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A SECOND PHILOSOPHY OF ARITHMETIC

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Abstract. This paper outlines a second-philosophical account of arithmetic that places it on a distinctive ground between those of logic and set theory.

In a pair of recent books,[1] I've proposed an austere naturalistic approach to philosophizing, characterized by the practices of an idealized inquirer called the Second Philosopher—an inquirer equally interested in all aspects of the world and our place in it, equally at home in physics, astronomy, biology, psychology, linguistics, sociology, anthropology, etc., and even logic and mathematics, as the need arises. The closest conventional classification is 'methodological naturalism,' described by one prominent taxonomist, David Papineau, as a family of views that 'is concerned with the ways of investigating reality, and claims some kind of general authority for the scientific method.'[2] I shy away from this succinct portrayal—and come at Second Philosophy roundabout, as what the Second Philosopher does—in order to highlight the fact that the Second Philosopher doesn't think of herself as marching under any banner of 'science method'; instead, she simply begins with ordinary perceptual experience, gradually develops more sophisticated means of observation and generalization, theory formation and testing, and so on. What she doesn't do is attempt any overarching account of her 'method';[3] when she turns her attention to the question of how best to investigate the world, she fully appreciates that her techniques of inquiry often end up needing revision and supplementation, an open-ended process that can't be foreseen and corralled in advance.[4] We describing her might use the rough label 'scientific' for her approach, but a true understanding the nature of Second Philosophy requires tracing her efforts in various particular cases and getting the hang of predicting how she would react in a new one.

In *Second Philosophy*, in one of those particular cases, the Second Philosopher turns her attention to logic, to the question of what grounds logical truth. Toward the end of that book, and as the main event in *Defending the Axioms*, she investigates the proper methods for higher set theory,[5] and the metaphysical and epistemological background that explains why these methods are the proper ones. The result is a sharp contrast between the robust worldly supports she identifies for logic, and the objective but metaphysically

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[1] Maddy (2007, 2011).

[2] See Papineau (2009).

[3] This would be to provide what's customarily described as a 'demarcation criterion' between science and nonscience.

[4] For example, in 1900, it wasn't unreasonable to think that the existence of entities like atoms could never be established, because they couldn't be observed. Perrin showed otherwise, introducing a subtle new method of confirmation into physical chemistry.

[5] By 'higher set theory,' I mean set theory that goes beyond number theory and into analysis, the study of reals and sets of reals, and beyond. From here on, I leave the 'higher' implicit.

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neutral account she offers in the case of set theory. Now what about arithmetic? Though this topic isn't the main focus of either discussion, a few glimpses turn up along the way: elementary arithmetic claims like $2 + 2 = 4$ are assimilated to those robust logical facts; a standard arithmetic of a potentially infinite natural number sequence[6] is not (where this is characterized by commonplace assumptions along the lines of informal Peano axioms—every number has a successor, proof by induction, etc.).[7]

One simple possibility would be to combine the classification of elementary arithmetic with logic and the classification of standard arithmetic with set theory, but I think this can't be quite right. In fact, I suspect there's something to the widely held belief that our grasp of the standard model of arithmetic is much more determinate than our grasp of V—which isn't to say that I lean toward the other extreme of lumping the notion of an infinite sequence[8] with the likes of $2 + 2$. The goal here is to give a second-philosophical account of the intermediate status enjoyed by standard arithmetic.

The discussion begins with a quick and sketchy recap of the two extremes represented by elementary logic and set theory (in §§I and II, respectively).[9] Much of this will be familiar to those hearty souls who've read the two books mentioned, but here the focus is on what I've sometimes called 'the ground of logical truth' or 'the ground of set theoretic practice.'[10] Very roughly, the thought is that the ground of a stretch of discourse is something extra-linguistic that guides and constrains what counts as proper or correct in that discourse, something extra-linguistic to which the discourse is responsive and responsible.[11] The plan is to characterize the contrast between logic and set theory in these terms, then to introduce the relevant point in between (in §III). Placing this proposal in a Kantian setting (in §IV) then serves to highlight the question of how and why arithmetic applies to the world (the topic of the final §V).

