) with $\mu = 1$ as a reduced form oligopsonistic competition among firms in an input market, when the relevant input supply curve takes a specific form. More broadly, they argue that the Tullock contest can be understood as competition among players in markets for "influence inputs". Interpreting the contest this way allows one to consider competition among players via alternative choice variables, such as quantities purchased or prices offered apart from total expenditures as in the Tullock contest. They show that the extent of rent-dissipation can depend on assumed choice variable for the competitors.

3.4.2 Bayesian foundations

We now turn to exploring the foundations for contest functions used to capture outcomes of litigation, advertising, lobbying and electoral competitions where competing parties put in efforts to persuade a relevant audience about the correctness of their competing views or assertions. In these contexts, the expended efforts are not bribes or transfers but help produce arguments and evidence towards supporting the contestants' cases. Skaperdas and Vaidya

(2009) show how the contest functions representing win probabilities can be justified as natural outcomes of an audience's Bayesian inference process in a trial setting involving a plaintiff (P), a defendant (D) and a judge.[9] While their model uses a judicial setting, the framework can be extended to other settings as well such as consumer choice and voting where a decision-maker needs to decide in favor of one party or another and is subject to persuasion from the competing parties. Below we discuss their approach in greater detail.

In Skaperdas and Vaidya (2009), a judge uses Bayesian inference to evaluate the guilt of D based on the evidence presented by the two competing parties, P and D. In the first stage, both P and D expend efforts e_p and e_d respectively to gather evidence favorable to their cause. As a result, via evidence production functions, each side may obtain a piece of evidence E_i (valued on a $(0, \infty]$ scale) in its favor where $i = p, d$, which it then presents to the court. The evidence production functions can be either deterministic or stochastic. In the second stage, based on the evidence pair observed, the court assesses the likelihood ratio of guilt L^g and using it along with its prior belief about guilt π, it determines its posterior probability of guilt π^* using Bayes' rule, so that:[10]

$$\pi^* = \frac{\pi L^g}{(1-\pi) + \pi L^g} \tag{16}$$

It is worth noting that the court is "limited world" Bayesian as it neither observes the efforts chosen by either parties nor the underlying evidence production functions. Hence the only additional information it receives over and above its priors is the evidence pair presented at the trial.[11] In contrast, it is assumed that P and D are aware of the court's inference process.

When the evidence production process is deterministic, the value of evidence obtained by contestant j, is given by $E_j = F_j(e_j)$ with $j = p, d$, where $F_j(.)$ is deterministically and monotonically increasing in resources. Apart from resources, the "truth" is also expected to play a role in evidence production so that for the same amount of resources, a contestant j arguing for the truth would have an edge so that $F_j(e_j) > F_i(e_i)$, $i, j = p, d$ with

[9] For an alternative axiomatic non-Bayesian approach to decision making by a trial court, see Daughety and Reinganum (2000b,a), where it is assumed that rules of evidence and procedure impose sufficient constraints on a court so as to not allow its judgment to be affected by priors. Their axioms are guided by the notion of procedural fairness and lead to an one-parameter family of functions that aggregates the submitted evidence into the court's assessment. For an analysis of "naive" Bayesian decision-making by juries, see Froeb and Kobayashi (1996).

[10] By definition, $L^g = \frac{\Pr(E_p, E_d | G)}{\Pr(E_p, E_d | I)}$. It is assumed that this ratio can be subjectively constructed directly as stated by Kadane and Schum (1996) (p.127) in making a holistic assessment of the probative force of evidence.

[11] This assumption is supported broadly by considerable amount of research in psychology and related areas (for an overview of research in this area, see Cialdini, 2001).

