Wittgenstein's Anti-Philosophy of Mathematics

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My goal in this paper is to outline a reading of Wittgenstein on mathematics. So much as been written on this topic, from so many different perspectives, that any claim to originality would be foolhardy. In this particular case, I will rely on the works of various commentators; my only clear contribution is the present arrangement of material and some connecting tissue and extensions here and there. Still, this reading contrasts dramatically with other popular approaches, including one championed by one of my own previous selves,[1] so I hope others may benefit from the novel vista as much as I have.

As a backdrop, I begin with a quick sketch of a contrasting reading. No one, so far as I know, has ever maintained this interpretation of Wittgenstein on mathematics in its pure form, but some have come close, and it provides a useful foil. I then turn to the alternative reading that is the real subject of this paper.

I. Wittgenstein as philosopher of constructivism

By now, all students of Wittgenstein are familiar with Kripke's influential reading of the late work that takes the rule-following considerations as central, sees them as presenting a skeptical paradox to the effect that "no course of action could be determined by a rule" (PI 201), and, finally, takes Wittgenstein to be presenting a skeptical solution to the paradox that incidentally implies the impossibility of a private language.[2] In more detail, the skeptical solution replaces the question "what must be the case for such-and-such to be a proper application of a particular rule?" with the twin questions "under what circumstances is such-and-such appropriate?" and "what is the role of this activity in our lives?" The answers to these questions show that rule-following is part of a shared practice, that our primitive inclinations to carry on as we do are held to shared criteria against which they can be judged correct or incorrect. On this analysis, my undertaking to follow a rule that assigns a name to my private sensation is pointless because thereafter

> ... whatever is going to seem right to me is right. And that only means that here we can't talk about "right". (PI 258)

In other words, my private naming ceremony does not create a criterion of correctness, and the purported rule remains empty.

How, then, *do* we manage to talk about our sensations? We consider it appropriate to say that someone is in pain when that person has been injured, when she cries, winces or groans, when she says, "I am in pain". Does this mean that the word "pain" refers to a complex of behavioral and environmental circumstances? No. According to this version of Wittgenstein, our error here is the assumption that "pain" refers at all:

> The paradox [of behaviorism] disappears only if we make a radical break with the idea that language always functions in one way, always serves the same purpose: to convey thoughts - which may be about houses, pains, good and evil, or anything else you please. (PI 304)

Some words, like "house", function descriptively: their referents are picked out by pointing; their criteria are direct criteriological descendants of the discredited truth conditions. Trouble arises, however, when we try to make all words fit this model. "Pain", for example, does not; despite the close criteriological connection between "pain" and pain behavior, "pain" does not refer to or describe behavior: "the verbal expression of pain replaces crying and does not describe it" (PI 244).

This non-descriptive model applies also to mathematics. The criteria for proper assertion of mathematical statements do not involve the observation of mathematical things, but calculations and proofs. But, though calculations and proofs are finite objects, finitism is an error parallel to behaviorism:

> Finitism and behaviorism are quite similar trends. Both say, but surely, all we have here is ... Both deny the existence of something, both with a view to escaping from a confusion. (RFM II 61)

The behaviorist, mistaking criteria for referents, takes sensation talk to be about behavior; similarly, the finitist, making the same error, takes mathematics to be about finitary objects like proofs. Again the answer is not that mathematical talk refers to finite arrays of symbols, but that mathematical talk does not refer at all.

A criteriological look at the institution of proof produces a highly non-traditional account. Logical inference is just a particular form of rule-following; which conclusions follow from which premises is determined by public criteria, not by objective fact. The "must" behind the mathematical proof, the sense that anyone who accepts the premises "must" accept the conclusion, is the blind "must" of rule-following in general, a "must" based in our stubborn, shared inclination to proceed as we do. In accepting a proof, I accept a new connection between concepts, that is, I accept new criteria for the application of those concepts.[3] The value calculated or the proposition proved now acts as a new law or paradigm.

But notice:

> The opposite of "there exists a law that p" is not: "there exists a law that ~p". But if one expresses the first by means of P, and the second by means of ~P, one will get int difficulties. (RFM V 13)

Of course, this is just what we do in stating the law of the excluded middle. Wittgenstein considers the typical intuitionist's example:

> We only see how queer the question is whether the pattern ϕ (a particular arrangement of digits e.g. "770") will occur in the infinite expansion of π, when we try to formulate the question in a quite common or garden way: men have been trained to put down signs according to certain rules. Now they proceed according to this training and we say that it is a problem whether they will *ever* write down the pattern in following the given rule.