§1. A second philosophy of logic. The Second Philosopher's view of logic begins from her common sense observation of the world: much of it consists of individual objects (stones, trees, cats, houses, planets), with various properties (cats can change their locations but trees can't), standing in various relations (houses are generally bigger than stones, sometimes even made of stones), and one situation involving those objects often depends on another (this billiard ball moves because that billiard ball collided with it, this flower blooms because that seed was well planted and nurtured). True to her inquisitive ways,

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6 The further assumption of a completed set of natural numbers invokes the axiom of infinity, perhaps *the* characteristic hypothesis of set theory. Cantor's bold move paid off in spades with the rise of modern set-theoretic mathematics (see, e.g., Maddy, 1988, p. 486 or 1997, pp. 51–52), but as noted, the plan here is to distinguish this type of grounding (discussed in §II) from that of standard arithmetic.

7 This shouldn't be taken as a rejection of Logicism (see footnotes 51 and 52).

8 Here and elsewhere, 'potentially' is left unstated.

9 For more on the topics of §I, see Maddy (2007, Part III) and also Maddy (2014, Unpublished); for §II, see Maddy (2011). For present purposes, I won't attempt to treat all the issues these positions raise, or even all the issues addressed in these other presentations.

10 I use the word 'ground' with some trepidation, recognizing that it has recently become a contentious term of art in a priori metaphysics. No entry into that discussion is intended here.

11 This is intended in a metaphysical, not an epistemological sense: adding 'this plant is a rose bush' to our store of botanical wisdom is 'proper or correct' when the plant in question in fact enjoys the fundamental botanical features common to roses, not when, given our current state of understanding, it seems to us that it does.

the Second Philosopher examines the credentials of this common wisdom. Her inquiry reveals that the atomic and molecular structures present in the apparent interior of stones and cats differ markedly from the surrounding air, that various constraints and forces of cohesion hold these structures together and keep them moving as a unit, that electromagnetic features of the tree's surface conspire with similar features of my hand to prevent the largely empty hand from moving through the largely empty trunk;[12] in other cases, like rainbows or mirages, this same inquiry reveals that apparent objects are sometimes only apparent. Similarly, she describes and confirms transfers of momentum between billiard balls and tests for the efficacy of such expedients as the use of Miracle Gro. Finally, she recognizes that few properties and relations are be fully determinate (there are tadpoles and frogs, juvenile and fully mature specimens of a certain biological kind, and just as surely there are borderline cases between these two).

Various forms of skepticism threaten this straightforward account. After all, the Second Philosopher's investigation itself relies on her beliefs about the properties and relations of many objects, including, for example, various measuring devices and detectors—what kind of evidence does she have for those beliefs, and for the beliefs on which that evidence relies, and so on? One such worry is that any inquiry beginning from beliefs involving objects with properties, standing in relations, with dependencies, will inevitably confirm the existence of such things. We've seen that this isn't true in the broadest sense: the Second Philosopher's inquiry does in fact lead to a rejection of some apparent objects, and she eventually uncovers a blanket failure of our usual ideas of objects with properties, standing in relations, when she looks into the 'particles' of quantum mechanics,[13] where her familiar style of dependencies also appears to fail.[14] But could such an inquiry ever lead her to conclude that the entire framework is flawed, at every level of description, that all such objects are ultimately illusory? It might seem that this couldn't happen, that the discovery that there are no ordinary objects would undermine the considerations that produced it in the first place, but perhaps this is too quick: in the wake of such a surprising discovery, the discredited ordinary objects might be reinterpreted as crude markers for the (nonobject-like) realities of fundamental physics, markers too crude to be legitimate entities that are simply constituted by those underlying realities.[15] In any case, at least it isn't obvious that conceptualization in terms of objects with properties, standing in relations, with dependencies, is trivially self-justifying. There remains the radical skeptical worry that all our theorizing about the external world is baseless, but this is too large a subject to broach here.[16]

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[12] Some (e.g., Eddington, Sellars, Ladyman and Ross) would distinguish the ordinary tree trunk from the largely empty tree trunk. For discussion, see Maddy (2014).