$i \neq j$ when $e_i = e_j$.[12] Using these functions, and assuming that the court's determination of L^g takes a *power-law form*:

$$L^g(E_p, E_d) = \lambda \left(\frac{E_p}{E_d}\right)^{\mu},$$

where $\lambda, \mu > 0$, the court's posterior probability of guilt is shown to be the additive form:

$$\pi^*(E_p, E_d) = \frac{\pi F_p(e_p)}{(1-\pi)F_d(e_d) + \pi F_p(e_p)}$$

when $\lambda = \mu = 1$.[13]

To go from π^* to the contest functions that capture the win probabilities as perceived by P and D, they consider two alternative decision rules that the court might use to arrive at its verdict in stage 3. One rule explored is that the court makes a probabilistic decision on guilt (akin to tossing an unfair coin) as follows:

$$\text{Choose } G \text{ with probability } \pi^* \text{ and } I \text{ with probability } 1-\pi^*. \qquad \text{(Rule 1)}$$

Another rule considered is "guilty beyond a reasonable doubt" or just "choose guilty if and only if there is a better than even chance of guilt," as given by:

$$\text{Choose } G \text{ if and only if } \pi^*(e_p, e_d) > \gamma \text{ where } \gamma \in (0,1). \qquad \text{(Rule 2)}$$

It is shown that under both the rules, the win probabilities of the contestants can take the additive asymmetric functional form in (5) where the bias parameters a_i are influenced by π under rule 1 and by the threshold γ under rule 2. To get to the additive contest function under rule 2, it is assumed that contestants have a uniform distribution over π, instead of knowing its realized value with certainty.

With rule 2, when both π and γ are known to the contestants, the contest function is shown to be an asymmetric perfectly-discriminatory one (or, the all-pay auction) as given by:

$$P_P(e_p, e_d) = \begin{cases} 1 & \text{if } \frac{F_p(e_p)}{F_d(e_d)} > \frac{(1-\pi)\gamma}{\pi(1-\gamma)}; \\ 0 & \text{if } \frac{F_p(e_p)}{F_d(e_d)} \leq \frac{(1-\pi)\gamma}{\pi(1-\gamma)}. \end{cases}$$

[12] As a specific example, consider the following production functions: $E_p = F_p(e_p) = \varphi f(e_p)$ and $E_d = F_d(e_d) = (1-\varphi)f(e_d)$ where, when the Defendant is innocent we have $\varphi \in (0, 1/2)$ and, similarly, when the Defendant is guilty, $\varphi \in (1/2, 1)$. The function $f(\cdot)$ is monotonically increasing in its argument and the parameter φ captures the impact of truth on evidence production.

[13] The power-law form is justified via research in psychophysical experiments, where it is well established that quantitative human perception (such as sensation of relative brightness of light or loudness of sound, as well as judgments concerning intensity of attitudes and opinions) of stimuli follows a power law. See Stevens (1966, 1975) and pages 127-133 of Sinn (1983) for a survey of these findings.

The above contest function differs from typical applications of all-pay auctions Hillman and Riley (1989); Kovenock, Baye, and de Vries (1996); Che and Gale (1998) only in that it is asymmetric and also it does not include an outcome that has a probability of 1/2 when the probability of guilt just equals γ.[14]

When the evidence production process is stochastic, the efforts by P and D determine each side's probability $h_j(e_j)$ of finding a favorable piece of evidence of a fixed value (E_j) where $j = p, d$ rather than deterministically affecting the value of the evidence itself with $h'_j(e_j) > 0$.[15] In such a setting, by applying the above two alternative decision rules, they show that the contest functions are of the difference-form variety. In particular, using rule 2 and some simplifying parametric restrictions, they show that when P and D have a uniform prior over π, the contest function takes the difference form as in (9) which closely resembles the piece-wise linear difference form contest explored by Che and Gale (2000) except that it can be non-linear in e_p and e_d with h_p and h_d interpreted as probabilities and bounded naturally between 0 and 1.

As the preceding discussion suggests, the theoretical research pertaining to contest functions is both diverse and mature. We now turn to a discussion of issues involved in empirically estimating such functions.

4 Some issues in econometric estimation

In contrast to the rich literature in contest theory, only a small body of the literature empirically estimates and tests the contest functions examined in the previous section.[16] There exist several difficulties to empirically test the

[14] This latter difference, of course, could simply be eliminated by having the court flip a fair coin when that is the case.

[15] These probabilities should of course be related to the truth. One way of explicitly incorporating the role of truth would be parameterizing the two functions so that $h_p(e_p) = \theta h(e_p)$ and $h_d(e_d) = (1-\theta)h(e_d)$, for some increasing function $h(\cdot)$ where $\theta > 1/2$ when D is guilty, and $\theta < 1/2$ when he is innocent.