> Of someone who is trained we can ask "How *will* he interpret the rule for this case?", or again "How *ought* he to interpret the rule for this case?" - but what if no decision about this question has been made? - Well, the answer is not: "he ought to interpret it in such a way that ϕ occurs in the expansion" or: "he ought to interpret it in such a way that it does not occur", but: nothing has so far been decided about this. (RFM V 9)

Until the rule has actually been followed, there is no right or wrong about how it must be followed:

> However queer it sounds, the further expansion of an irrational number is a further expansion of mathematics. ... when I calculate the expansion further, I am deriving new rules which the series obeys. (RFM V 9, 11)

It is only a platonistic picture that disposes us to think otherwise:

> ... if you say that the infinite expansion must contain the pattern ϕ or not contain it, you are so to speak shewing us the picture of an unsurveyable series reaching into the distance. (RFM V 10)

This picture Wittgenstein naturally rejects:

> But what if the picture began to flicker in the far distance? (RFM V 10)

> What harm is done e.g. by saying that God knows *all* irrational numbers? Or: that they are already all there, even though we only know certain of them? Why are these pictures not harmless? ... For one thing, they hide certain problems. ... Even God can determine something mathematical only by mathematics. Even for him the mere rule of expansion cannot decide anything that it does not decide for us. (RFM VII 41)

No picture, no supreme being, or no ideal mathematical sequence can give substance to the claim that ϕ either appears or doesn't appear at some point in the expansion of π before the expansion is actually carried out.

Thus, in his own idiosyncratic style, Wittgenstein arrives at a position on the law of the excluded middle that appears indistinguishable from the intuitionist's. In a similar spirit, he sounds other familiar intuitionistic themes. For example, on the subject of non-constructive proofs, he writes:

> A proof that shews that the pattern "777" occurs in the expansion of π, but does not shew where. Well, proved in this way this "existential proposition" would, for certain purposes, not be a *rule*. ... is it reasonable to say of the proof concerned: it proves the existence of "777" in the expansion? This can be simply misleading. (RFM VII 41)

He even repeats the favorite image of the original intuitionists:

> The mathematician is an inventor, not a discoverer. (RFM I 168)

The criteriological view provides clear support for such a position: nothing predetermines the correct results of our mathematical rule-following; correctness only

arises when we have followed the rule as our training inclines us to do, in agreement with the inclinations of our colleagues. The source of the mathematical "must" is not an objective reality independent of us (ideal or otherwise), but the blind conviction with which we proceed as we do. In fact, this Wittgensteinian view is more radical than the intuitionist's, because it denies what the intuitionist would assert: that, for example, direct methods of calculation predetermine whether or not a given large number is prime or not, even if no one has ever checked this particular case.[4]

This might be a viable reading of Wittgenstein, were it not for the prominence of remarks like this:

> Philosophy may in no way interfere with the actual use of language ... It leaves everything as it is. ... It also leaves mathematics as it is ... (PI 124)

Surely, one cannot deny the law of the excluded middle or rule out non-constructive existence proofs and at the same time leave "mathematics as it is". But what is the motivation for this prohibition? If philosophy provides compelling reasons to abandon the platonistic picture, if current mathematical practice is based on that picture, why shouldn't the result of philosophical analysis be allowed to reform that practice? Mightn't Wittgenstein's reluctance be some form of false modesty?

This reading of Wittgenstein's late views uncovers a tension between the upshot of his philosophical views and his insistence that philosophy alters nothing.[5] It tempts us to downplay the non-interference remarks in favor of the presumed payoffs of his contentful philosophical conclusions. A directly opposed approach - my focus in this paper - would give pride of place to the non-interference claims and adjust the reading of the rest to match.

II. Wittgenstein as anti-philosopher

My goal now is a very different reading of Wittgenstein, a reading that highlights his claims that philosophy should not interfere with mathematics (or language in general).[6] Philosophy will not interfere because any philosophical theses would be too obvious to debate:

> If one tried to advance *theses* in philosophy, it would never be possible to question them, because everyone would agree to them. (PI 128)

Emphasizing these remarks gives full attention to a strain in Wittgenstein's thought that goes back to the *Tractatus*:

> The totality of true propositions is the total natural science ... Philosophy is not one of the natural sciences. ... Philosophy is not a theory, but an activity. ... The result of philosophy is not a number of "philosophical propositions", but to make propositions clear. (TLP 4.11 - 4.112)

By the period we are concerned with, the picture has darkened to a stark denial:

> Philosophy simply puts everything before us, and neither explains nor deduces anything. ... we may not advance any kind of theory. ... We must do away with all *explanation*, and description alone must take its place. (PI 126, 109)

This qualifies as "anti-philosophy" in the sense that modern novels have "anti-heros": the anti-hero is the protagonist of the story, just as the hero once was, but she lacks the usual attendant virtues (nobility, strength, courage, etc.). Antiphilosophy, though done by professional philosophers, makes no attempt at explanations or true philosophical theories. Before examining the rule-following considerations from this perspective, we need to understand the thinking behind this view of the philosophical enterprise.

What Wittgenstein offers here is not strictly speaking a meta-philosophy of anti-philosophy. If philosophy states no controversial theses, neither does meta-philosophy, as the two are one and the same:

> One might think: if philosophy speaks of the use of the word "philosophy" there must be a second-order philosophy. But it is not so: it is, rather, like the case of orthography, which deals with the word "orthography" among others without then being second-order. (PI 121)

Instead of a set of meta-philosophical theses, Wittgenstein gives us hints of a general approach, gleaned, as it were, from what he takes to have worked in various particular cases. These hints might best be described as a psychology of philosophy, though surely not the sort of empirical psychology that serves as one of his most frequent targets. It would more properly be compared to Freudian psychology, as Wittgenstein understands it:

> Take Freud's view that anxiety is always a repetition in some way of the anxiety we felt at birth. He does not establish this by reference to evidence - for he could not do so. But it is an idea which has a marked attraction. It has the attraction which mythological explanations have, explanations which say that this is all a repetition of something that has happened before. And when people do accept or adopt this, then certain things seem much clearer and easier for them. (LC, p. 43)

In such an analysis, there is no science, there are no laws, there is only "speculation" (LC, p. 43), but the speculation is beneficial to those inclined to accept it:

> [it] makes it easier for them to go certain ways: it makes certain ways of behaving and thinking natural for them. They have given up one way of thinking and adopted another. (LC, pp. 44f.)