[13] I allude here to such phenomena as the twin-slit experiment (particles don't appear to traverse continuous trajectories), the challenge to individual identity from quantum statistics (the state with particles *a* and *b*, in that order, doesn't appear to differ from the state in which they are switched), and Stern-Gerlach experiments (a particle can't have vertical and horizontal spin properties at the same time). See Maddy (2007, §§III.4.i and ii), and the references cited there.

[14] This time I have in mind EPR-type phenomena, where measuring a particle here seems to affect the properties of a particle over there with no intervening mechanism. See Maddy (2007, §III.4.iii), and the references cited there.

[15] As I understand it, this is the position of Ladyman & Ross (2007). Though I obviously don't endorse their line of thought (see footnote 12), it does illustrate how this sort of discovery could go.

[16] Unlike some naturalists, the Second Philosopher doesn't simply dismiss the external world skeptic. See Maddy (2007, §§I.1 and I.2, 2011a) for her take on the dream argument, and Maddy

Thus, in telegraphic summary, the relevant metaphysics. On the epistemic side, developmental psychologists have confirmed that we come to detect the presence and absence of these medium-sized objects and their properties and relations quite early on, prelinguistically, along with conjunctions and disjunctions of these and simple transfers of momentum. Indeed, we share these abilities with a wide range of animals, from birds to monkeys, which suggests an evolutionary origin.

To summarize, then, the Second Philosopher concludes that much of the world displays a familiar abstract template—a domain, properties and relations, dependencies—amenable to a strong Kleene three-valued scheme;[17] let me call this formal structuring KF, in honor of Kant and Frege, from whom it derives. Any KF-world validates most classical inferences involving 'and,' 'or,' 'not,' 'all' and 'exists': for example, the DeMorgan laws, the distributive laws, double negation elimination, universal instantiation, etc. Exceptions arise, for example, with the Law of Excluded Middle (if *p* is indeterminate, so is *p*-or-not-*p*), modus tollens (if-*p*-then-*q* and not-*q* only imply that *p* isn't true, but it might be indeterminate), and (for similar reasons) reductio ad absurdum. This much rudimentary logic is reliable in any domain with KF-structuring, and thus valid in our world insofar as it enjoys this structure.

Consider, for example, a simple logical truth: if the book is either red or green, and it's not red, then it's green. This claim is true in a given bibliographical situation not because the book has a certain physical composition or light behaves a certain way, but simply because the book is an individual object with properties, and red and green are such properties.[18] As long as this structure is present, the inference is reliable: the rare books are bound in a special red binding and shelved on the third floor with the green books; this book is shelved on the third floor, but it's easily available from Amazon; therefore, this book is green. Here our particular interest is in what grounds this sort of inference, that is, what makes it reliable, or in my more general terms, what makes it a proper or correct move to make in this stretch of discourse. We now have the Second Philosopher's simple answer: all that matters is the presence of KF-structuring. This is not unlike the sense in which the interactive behaviors of a given chemical compound are grounded in the arrangements of its molecules and so on, not in the cost of a beaker of the stuff.

So the Second Philosopher's position is that (1) this rudimentary logic applies to any aspect of the world with KF-structuring, which many of the world's aspects, at various levels,[19] do enjoy, (2) human beings tend to trust its simpler inferences because their most primitive cognitive mechanisms detect and represent many of these structures,[20] and (3) those primitive cognitive mechanisms are as they are because human beings (and their evolutionary ancestors) interact almost exclusively with aspects of the world that do enjoy this structure.

(2011b) for the argument from illusion. There remain arguments based on the infinite regress of justifications and the closure principle, which I hope to discuss one day.

17 The exception is the rudimentary conditional, which retains some nontrivial dependency relation between antecedent and consequent (see the transition to classical logic below).