[16] It is worth noting that the empirical literature on tournaments is closely related to the empirical studies of contests. Briefly, the tournament literature aims to further our understanding of compensation and incentives within organizations. It shows that a firm may motivate employees by running competitions for rewards (e.g., promotions). This can sometimes be preferable to individualistic schemes (e.g., bonuses). For instance, if accurately measuring individual performance is costly, the firm may economize by measuring only rank ordering of performances. Tournament theory generates many predictions. For example, it predicts that larger prizes motivate more effort and performance and a greater effect of extra effort on the chance of winning brings greater motivation. Many of the empirical researches on tournaments concentrate on verifying that tournaments work as suggested by the theory by using data from contexts such as golf (Ehrenberg and Bognanno, 1990a,b), auto racing (Becker and Huselid, 1992), lab experiments (Bull, Schotter, and Weigelt, 1987), agricultural (poultry) production (Knoeber and Thurman, 1994), and salary structures of CEO's (Main, O'Reilly, and Wade, 1993). However, very few tournament studies examine the mechanism that translates an individual's effort into her

contest models against real data. First of all, although micro-foundations of various conflict models are laid out, most influential contributions have been derived under very restrictive assumptions about the underlying process. For example, it has been known that the generalized ratio model is isomorphic to the logit model up to a logarithmic transformation, thus both models preserve the independence of irrelevant alternatives property. However, some empirical research suggests that the IIA property fails to hold in many real world contests[17]. Some authors prefer the probit in order to overcome this technical difficulty. Second, contestants' efforts are generally unobservable to researchers. Proxies are usually used to restore the information about players' effort levels devoted to the contest. Since these proxies at best approximate to the real value, a potential measurement error issue rises from any attempts to estimate the contest models. As a consequence, the accuracy of corresponding estimates is in doubt. Third, since all contest models are highly nonlinear empirically comparing contest models becomes a difficult task. In this section, we review some empirical studies of contests and discuss three main issues of concern.

4.1 Data issues

To analyze contest models, Sunde (2003) sets up several standards to choose an appropriate data set. Although his argument is based on sporting competitions, it actually applies to all the empirical studies of contests. Unfortunately, these standards are often violated by many contest data sets.

First, the data set should satisfy the Ford condition (see Hunter, 2004): in every possible partition of the competing parties into two nonempty subsets, some party in the second set beats some party in the first set at least once. This is because, as noted by Ford (1957), if it is possible to partition

probability of winning, and therefore are beyond the scope of the current paper.

[17]When testing the IIA condition, researchers generally use a multinomial logit model (MNLM), which is given in (11), to capture contests among more than two contestants. There are several tests available for violations of IIA, which compare the results from the full MNLM estimated with all possible outcomes to the results from a restricted estimation that includes only some of the outcomes. IIA holds when the estimated coefficients of the full model are statistically similar to those of the restricted one. If the test statistic is significant, the IIA property is not supported by the data. Hence neither the generalized ratio model nor logit model properly characterizes the contest. The first test of IIA was proposed by McFadden, Train, and Tye (1981). This likelihood ratio test, hereafter MTT, compares the value of the log-likelihood equation from the restricted estimation to the value obtained by substituting the estimates from the full model into the log-likelihood equation for the restricted estimation. Small and Hsiao (1985) demonstrated that the MTT test is asymptotically biased and proposed an alternative likelihood ratio test, known as the Small and Hsiao test, that eliminates this bias. A third IIA test, proposed by Hausman and McFadden (1984), compares the estimates from the full and restricted model. The most commonly used tests are the Hausman and McFadden (HM) test and the Small and Hsiao (SH) test, which are frequently discussed in many econometrics texts (e.g., Greene, 2008; Train, 2003).

the set of contestants into two groups A and B such that there are never any intergroup comparisons, then there is no basis for rating any contestant in A with respect to any contestant in B. On the other hand, if all the intergroup comparisons are won by a party from the same group, say group A, then if all parameters belonging to A are doubled and the resulting vector renormalized, the likelihood must increase; thus, the likelihood has no maximizer. The Ford condition eliminates these possibilities.