This could just as easily describe philosophy on Wittgenstein's later approach.[7]

This "psychology" begins with an analysis of how philosophical problems arise out of linguistic confusion:

> Philosophy is a battle against the bewitchment of our intelligence by means of language. (PI 109)

We are so bewitched when we inadvertently allow ordinary forms of expression to be lifted from their proper context, after which, we attempt to apply them in a context - free or "philosophical" sense. When this happens, the contextual backing that gives sense to those expressions is missing, and we are left at a loss as to whether they apply or not.

> ... philosophical problems arise when language *goes on holiday*. (PI 38) The confusions which occupy us arise when language is like an engine idling, not when it is doing work. (PI 132)

As a result, we "don't know [our] way around" (PI 123), and we feel ourselves up against a very deep problem indeed:

> The problems arising through a misinterpretation of our forms of language have the character of *depth*. They are deep disquietudes; their roots are as deep in us as the forms of our language and their significance is as great as the importance of our language. ... that is what the depth of philosophy is. (PI 111)

How does this happen? Various things can push us in this unhealthy direction. For example, we often come to a given subject with philosophical prejudices that cloud our vision:

> The ideal, as we think of it, is unshakable. You can never get outside it; you must always turn back. There is no outside; outside you cannot breathe. - Where does this idea come from? It is like a pair of glasses on our nose through which we see whatever we look at. It never occurs to us to take them off. (PI 103)

This can keep us from examining the details of a given practice because we are already convinced that they will be of no help with the question we want answered:

> If I am inclined to suppose that a mouse has come into being by spontaneous generation out of grey rags and dust, I shall do well to examine those rags very closely to see how a mouse may have hidden in them, how it may have got there and so on. But if I am convinced that a mouse cannot come into being from these things, then this investigation will perhaps be superfluous. .... But first we must learn to understand what it is that opposes such an examination of details in philosophy. (PI 52)

Sometimes, we are simply mislead by grammatical analogies:

> How does the philosophical problem about mental processes and states and about behaviorism arise? - The first step is the one that altogether escapes notice. We talk of processes and states and leave their nature undecided. Sometime perhaps we shall know more about them - we think. But that is just what commits us to a particular way of looking at the mat-

> ter. For we have a definite concept of what it means to learn to know a process better. (The decisive movement in the conjuring trick has been made, and it was the very one that we thought quite innocent.) - And now the analogy which was to make us understand our thoughts falls to pieces. So we have to deny the yet uncomprehended process in the yet unexplored medium. And now it looks as if we had denied mental processes. And naturally we don't want to deny them. (PI 308)

But whatever the initial cause, the result is a sort of puzzlement that cannot be resolved by ordinary means, simply because it has foolishly divorced itself from ordinary means, and we are off on our misguided business of formulating and debating controversial philosophical theses.

On this reading of the circumstances, to put forward yet another philosophical theory is simply to exacerbate the problem. Similarly, it is not enough to find errors of fact or logic in the premises of one's opponent's arguments. What is needed, rather, is treatment:

> Thus, for example, what a mathematician is inclined to say about the objectivity and reality of mathematical facts, is not a philosophy of mathematics, but something for philosophical *treatment*. ... The philosopher's treatment of a question is like the treatment of an illness. (PI 254f.)

Rather than devising theories, the philosopher should concentrate on uncovering the linguistic sleight of hand that started us philosophizing in the first place.

> What *we* do is to bring words back from their metaphysical to their everyday use. (PI 116)

There is no single way to do this:

> There is not a philosophical method, though there are indeed methods, like different therapies. (PI 133)

But whatever method is used, the goal, the standard of success, is the same:

> The real discovery is the one that makes me capable of stopping doing philosophy when I want to. - The one that gives philosophy peace, so that it is no longer tormented by questions which bring *itself* into question. (PI 133)

To succeed in philosophy is not to present a true theory, but to put a pseudo-problem to rest.

This may seem to recommend a reform of language, but recall that Wittgenstein's anti-revisionary stance applies only to "the actual use of language". In philosophy, language is not being used; it is "idling". What this therapy aims at, then, is the removal of these pockets of confusion:

> For the clarity we are aiming at is indeed *complete* clarity. But this simply means that the philosophical problems should *completely* disappear. (PI 133)

When philosophy has been excised, ordinary language, indeed ordinary mathematics, should remain unchanged. They, by themselves, do not force us into philosophy; for this move, some form of linguistic confusion is required, the very confusion the philosopher should strive to eliminate rather than to exacerbate.