18 Of course color properties are more complex than this suggests, but I hope I may be forgiven this oversimplification for the sake of illustration.

19 For example, it appears among various decks of cards with various back designs standing in various spatial relations, but also among various individual cards of various different suits standing in ordering relations within a single deck. Indeed, the relevant structures can crosscut at a single size-scale, as with Quine's rabbits and rabbit stages.

20 There are no doubt many aspects of the world with this structure that we can't or don't notice.

In stark contrast to the status of logical truth and validity on most philosophical treatments, here rudimentary logic is contingent on the way many aspects of our world happen to be, as is clear from the fact that it doesn't work, isn't reliable, in the quantum world, where the requisite structure is missing. The distributive law is a familiar example: it's valid in rudimentary logic, but in the quantum world, it may be that a given electron has vertical spin up or vertical spin down, and horizontal spin right or horizontal spin left, even though none of the obvious conjunctions—vertical spin up and horizontal spin right, vertical spin up and horizontal spin left, vertical spin down and horizontal spin right, vertical spin down and horizontal spin left—in fact obtains.

Despite this, the conviction that the world must exhibit KF-structuring, that all possible worlds have (at least) that much structure, remains strong, so strong that some less naturalistically minded philosophers are willing to reject quantum mechanics! This reaction is understandable: KF-structure is embedded in our most primitive ways of thinking; a modern day neo-Kantian might even claim that it's impossible for us to cognize, either perceptually or theoretically, without it. The Second Philosopher sees no firm ground on which to draw this strong conclusion, but there's no doubt that the sway of these forms runs very deep, so deep that we've enshrined them in our very conception of a 'possible world.' Little wonder, then, that logical laws strike us as necessary, and that quantum mechanics is so baffling.

Another orthodox view is that our knowledge of logic is a priori. If in fact those primitive cognitive mechanisms are instilled by evolution, written into our genetic code, then it could be that our tendency to trust the simpler inferences of rudimentary logic comes to us with no input from experience—and a sufficiently externalist epistemologist might argue that this counts as a priori knowledge. To defend our belief, however, to collect sufficient evidence for the reliability of rudimentary logic, requires empirical investigation of the sort summarized here, so an internalist epistemologist might call the case the other way. The Second Philosopher sees no special need to pass judgment on the 'concept of knowledge'—she may doubt that anything sufficiently precise actually answers to our undeniably effective use of the word 'know'—so she rests content with describing the relevant facts of the case.

Finally, there is the venerable philosophical notion of analyticity, so often affirmed on logic's behalf: in modern form, logic is said to be true by virtue of the meanings of the logical particles. So, for example, it might be claimed that the distributive law can't fail, because its validity follows from, or is somehow contained in, the meanings of 'or' and 'and.' This may be true—just as the validity of the inference from any p to any q follows from, or is contained in, the meaning of 'tonk'[21]—but no amount of truth by virtue of meaning will make the distributive law viable in application to that electron considered a moment ago. In other words, perhaps one can make whatever one likes true by adopting the appropriate meanings, but it remains a contingent matter, an empirical question, which meanings are suitable, which are effectively applicable, in which worldly situations. So even if rudimentary logic is analytic in some sense, its successful use remains answerable to the presence of the worldly structures identified by the Second Philosopher.

One final step is needed, because rudimentary logic, a rather weak and ineffective instrument, lies at some distance from the full force and glory of classical logic. What bridges the gap? The Second Philosopher's answer is that various idealizations are brought into play: we pretend that every referring expression of the language in question succeeds

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[21] Begin with p. By the introduction rule for 'tonk,' it follows that 'p tonk q.' Then, by the elimination rule for 'tonk,' we arrive at q.