Secondly, data sets of individualistic contests (such as tennis, golf, and boxing) are preferred to the ones on team competitions (such as soccer, football, and basketball). In contests among teams, the players act as agents on behalf of the team–which may be an actual employer (e.g., a club) or some delegated authority (e.g., a national sports team). The difference between individualistic and team contests is important for two reasons. On the one hand, in a contest among teams, the team performances depend on not only individual players' effort levels, but also the aptness of the team members coordinating with each other. In sports competitions, we call this productive interactions among team members "the chemistry" of a team. On the other hand, the classic free-rider problem arises, where agents fail to internalize the benefit that accrue to other members of the team when making effort decisions. Both reasons potentially add extra dimensions to the empirical study of contest models and thus make them more difficult to analyze. A recent study by Ahn, Isaac, and Salmon (2011) conducts a series of economic experiments to examine the behavior of groups and individuals competing against each other in rent-seeking contests. By fitting their data against the Tullock model (12) with $\mu = 1$, their results show that, in contests between groups, individuals show decreased effort levels when in groups rather than when playing as individuals, which can be explained by the classical free-rider argument. Szymanski (2003) gives an excellent discussion on the difference between individualistic and team contests.

Thirdly, Sunde (2003) further suggests that the data should have only two parties involved in each contest. Specifically, the consideration of evidence for contests between only two players has several advantages. The most important one is that the strategic interactions in contests of only two parties are clear, no complex issues such as coalition, sabotage, and doping have to be considered.

The fourth standard of choosing data set requires that the structure of information available in contests should closely resemble the structure and requirements of information in theoretical models, i.e., both prizes and the characteristics of participants should be known *ex ante* by every player involved.

Finally, dynamic effects or long-term benefit streams over uncertain time horizons, which may be embedded in the real-world data need to be filtered out.

As already mentioned, another issue pertaining to the data used in empirical study of contest models is the potential unobservability of efforts. Estimating a contest model requires one to have measures of effort levels of all contestants. However, measuring effort could be a very difficult task, if it is possible at all. In general, effort is defined as a conscious exertion of power. The word "conscious" reveals the subjective nature of this concept. Hence it is reasonable to argue that contestants' efforts are generally unobservable to researchers. In order to overcome this difficulty, researchers have suggested to use the resources devoted to the contest to capture the essence of efforts. For example, in the conflict literature, Collier and Hoeffler (1998) suggest using casualties to proximate the efforts devoted to civil wars. Rotte and Schmidt (2002) and Hwang (2009) propose using personnel strengths as proxies for the efforts of conflicting parties. In laboratorial experiments, Fonseca (2009) and Ahn et al. (2011) use monetary contributions to the game to represent the elicited efforts.

Even with the aid of this approximation, there are still some difficulties left in the empirical estimation of contests. The main difficulty arises due to the unquantifiable nature of some resources. For instance, as the literature on military science suggests that, morale, intelligence, and logistics are key factors in determining battlefield success. They certainly should be counted as part of the resources devoted to the conflict. Since these factors are difficult to be quantified, one has no way to aggregate them into the total resources expended. All these practices introduce measurement error and hence endogeneity into the contest models, which will be discussed in details next.

4.2 Endogeneity

Among many difficulties of empirically estimating the contest models, the potential for endogeneity calls for a special attention. In an econometric model, a parameter or a variable is said to be endogenous when there is a correlation between the parameter or the variable and the error term. Generally speaking, a loop of causality between the independent and dependent variables of a model leads to endogeneity. As highlighted in the economic literature (e.g., see Miguel, Satyanath, and Sergenti, 2004), endogeneity generates inconsistent and biased estimates of the unknown parameters, which adversely affects the explanatory power of the contest models. In general, endogeneity arises in various situations including measurement error, autoregression with autocorrelated errors, simultaneity, omitted variables, and sample selection errors. For the study on contests, endogeneity mainly comes in two possible ways.

One channel through which endogeneity affects the contest models is the measurement error. As mentioned in 4.1, it is often very difficult to disentangle a player's effort with her performance. The generally accepted method of restoring the efforts data is to approximate them by using the resources devoted to the contests. This approximation introduces measurement error into the estimation and causes endogeneity problems.