III. An anti-philosophy of rule following

To see how this approach works, let's see how it applies to our familiar case of following a rule:

> Now - judged by the usual criteria - the pupil has mastered the series of natural numbers. Next we teach him to write down other series of cardinal numbers and get him to the point of writing down series of the form
>
> 0, n, 2n, 3n, etc.
>
> at an order of the form '+n', so at the order '+1' he writes down the series of natural numbers. - Let us suppose we have done exercises and given him tests up to 1000. ... Now we get the pupil to continue a series (say +2) beyond 1000 - and he writes 1000, 1004, 1008, 1012. (PI 185)

At this point, in actual practice, I correct him, give more examples, explanations, etc., and this usually works in time. If it doesn't, I give up and declare this pupil (for whatever reason) incapable of understanding arithmetic; nothing can force him to understand, after all. This is a shame - it doesn't bode well for the pupil - but I surely don't take it as undermining my conviction that "1002" is the correct response to the order. In practice, the case is as simple as that.

But when I come across this discussion in a philosophy book, my reaction is different. The pupil's recalcitrance makes me realize, as it did a moment ago, that my examples and explanations cannot force him to understand, but this time I focus on the fact that I myself *do* understand. This understanding of mine goes beyond the particular examples I give, so there must be a gap between my understanding of what "plus 2" means and what I can tell the pupil. Now I am off after the goal of specifying what understanding consists in. I might begin by suggesting that my understanding consists of my fixing on a certain interpretation of the phrase "plus 2", along the lines of my ill-fated debate with the Wittgensteinian skeptic, as described by Kripke. After reading Wittgenstein, I might be tempted to say it consists in having a brute inclination to go on in accordance with the public criteria of my community.

But this is not what Wittgenstein says. In another rule following discussion, he writes:

> I can train someone in a *uniform* activity. ... Now I ask myself, what is it that I want him to do, then? The answer is: He is always to go on as I have showed him. And what do I really mean by: he is always to go on in that way? (RFM VI 17)

If the line of thought in the previous paragraph were correct, we should reply: I mean that he should go on as I and the rest of our community goes on, in accord with our shared criteria. But Wittgenstein continues:

> The best answer to this that I can give myself, is an example like the one I have just given. I would use this example in order to shew him, and *also* to shew myself, what I mean by uniform. (RFM VI 17)[8]

The best answer is another example. And notice: this is just the answer I gave earlier, when I approached the question in an ordinary, rather than a philosophical context. Here Wittgenstein is advocating just that ordinary, mundane answer, the answer that actually works in practice. What makes us think this isn't enough?

The questions we want answered are ordinary questions - does he understand?, what does it mean to say he must go on like this?, when does a rule determine which response is correct? - and the fact is that we have answers. We can tell when someone understands and when she doesn't. We can give examples and explanations that show what we mean by "go on like this". We can tell a rule that does determine a correct response from one that doesn't (e.g. $y = x^2$ versus $y \neq x^2$ in PI 189). But in a philosophical mood, we won't accept those answers; they are the rags that we simply know cannot produce a mouse, so we don't even bother to look at them.

And how do we know that they cannot produce a mouse? Having imagined a case in which someone's behavior was not a good indicator of understanding, we conclude that no amount of behavior should convince us that someone understands, that there is always a doubt about whether or not someone understands. That this isn't true in practice doesn, t bother us; we are operating now in the realm of philosophical imagination. We confront a fearful gap between behavior and understanding, and we demand an account of understanding that bridges this gap, not under ordinary conditions, but under any conditions we can imagine. Obviously, an account of understanding as it functions in actual life will not do the job; the rags cannot produce such an extraordinary mouse. We have elevated the ordinary questions, which have ordinary answers, into philosophical questions, at which point, ordinary answers no longer suffice. Unsurprisingly, we no longer know our way around.

The solution is not to propose a philosophical theory of understanding, even a communal one; the solution is to refuse to elevate the ordinary questions. When asked what understanding consists in, we must insist

> ... that there is a way of grasping a rule which is *not* an *interpretation*, but which is exhibited in what we call "obeying the rule" and "going against it" in actual cases. (PI 201)

We must examine the rags, for the answers to our original questions, the legitimate questions, were there all along. And notice: the true theses of this answer will state only "what everyone admits" (PI 599).

IV. An anti-philosophy of private language

Our attention to the actual practice of following rules has so far concentrated on linguistic and mathematical training, and similar communal activities. These mundane observations have managed to answer our questions about understanding, but the interlocutor now has another worry: aren't there cases of rulefollowing that don't involve any training or indeed any interaction with our fellow language users?

> ... could we ... imagine a language in which a person could write down or give vocal expression to his inner experiences - his feelings, moods, and the rest - for his private use? ... The individual words of this language are to refer to what can only be known to the person speaking; to his immediate private sensations. So another person cannot understand the language. (PI 243)

If such a private language is possible, then rules can be understood, and followed, in ways other than the mundane and obvious ones championed in the previous section, and our humdrum analysis is incomplete.[9]

Wittgenstein begins the discussion with a series of observations about our actual sensation talk: that sensation words are learned via their connection with the natural expressions of pain (PI 244); that we can tell when other people are in pain; that you, not I, can doubt whether I am in pain (PI 246). He then pauses to remind us that

> The philosopher's treatment of a question is like the treatment of an illness. (PI 255)

suggesting that the very idea of a private language is a confusion that needs treatment. Then comes an analysis of what would actually be involved in a truly private language (PI 258ff).