in picking out an individual, that its predicates aren't vague or otherwise indeterminate, that the dependence of one situation on another can be adequately represented by the material conditional. As with any scientific idealization, these should only be employed where and when they are both effective and benign, and care must always be taken not to abuse them: just as neglecting friction too broadly leaves us unable to explain how we manage to walk, neglecting vagueness too broadly can leave us with a sorites problem. Many of the well-known nonclassical logics—free logic, various conditional logics, the many attempted logics of vagueness, etc.—can and, from the second-philosophical point of view, should be viewed as arising from the conviction that the corresponding idealization is either not effective or not benign in some or all situations, and that the proposed alternative is preferable. It isn't enough to show that a certain corner has been cut or falsification introduced—on the proposed account, the classical logician admits as much—the more difficult task is showing where and when this amounts to a damaging distortion and how an alternative logic (not just more careful use of classical logic) does a better job. This is no doubt a worthy project, but at least for now, I think it's not clear that such a case has been convincingly made.

Thus, in quick summary, the Second Philosopher's account of logic: rudimentary logic, where it holds, is grounded in facts of the world's contingent structure[22]; classical logic results from a string of (apparently) beneficial and benign idealizations. Now we turn to the other extreme, the case of set theory.

§2. A second philosophy of set theory. As the Second Philosopher's investigation of the world becomes more sophisticated, she eventually feels the need for mathematical tools beyond logic and arithmetic, just as Newton felt the need for the calculus. Recapitulating the developments of the 18th century, she devises ever-improving methods of analysis, following Euler and his contemporaries; retracing the progress of the 19th century, she begins to see the wisdom in pursuing pure mathematics, if only to provide a greater variety of potential abstract models for physical situations.[23] At that point, her mathematical inquiries often aim at internal mathematical goals, like Cantor's desire to extend the theory of trigonometric representations, or Dedekind's interest in representation-free definitions, or Zermelo's hope of analyzing fundamental mathematical notions like 'number' and 'order' and 'function' in their simplest form, or the modern set theorist's pursuit of a mathematically rich theory of sets of real numbers. Given the relevant mathematical goals, she isolates and evaluates the methods of contemporary set theory, assessing them as proper insofar as they facilitate the attainment of those goals.

But knowing how to go about set theory isn't all there is to her study; she also wants to understand what this particular human practice is doing: does it have a subject matter of its own, like physics or botany? If so, how do its methods manage to track the truth about that subject matter? Phrased in slightly nontraditional terms, these are the classic metaphysical and epistemological questions in the philosophy of mathematics.

The answers proposed in *Defending the Axioms* proceed in three stages: description of two apparently second-philosophical takes on set theory, depending on whether or not it's regarded as a body of truths, followed by an account premised on an analysis of the relationship between these two. At stage one, then, we suppose the Second Philosopher

22 To be clear, the claim isn't just that rudimentary logic holds where there's KF-structuring—a near tautology—but also that many aspects of the physical world do in fact have this structure.

23 This isn't the only reason, as should become clear by the end of this section.

takes set theory to be a body of truths. Assuming those truths say what they seem to say, it follows that many sets exist, and along the way, many of their properties and relations are catalogued. The standard practices invoke no dependence on us, no positioning in space or time, no causal interactions, so according to the only methods so far identified as reliable for finding out about sets, our ontology satisfies the usual negative criteria for abstractness.

This may seem to raise the familiar Benacerrafian epistemological challenge—how do we come to know anything about entities of this abstract variety?[24]—but notice that the line of thought behind that challenge runs something like this: set theory is an a priori discipline concerned with the features of a domain of abstract objects; we humans live in the physical world, which is entirely separate from the world of abstracta; how do our mundane procedures of selecting axioms and proving theorems manage to track what's going in that entirely separate world? This style of reasoning—from a metaphysical account of the nature of set theory to a question about the propriety of its actual methods—reverses the Second Philosopher's approach: she begins by isolating and analyzing those actual methods and assessing their rationality as means toward the goals of the practice; when she comes to regard set theory as a body of truths, she judges that the very methods that accomplished this are (largely)[25] reliable in tracking the facts about the relevant domain. For her, the question of set theory's metaphysics comes last, in a form with a faintly Kantian flavor—what are sets that we can know them in these ways?—and she adopts the simplest hypothesis available: that is precisely what sets are, things that can be known about in these ways. She resists the temptation to make any claims about sets that go beyond what set theory tells us.