The other possible cause of endogeneity comes from the suspicion that the dependent variable (which is the contest outcome) may have some feedback effect on the explanatory variable (such as efforts) invested in the game. In many contests, a success is achieved through a series of strategic interactions. During this period, each competing party keeps updating the information about its standing. It then makes the corresponding moves to reinforce its possibility of winning. It is therefore clear that both competing parties' total efforts invested in the contest could be affected within the time frame of the game.

Inadequately addressing endogeneity among economic variables usually fails to establish a convincing causal relationship. It often leads to biased estimates of parameters of interest and false policy implications. In order to remove the endogeneity problem from the conflict model, researchers are advised to do two things: (1) including instrumental variables (IV) to alleviate the potential measurement error problem and (2) carefully scrutinize the data and remove observations which could be possibly affected by the feedback effect. Generally speaking, an instrumental variable is a variable that does not itself belong in the explanatory equation and is correlated with the endogenous explanatory variables, conditional on the other explanatory variables. Formal definitions of instrumental variables, using counterfactuals and graphical criteria, are given in Pearl (2000). Heckman (2008) also gives a thorough discussion about the relationship between instrumental variables method and causality in econometrics. As a rule of thumb, there are two main requirements for using an IV:

1. The instrument must be correlated with the endogenous explanatory variables, conditional on the other explanatory variables.

2. The instrument cannot be correlated with the error term in the explanatory equation, that is, the instrument cannot suffer from the same problem as the original predicting variable.

In the contest literature, researchers pick variables such as precipitation (e.g., Miguel et al., 2004) as an IV to rectify the bias caused by endogeneity. The second treatment, data cleansing, requires the researchers to identify the potential feedback effect between the error terms and the explanatory variables. In the current context, any contest observation involving strategic interactions between two competing parties should be removed from the data set. For example, Jia (2008a) explains why certain observations have to be removed from the data set when he attempts to estimate and compare various contest models using National Basketball Association data.

Specifically, Jia (2008a) argues that there are two potential channels through which the feedback effect operates. The first channel can be described as follows. Consider in particular the following two polar cases that might arise in some stage of the regular season play. (1) A team has played sufficiently well so as to secure its first standing in its own division. In this case, the team may "shirk," i.e., exert less effort and hence throw away some of the remaining games. (2) A team has played so poorly that it loses any hope of entering the playoffs. In such a case, it is also likely that the team will shirk in the following games. In both cases, the previous game results affect the current and future effort levels.

The second channel mentioned in Jia (2008a) is on a more micro level. Consider the case when one team holds a significant advantage over its opponent at the end of the third quarter so that the outcome of the game has already been decided. It is fairly common to observe that the coaches of one or both teams replace their best players with substitutes during the fourth quarter, which is usually known as "garbage time." This practice serves to give those substitutes playing time experience in an actual game situation, as well as to protect the best players from the possibility of injury.

In view of these two potential feedback effects between the error terms and the explanatory variables, Jia (2008a) deletes all the games in which one team holds a significant advantage (≥ 20 points) over the other at the end of the third quarter, which guarantees there is no "garbage time" exists in the resulting data set. He further deletes the observations in the final stage of the regular seasons – specifically, the regular season NBA games after March 31. Each NBA team has about ten additional games following this March 31 cutoff before the end of the regular season. Omitting those observations after March 31 largely removes the possibility that any team knows its final standing for sure.

4.3 Model Comparison

As shown in the previous discussion, there are many ways to model contest situations in terms of the (relative) resources invested by the parties involved. In particular, three models are prominent in the literature. They are generalized ratio or "Tullock"model (3), logit model (7), and probit model (8). The main difference among these three models lies in the assumptions made on the distributions of the error terms. McFadden (1974) shows that the logit model is based upon the assumption that the error terms are extreme value distributed. As shown by Jia (2008b), the generalized Tullock model assumes the error terms have an inverse exponential distribution, which is isomorphic to the logit model up to a logarithmic transformation. The probit model, on the other hand, relies on the assumption that the error terms are normally distributed. It is therefore natural to wonder what model best captures the characteristics of a particular contest and gives the most accurate predictions.