First, clearly, the words for the sensations of a private language cannot be systematically linked to natural expressions of sensation, because in that case, they would function just as our public sensation words do, and other people could understand them. So a truly private language would have to go something like this:

> I want to keep a diary about the recurrence of a certain sensation. To this end I associate it with the sign "E" and write this sign in a calendar for every day on which I have the sensation. (PI 258)

But how is this "association" accomplished?

> I can give myself a kind of ostensive definition.
> - How? Can I point to the sensation?
> Not in the ordinary sense. But I speak, or write the sign down, and at the same time I concentrate my attention on the sensation - and so, as it were, point to it inwardly. (PI 258)

So the association is to be set up on the model of an ostensive definition performed privately.

Now the topic of ostensive definition is a familiar one by this point in the *Investigations;* it was discussed at length in the opening sections (1 - 38). There Wittgenstein considers a view of language that he traces to Augustine:

> It is this: the individual words in language name objects - sentences are combinations of such names. (PI 1)

Augustine describes our learning of such a language:

> When they (my elders) named some object, and accordingly moved towards something, I saw this and I grasped that the thing was called by the sound they uttered when they meant to point it out. (PI 1)

This process is familiar: to teach someone the meaning of a word, I point to what it signifies; I use what we've been calling a "definition by ostension". Now, with Augustine, we see language as founded on the relations of things to their names, and we see Augustine, as a child, being introduced into language by watching his elders point at objects while reciting their names, that is, by a series of ostensive definitions.

One obvious problem with this picture of language is that many words in our language are not names at all, a point Wittgenstein emphasizes, but our concern is with its appeal to ostensive definition. Wittgenstein begins, characteristically, by pointing out what we all realize, that ostensive definitions can fail; my student

> ... might ... take the name of a person, of which I give an ostensive definition, as that of a colour, of a race, or even of a point of the compass. (PI 28)

So what disambiguates the ostensive definitions my elders use in teaching me? In practice, we often resolve such ambiguities by further explanations - "No, the person, not the color!" - but these require that we understand other words, and thus, cannot serve as primitive introductions into language. Wittgenstein's interlocutor makes a series of proposals:

> ... all you need - of course! - is to know or guess what the person giving the explanation is pointing to. (PI 33)

> 'I always do the same thing when I attend to a shape [as opposed, say, to the color]: my eye follows the outline and I feel ... ' (PI 34)

> ... a *spiritual* (mental, intellectual) activity corresponds to [pointing to the shape]. (PI 36)

> ... hearing the name calls before our mind the picture of what is named. (PI 37)

Ultimately, none of these efforts is any more successful than the similar moves in the rule-following discussion, and we end up with

> ... the conception of naming as, so to speak, an occult process. Naming appears as a *queer* connexion of a word with an object. (PI 38)

At this point, we are faced with a deep philosophical question: how do words connect with things?

Ostensive definition, which seemed at first one of the most common and unproblematic processes in our practice of language, now appears hopelessly mysterious and "queer", a deep philosophical problem. The anti-philosopher suspects a wrong turn was taken somewhere, and indeed, it isn't hard to find. Ostensive definition is unproblematic in the everyday context of our linguistic practices, against the shared background of our training and our reactions to that training. But the Augustine-inspired philosophical account of "language in itself" takes this simple process out of its context within our language and asks it to serve as an ultimate foundation for all of language. But there, out of context, it cannot take hold:

> ... ostensive definition explains the use - the meaning - of the word when the overall role of the word in language is clear. ... "I set the brake by connecting up rod and lever" - Yes, given the whole of the rest of the mechanism. Only in conjunction with that is it a brake-lever, and separated from its support it is not even a lever; it may be anything, or nothing. (PI 30, 6)

We have, then, a typical philosophical confusion: a perfectly workable commonplace - ostensive definition - is lifted out of its context and asked to perrorm a philosophical task. When it is (inevitably!) found wanting, the result is a deep philosophical problem. We erred - a error that ultimately produced this deep philosophical problem - when we undertook the philosophical task of finding a foundation for language in itself - a foundation divorced from our actual linguistic practices - an undertaking doomed to failure.

Returning to the question of private language, we can now summarize what is required for a language to be truly private, that is, for a language to be incomprehensible to anyone but the private linguist himself. First, as noted a few pages back, the private experience named must lack the sorts of natural expressions and outward signs available in our public language, because such expressions and signs would allow someone else to understand. Second, the ostensive definition by which (we assume) the private name is introduced must function without the usual public context that makes our familiar ostensive definitions work:

> When one says "He gave a name to his sensation" one forgets that a great deal of stage-setting in the language is presupposed if the mere act of naming is to make sense. (PI 257)

Having reminded us of this observation from the early pages of the *Investigations*, Wittgenstein sets out, in the following section, to investigate just how this private naming might be accomplished.

I attempt to name a certain sensation of mine by associating it with the sign "E" and undertaking to write "E" in my diary every time I experience that sensation.