We have here two forms of realism about the domain of set theory: a Robust Realism that begins from the metaphysical end, argues that a nontrivial epistemology is needed, then presents a challenge to the actual methods of set theory; a Thin Realism that begins from the methodological end, argues that the actual methods are (largely) reliable, then posits a minimal metaphysics consistent with that conclusion. The Robust Realist worries over the determinacy of the Continuum Hypothesis (CH), as a matter of metaphysics—is V or the concept of 'set' (or some such) sufficiently determinate?—then considers that excluded middle may be mis-applied in this case; the Thin Realism trivially derives the determinacy of CH from the centrality of excluded middle among set theory's effective methods.

But this Thin Realism may seem too easy. Shouldn't an objective ontology imposes some constraints on our practices? And if it does, don't we fall back into demanding the sort of nontrivial epistemology characteristic of Robust Realism? The Second Philosopher's answer is that set-theoretic methods track sets, of course, and that its methods *are* objectively constrained. Just as the concept 'group' was shaped by the kinds of mathematical goals that abstract algebra in general, and the developing group theory in particular, aimed to serve, the choice of axioms for set theory is guided by the various mathematical jobs the theory is intended to do: extending the theory of trigonometric representations (as in Cantor), facilitating representation-free definitions and abstract reasoning (as in Dedekind), clarifying the notions of 'real number' and 'continuity' (as in Dedekind) and of 'number' and 'order' and 'function' (as in Zermelo)—and eventually, providing an intra-mathematical style of foundation for classical mathematics,[26] enriching the theory

[24] See Benacerraf (1973).

[25] Of course there's always room for discovering and correcting errors—errors of judgment in the selection of concepts or axioms, as well as logical errors in proofs—and improving those methods.

[26] I have in mind here neither a metaphysical foundation—which tells us what mathematical objects 'really are'—nor an epistemological foundation—which explains how we come to know

of reals and sets of reals (as in determinacy theory), and perhaps one day settling the Continuum Hypothesis. The suggestion is that all these can be seen as ways of promoting the kind of thing often loosely classified as 'deep' mathematics,[27] and that 'depth' here is an objective mathematical feature, not just a matter of passing fashion or subjective whim.[28] So the Thin Realist's methods are indeed answerable to sets, but these sets turn out to be markers for the objective currents of mathematical depth.

Thin Realism, then, is the second-philosophical account of set theory that results when set theory is regarded as a body of truths. The other option is to see the set theorist as engaged in a largely successful practice of developing a theory of sets in pursuit of important mathematical goals of the sort listed a moment ago—not in the business of uncovering truths, but of devising effective ways of doing various pressing mathematical jobs, just as group theory, when the concept was being formed, was out to devise effective ways of doing important mathematical jobs of its own. On this picture, no metaphysics, no abstract ontology, is differentiating correct from incorrect decisions on axiom or concept choice; nevertheless, it's emphatically false to say that 'anything goes': those decisions are sharply constrained by the demands of the mathematical aims in play. Call the Second Philosopher who follows this line of development an Arealist.

The Thin Realist and the Arealist may appear far apart—one holds that sets exist and set theory describes them, the other holds that sets don't exist and set theory isn't in a descriptive line of work at all—but in fact the similarities are more striking than the differences: at a fundamental level, the objective reality that guides both the Thin Realist's description of sets and the Arealist's development of set theory is precisely the same strains of mathematical depth; the considerations that lead the Thin Realist to think a given axiom candidate is true are precisely the same considerations that lead the Arealist to think that axiom candidate is a good one to add to the official list of axioms. As far as the pursuit of set theory goes, the practice of actually doing set theory, the two are entirely indistinguishable. The difference only comes in the aftermath, where one adds the words 'true,' 'exist,' 'know,' etc., and the other feels no need to so embellish.[29] The real contrast, we come to