The effort to decide among these three models has been frustrated by the fact that despite the three models entail quite different theoretical consequences, they are practically indistinguishable with data. For instance, Burke and Zinnes (1965) compare a probit model ($\mathcal{T}$) and a logit model ($\mathcal{L}$) and claim:

> Unfortunately, the nature of the solutions makes it very difficult to design an experiment for deciding between the theories... For the Gulliksen-Tukey (1958), Guilford (1954), and Thurstone (1959) data, the $\mathcal{T}$ predictions are considerably better than the $\mathcal{L}$ predictions.

Yet Hohle (1966) finds that

> (a) neither model provided uniformly satisfactory representations for the data, and (b) (for) all six sets of data were more accurately represented by Model II ($\mathcal{L}$) than by Model I ($\mathcal{T}$).

In a comprehensive survey, Batchelder (1983) concludes that it would require an unrealistic amount of data to achieve any reasonable power in testing between them statistically. Stern (1990) uses Gamma distribution to approximate the probit and logit models and compares their performances against the game results from the 1986 National League baseball season. He also concludes it is disturbing to find so little difference between these two models.

This problem arises because of two reasons. Firstly, all three models are highly nonlinear, and yet a commonly accepted goodness-of-fit measure is unavailable to achieve a convincing conclusion. Secondly, the generalized ratio and the logit models are isomorphic and thus nested together. The classic econometric theory fails to cope with these two problems.

One possible remedy is provided by Bayesian Econometrics. As opposed to classic methods, the Bayesian approach treats any two candidate models as hypotheses. Rather than artificially designing some goodness-of-fit statistics, Bayesians choose a natural criterion, the Bayes factor, to compare alternative models. The Bayes factor for model A_1 versus model A_2 can be defined as

$$B_{12} = \frac{f(y|A_1)}{f(y|A_2)},$$

where

$$f(y|A_i) = \int_{\Theta_i} f(\theta_i|A_i) f(y|\theta_i, A_i) d\theta_i$$

is the marginal likelihood of model i, $i = 1, 2$ (see Kass and Raftery, 1995).

The interpretation of the Bayes factor is given by (Jeffreys, 1961, Appendix B), and Kass and Raftery (1995). Jeffreys suggests the following criterion as the "order of magnitude" interpretation of B_{12}:

$$\begin{array}{llllll} 1 & < & B_{12} & < & \infty, & \text{evidence supports } A_1, \\ 10^{-1/2} & < & B_{12} & \leq & 1, & \text{very slight evidence against } A_1, \\ 10^{-1} & < & B_{12} & \leq & 10^{-1/2}, & \text{slight evidence against } A_1, \\ 10^{-2} & < & B_{12} & \leq & 10^{-1}, & \text{strong evidence against } A_1, \\ 0 & < & B_{12} & \leq & 10^{-2}, & \text{decisive evidence against } A_1. \end{array}$$

The key step in the Bayesian model comparison is computing a good approximation to the marginal likelihoods. For our nonlinear regression models (generalized ratio, logit, probit), the main difficulty is that the marginal likelihood functions cannot be expressed directly as some posterior moments, and consequently the computation cannot be interpreted directly as a special case of the simulation-consistent approximation of posterior moments. Fortunately, there are computational methods specifically tailored to overcome these kinds of problems. Interested readers can find a detailed description of two different methods to compute the marginal likelihoods of probit and logit models respectively in Appendix of Jia and Skaperdas (2011).

5 Concluding remarks

Many interesting real world phenomena such as sports, advertising competition, research or labor tournaments, or court battles are adversarial settings where the outcome is typically uncertain but it does depend on the efforts expended by adversaries. We have considered contest functions to be probabilistic choice functions that depend on the efforts of adversaries. We have examined both their theoretical foundations and several issues in estimating them empirically. It is of some interest that all four types of theoretical foundations can yield under some conditions the ratio functional form, which also happens to be the most used functional form in applications. Whereas research on theoretical foundations is rather mature, there is much less empirical research on the topic - that is clearly a promising area for future research.

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