The first point to notice is that anything at all is similar to the original sensation in some way or other, so the sensation itself is not enough to determine whether something new is the same or different, another case of E or not. If my private ostension only picks out this particular sensation, that by itself does not settle what is to count as E again:

> ... in the present case I have no criterion of correctness .... a note has a function, and this "E" has none so far. (PI 258, 260)

At this point, I am free to use "E" whenever I like:

> ... whatever is going to seem right to me is right. And that only means that here we can't talk about "right". (PI 258)

So, there must be more to the ostension that mere picking out the object; otherwise, no correct use for the expression "E" will have been determined.

What more went on when I set up the association? I must have concentrated on certain aspects of the original sensation; these aspects must determine whether the next thing is E again or not. In fact, one such aspect is has been explicit from the start; I intend to use "E" as the name of a sensation. But:

> What reason have we for calling "E" the sign for a *sensation*? For "sensation" is a word of our common language, not of one intelligible to me alone. So the use of this word stands in need of a justification which everybody understands. (PI 261)

Rules from the public language govern the use of sensation words, so the rules of use for a sensation word cannot be determined solely by my own private decisions. I retreat: what I ostend is not necessarily a sensation, but surely a private linguist has something!

> "Has" and "something" also beiong to our common language. (PI 261)

Even these words are governed by public rules, and thus disallowed in determining the workings of my private language. Obviously, I am left with nothing to say:

> ... when one is doing philosophy one gets to the point where one would like just to emit an inarticulate sound. (PI 261)

An inarticulate sound is one that has no role in a language.

We thought we could imagine a private language, but when we set out to do so, we come down to the image of a person emitting an inarticulate sound. Nothing precludes that sound actually playing a role in a practice closed to us, but this idea raises more questions:

> ... such a sound is an expression only as it occurs in a particular language-game, which should now be described. ... Is it to be assumed that [the pri-

> vate linguist] invent[s] the technique of using the word; or that [he] found it ready-made? (PI 261, 262)

So, while we can easily "imagine human beings who [speak] only in monologue" (PI 243), it is not as clear as it once seemed that we can imagine someone speaking a private language. This doesn't prove that a private language is impossible, but the purported example of private language can hardly count as a serious threat to Wittgenstein's homespun observations on rule-following until it is given more substance than this.

Here, once again, Wittgenstein has treated our philosophical confusion (our faith in the idea of a private language) rather than defending his own controversial philosophical theses (a private language is impossible). He does this by the simple expedient of asking for the details, asking precisely how a private language would work, and pushing these questions till we see that the idea is much murkier than we originally thought.

V. An anti-philosophy of mathematics

The anti-philosophical perspective likewise produces a very different reading of Wittgenstein's intuitionist-sounding remarks about the law of the excluded middle[10] and non-constructive proof. For example, when he writes:

> ... if you say that the infinite expansion must contain the pattern ϕ or not contain it, you are so to speak showing us the picture of an unsurveyable series reading into the distance. (RFM V 10)

he is criticizing not the law of the excluded middle, but the platonist's idea that the use of the law can be justified. Our use of the law can no more be justified than our conviction that 1002 follows 1000 in the series plus 2, but this is not to say that our use is in any way incorrect.

But whatever the platonist's errors, the intuitionist is no less benighted. Having argued, side-by-side with Wittgenstein, that the platonist's use of the law cannot be justified, the intuitionist concludes that it is incorrect, while in fact, the law of the excluded middle is bedrock in our logic of propositions:

> I need hardly say that where the law of the excluded middle doesn't apply, no other law of logic applies either, because in that case we aren't dealing with propositions of mathematics. (Against Weyl and Brouwer) (PR XII 151)

> The word "proposition", if it is to have any meaning at all here, is equivalent to a calculus: to a calculus in which p v ~p is a tautology (in which the law of the excluded middle holds). (PG, p. 368)

The intuitionists think they are managing to do without it only because they have inadvertently changed the subject:

> If "you do it" means: you must do it, and "you do not do it" means: you must not do it - then "Either you do it, or you do not" is not the law of the excluded middle. (RFM V 17)

So Wittgenstein is attacking as needless philosophy both the platonist's attempt to justify the law of the excluded middle and the intuitionist's attempt to undermine and replace it. The law is an integral part of our linguistic practice; that is all there is to say.

The case of non-constructive proofs is somewhat different because mathematical existence is not as fundamental as negation and disjunction:

> We have no concept of existence independent of our concept of an existence proof. (PG, p. 374)

So again, both the platonist and the intuitionist err in thinking their positions can be justified at a deeper level than that of a choice between concepts of existence. "The disastrous invasion' of mathematics by logic" (RFM V 24) blinds the platonist to the difference between constructive and non-constructive existence proofs:

> The harmful thing about logical technique is that it makes us forget the special mathematical technique. ... When a proof proves in a general way that *there is* a root, then everything depends on the form in which it proves this. On what it is that here leads to this verbal expression, which is a mere shadow, and keeps mum about *essentials*. Whereas to logicians it seems to keep mum only about incidentals. (RFM V 24, 25)

The important differences between types of proofs are masked by their shared logical expression: there is an x, such that ...