mathematics—nor a conceptual foundation—which provides a conceptual framework in which all mathematics should take place (for more, see Maddy, 1997, §I.2). As sketched in Maddy (2008) and Part I of Maddy (2011), the rise of pure mathematics at the end of the 19th century produced a new methodological question for mathematics: how can the legitimate mathematical structures and proofs be separated from the problematic ones, now that the relation to the physical world no longer guides us? Set theory was designed, in part, to fill this particular need, a process I like to call 'queasiness removal.' More disclaimers: this isn't to say that the other types of foundation are wrong-headed or impossible, just that queasiness removal is all I take set theory to aspire to; the suggestion is that this is one of the goals of set theory, and thus properly bears on how set theory should be conducted, not that set theory is necessarily the only way it could be achieved; seeing set theory as this sort of foundation isn't to limit mathematics to what set theory can found—it's set theory's job to be broad enough, not mathematics' job to be narrow enough.

27 In Maddy (2011), where 'depth' is introduced, no definition or analysis of the notion is attempted; instead I provide a range of examples and undertake to explain how and why they qualify, in terms of means/ends reasoning based on the goals of set-theoretic practice.

28 Of course, even natural science is tied to human interests and abilities, in that we are drawn to certain areas of inquiry by our interests, hampered or helped by certain of our abilities, and so on. The question is whether mathematical depth is tied to our interests and abilities in some more fundamental way, and my claim here is that it isn't.

29 Even the Arealist may indulge in 'truth' and 'existence' talk inside of set theory: a set 'exists' in the sense that a certain existential statement follows from the axioms, or a theorem is 'true' in the sense of holding in V. Where the Thin Realist goes further is in applying external notions of

see, is between the Thin Realist and the Arealist, on the one hand, and the Robust Realist, on the other. Of the three, only the Robust Realist thinks that the methods of set theory need ratification from an extra-mathematical source.

Thus stages one and two. At the final stage three, the Second Philosopher called upon to adjudicate: who's right, the Thin Realist or the Arealist? They begin from the same second-philosophical starting point—investigating the world in ordinary empirical ways, gradually correcting and improving their methods—they both come to applied, then to pure mathematics by the same route—recognizing the usefulness of the former in their theorizing, coming to appreciate the value of pure mathematics, if only for its unexpected applications[30]—they both ratify the actual methods of set theory and pursue it in exactly the same ways, guided by the same mathematical values. The only point of contention arises when they're faced with deciding whether or not to classify set theory along with physics, chemistry, botany, astronomy, linguistics, psychology and so on.[31] The Thin Realist is particularly impressed by the similarities between sets and concreta, for example, in the logically relevant features of enjoying properties, standing in relations, exhibiting dependencies, and in the use of logic, theory formation, and means-ends reasoning in their study; overall, she concludes that set theory is yet another body of truths, that her step-by-step investigations of the world have revealed the presence of abstract as well as concrete entities. Meanwhile, the Arealist is more impressed by the differences between abstracta and concreta, between set-theoretic methods and her more familiar ways. She sees set theory as a new sort of practice, different in kind from her investigations of the world, and feels no need to fit it into that mold. So who's right?

Tempting as it is to think there must be a right and a wrong answer here, my suggestion in *Defending the Axioms* is that this is an illusion, that no fact of the matter determines whether or not the notions of truth and existence, quite at home in empirical science, should be extended to set theory and the rest of mathematics. The case runs more-or-less parallel, I propose, to that of amorphous ice, a nice example of Mark Wilson's:[32] when water is cooled very quickly, it doesn't form the crystalline structure of ordinary ice, but a less organized solid structure more like that of ordinary glass. So, is amorphous ice really ice? Some chemists describe it as a kind of ice, others describe it as an ice-like solid, but once the underlying facts are clear, either way of speaking seems acceptable. In this way, I propose that once we understand how set theory arose, what set-theoretic practice aims at, what its methods are designed to track, the Thin Realist's way of describing it and the Arealist's way of describing it are both acceptable. The choice is an inconsequential matter of preference.