The intuitionist, on the other hand, errs in thinking that the concept of existence itself forces us to reject the platonist,s non-constructive existence proofs:

> When the intuitionists and others talk about this they say: "This state of affairs, existence, can be proved only thus and not thus." And they don't see that by saying that they have simply defined what *they* call existence. For it isn't at all like saying "that a man is in the room can only be proved by looking inside, not by listening at the door." (PG, p. 374)

The intuitionist's version is no better than the platonist's; neither can be independently justified, nor need they be. In contrast with the reforming constructivist of section I above, the anti-philosophical Wittgenstein does indeed leave mathematics as it is; his only interest is to relieve us of the notion that the mathematical practices we have can be defended by philosophical arguments.

There is, however, yet another strain in Wittgenstein's writings on mathematics. To Hilberts famous cry - "No one shall drive us out of the paradise Cantor has created for us"[11] - Wittgenstein is quoted as replying:

> I would say, "I wouldn't dream of trying to drive anyone out of this paradise." I would try to do something quite different: I would try to show you

> that it is not a paradise - so that you'll leave of your own accord. I would say, "You're welcome to this; just look about you." (LFM, p. 103)

And this passage is not unique; there are similar passages in other places:

> What I am doing is, not to show that calculations are wrong, but to subject the *interest* of calculations to a test. (RFM II 62)

> Philosophical clarity will have the same effect on the growth of mathematics as sunlight has on the growth of potato shoots. (In a dark cellar they grow yards long.) (PG, p. 381)

A constructivist might think that the law of the excluded middle and non-constructive proofs are the sorts of things philosophical clarity will prompt us to forego, but from the anti-philosophical point of view, this interpretation overlooks the non-reformist reading just rehearsed and puts an unbearable strain on the maxim that philosophy leaves mathematics as it is. But the question remains: where are these unlit potato shoots?

Wittgenstein's answer isn't hard to find:

> I want to say: it is essential to mathematics that its signs are also employed in *mufti*. ... It is the use outside mathematics, and so the *meaning* of the signs, that makes the sign-game into mathematics. (RFM V 2)

From this point of view, the parts of mathematics without application are just empty games with meaningless signs, a classic example of "language on holiday". Pure mathematics is

> ... a piece of mathematical architecture which hangs in the air, and looks as if it were, let us say, an architrave, but not supported by anything and supporting nothing. (RFM II 35)

As mathematicians realize this, the face of mathematics will change:

> What will distinguish the mathematicians of the future from those of today will really be a greater sensitivity, and *that* will - as it were - prune mathematics; since people will then be more intent on absolute clarity than on the discovery of new games. (PG, p. 381)

Applied mathematics can be made clear - it has a real use - but pure mathematics should be pruned away.

But what exactly is wrong with pure mathematics? To see this, we must first distinguish what a mathematician actually does from

> ... what a mathematician is inclined to say about the objectivity and reality of mathematical facts. (PI 254)

The former we leave as it is (PI 124), but the latter is "something for philosophical *treatment*" (PI 254). Philosophical treatment, as we've seen, involves showing that a form of words we're inclined to use actually fails to function, most often because

we have removed those words from the context that gives them their sense. In this case, we have appropriated the language appropriate to natural science:

> ... if you forget where the expression "a reality corresponds to" is really at home - ... What is "reality"? We think of "reality" as something we can *point* to. It is *this, that*. ... Professor Hardy is comparing mathematical propositions to propositions of physics. This comparison is extremely misleading. (LFM, p. 240)

This leads to the idea of mathematics as "the natural history of mathematical objects themselves" (RFM II 40), of mathematicians as studying "mathematical *objects* and their queer properties" (RFM V 5). We are then confronted with the numerous deep philosophical quandaries that haunt a platonist ontology: what are these entities? where are they? how do we know about them? But as usual

> The feeling of something *queer* here comes from a misunderstanding. (RFM V 6)

That misunderstanding is the supposition that mathematics is a science.

The treatment, as always, is to avoid the slip that led us into confusion:

> ... what is caused to disappear by such a critique are names and allusions that occur in the calculus, hence what I wish to call *prose*. It is very important to distinguish as strictly as possible between the calculus and this kind of prose. (WVC, p. 149)

This treatment brings out the crucial difference between applied and unapplied mathematics: a piece of applied mathematics will retain its use even if the "prose" is eliminated; its interest derives from its effectiveness in science. The interest of a piece of unapplied mathematics, on the other hand, cannot be traced to its use in science, and this inclines the mathematician to imagine an application in a purely mathematical realm:

> One would like to say of it, e.g.: it introduces us to the mysteries of the mathematical world. *This* is the aspect against which I want to give a warning. (RFM II 40)

This dangerous prose proposes an illusory "application":

> ... *what* mathematicians take for their application - is quite fantastic ... So that ... one is doing a branch of mathematics of whose application one forms an entirely false idea. (RFM V 5)
>
> What we have then is the imaginary application. The fanciful application. (RFM V 29)

This deceptive prose gives false interest to mathematics without real application.

> The misunderstandings we are going to deal with are misunderstandings without which the calculus would never have been invented, being of no

> other use, where the interest is centered entirely on the words which accompany the piece of mathematics you make. (LFM, pp. 16-17)

Thus, philosophical clarity - the elimination of the murky aura of prose surrounding actual mathematical activity - will leave applied mathematics intact while undermining the sole motivation for the pursuit of pure mathematics.

Wittgenstein's critique of set theory fits this outline exactly; set theory is his paradigm of a branch of mathematics pursued for purely fanciful reasons. By way of contrast, he considers the introduction of imaginary points into geometry, in particular, the idea that every line intersects a given circle, though some intersect it at imaginary points:

> The proof has a certain charm if you like that kind of thing; but that is irrelevant. The fact that it has this charm is a very minor point and is not the reason why those calculations were made. - That is colossally important. The calculations here have their use not in charm but in their practical consequences. (LFM, p. 16)

- Set theory suffers in the comparison.

> It is quite different if the main or sole interest is this charm - if the whole interest is showing that a line does cut when it doesn't, which sets the whole mind in a whirl, and gives the pleasant feeling of paradox. If you can show there are numbers bigger than the infinite, your head whirls. This may be the chief reason this was invented. (LFM, p. 16)

When we realize that our picture of a magnificent world of infinite numbers is a philosophical illusion, the usual consequences are predicted:

> ... this doesn't mean that certain mathematical propositions are *wrong*, but that we think their interest lies in something in which it does not lie. I am *not* saying transfinite propositions are *false*, but that the wrong pictures go with them. And when you see this the result may be that you lose your interest. (LFM, p. 141)

The mathematician of the future "will laugh at this hocus pocus" (RFM II 22).

Though other branches of mathematical logic share Wittgenstein's disapproval, his treatment of these cases differs from that of set theory. Here the misguided motivation is not the description of a fantastic world of transfinite numbers, but the intent to provide a foundation for mathematics. Consistent with his discussion of rule-following in general, Wittgenstein holds that mathematical practice needs no foundation, no ultimate justification. By undermining the search for foundations, Wittgenstein claims to undermine the sole motivation for these branches of mathematics:

> ... what was the attempt [to formalize logic] made for at all? (What was it useful for?) Did not this need, and the idea that it must be capable of satisfaction, arise from a lack of clarity in another place? ... The question "What was it useful for?" was a quite *essential* question. For the calculus was not

> invented for some practical purpose, but in order "to give arithmetic a foundation". (RFM III 85)

Here again, with their motivation removed, mathematicians might be expected to lose interest in these studies.

Still, however draconian the changes Wittgenstein predicts for the mathematical landscape, it must be noted that the critique described here is not that of the constructivist. The use of the law of the excluded middle is irreproachable (though also indefensible) in proper mathematics - that is, in applied mathematics - as is proof by non-constructive means. Though Wittgenstein and the constructivist both disapprove of set theory and foundational studies, they do so from perspectives so different that it would be a distortion to classify the anti-philosophical Wittgenstein as a constructivist in the usual sense. His sole point of contact with constructivism is his intent to curtail classical mathematics, albeit indirectly.[12]

Notes

1 See my (1984) and (1986)

2 See Kripke (1982). A similar reading appears in Fogelin (1987). (The first edition of Fogelin's book was published in 1976; in the second edition, he compares his view with Kripke's.) I give only the barest sketch here, assuming most readers will be familiar with the wealth of details.

3 For a illuminating treatment of this point, see Chihara (1963).

4 This explains why Dummett refers to Wittgenstein's position as "an extreme version ... of constructivism" ([1959], pp. 169). For a tamer reading, see Dummett (1959), pp. 178f., or (1978), p. 115.

5 Though Kripke draws no conclusions about Wittgenstein's views on mathematics beyond the rejection of finitism, Fogelin (1968) comes quite close to a constructivist reading. Kripke does, however, express open suspicions about the non-interference claims ([1982], pp. 64-66, 69f.). Wright explicitly wonders why the philosopher should abstain form allowing "the fruits of philosophical insight to filter back into a 'straightened out' grammar" ([1980], p. 278).

6 The approach of this section and the next draws on Diamond [1991] (especially the introduction and chapter 1 and Goldfarb [1983] and [1985]. Diamond emphasizes the "rags and mice" passage (PI 52) and both Diamond and Goldfarb feature RFM VI 17.

7 In conversations with Rush Rhees in the 1940s, Wittgenstein described himself as "a disciple of Freud" (LC, p.41).

8 Cf. PI 210: 'Every explanation which I can give myself I give to him too.'

9 Notice the contrast with our earlier interpretation of the so-called "private language argument". There we saw Wittgenstein arguing for some controversial claims about what is involved in following a rule, and then inferring the impos-

sibility of private language from its failure to conform to this account of rule-following. Here our "account of rule-following" consists of some commonplaces about how training works, and private language arises as part of an argument that these commonplaces are not enough.

The treatment of private language in this section follows Stroud [1982]. (Goldfarb seconds in [1985], p. 488, and Stroud supports the "meta-philosophy" of section II above.) Diamond gives a related reading ([1989], p. 21), and Fogelin ([1987], pp. 172 - 5) makes the connection with Wittgenstein's treatment of ostensive definition early in PI without identifying this as the central argument. Goldfarb [1983] gives a more extended discussion of those early sections of PI.

[10] Th following anti-philosophical treatment of the law of the excluded middle draws on Lear (1982).

[11] Hilbert (1926), p. 191.

[12] I question Wittgenstein's prediction that set theory will lose its interest without the support of an extra-mathematical Platonism in 'Naturalizing mathematical methodology'.

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