So in the end, the Second Philosopher differs from both the Thin Realist and the Arealist on ontological issues—she thinks we don't go wrong speaking in either idiom—but our central concern here is the ground of set-theoretic practice: what guides and constrains our choice of new axioms?, to what the extra-linguistic something is the discourse responsible and responsive? The answer, it turns out, is that common core of both Thin Realism and Arealism: those underlying facts of mathematical depth, facts analogous to the ones

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'truth' and 'existence,' notions available at the level of the Second Philosopher's discussion of set theory as a human practice. This is the point at which the Arealist demurs.

30 For example, group theory (see Maddy, 2007, pp. 330–331). As remarked in footnote 23, this isn't the only motivation (as comes out in a moment).

31 I put the question this way, rather than simply asking whether or not set theory counts as a science, because neither the Thin Realist nor the Arealist, both Second Philosophers at heart, employs a demarcation criterion for separating science from nonscience.

32 See Wilson (2006, pp. 55–56).

about the behavior of water when quickly cooled. If this is right, and for present purposes I assume that it is, we now see why its surprising applications aren't the only reason the Second Philosopher has for pursuing pure mathematics: the phenomenon of deep mathematics is as real as any other she studies, and her curiosity naturally extends to tracing its contours and understanding its nature. What we have here is a form of Objectivism without objects, and even without truth. We've taken Kreisel's famous aphorism—that what grounds mathematics isn't mathematical objects, but the objectivity of mathematical truth—and gone one better: what grounds it isn't even truth, it's mathematical depth.

§3. The ground of arithmetic. At this point, the stark differences between rudimentary logic and higher set theory are obvious. Both are grounded in objective phenomena—KF-structuring and mathematical depth, respectively—but those objective realities are quite different, and the two practices are guided and constrained by them, responsive and responsible to them, in different ways. On the one hand, KF-structuring is a straightforward feature of (some aspects of) the physical world, and it exercises its influence in equally straightforward fashion: 'if the book is either red or green, and it's not red, then it's green' is true because the book is an object and colors are properties, just as 'compound X interacts with Y to form Z' is true because of the molecular structures of X, Y, and Z.[33] The facts of mathematical depth, on the other hand, certainly don't appear to be ordinary physical facts; the attractions of the concept of group or an axiom of large cardinals derive from their role in the fruitful pursuit of abstract algebra or higher set theory, in their purely mathematical virtues. These facts of mathematical depth don't make a given large cardinal axiom true—or at least they do so only in the entirely optional usage of the Thin Realist—what they do is make it a good axiom to add to our list; they 'guide and constrain' the practice of set theory, the practice is 'responsible and responsive' to them, but in the looser sense of determining proper or correct development, not in sense of generating truths. In the terminology of §II, the Second Philosopher's account of logic in §I counts as a robust form of realism—the developmental story anchors a nontrivial epistemology, the tell-tale mark of robustness—while the realism of §II is thin at best.

Now, at last, we're in a position to take up our central question: where does arithmetic fit into this picture? Let's begin from the logical end of the spectrum, where the connections are closest. So far we've talked about cases like our simple bibliographical example, but if we're studying logic itself, not library science, our interest isn't in claims about individual books and their colors, but in generalizations like disjunctive syllogism: for any objects, any properties, an inference from *Pa*-or-*Qa* and not-*Pa* to *Qa* will be valid. This is a claim about any KF-structure: if the first two hold in such a structure, so will the third.

Compare this with a claim of elementary arithmetic, like $2 + 2 = 4$. In a particular case—two apples and two oranges on the table make four fruits—this is just another fact of rudimentary logic, a valid inference only somewhat more complex than the one about the book and its color: from 'there is a thing, and another thing, both apples, both on the table, and a thing, and another thing, both oranges, both on the table' and 'nothing's both an apple

[33] I don't think either of these presupposes a more-than-disquotational brand of truth: 'if the book ...' is true iff (by disquotational truth) if the book ... iff (by the grounding of logic) the book is an object and colors are properties; 'compound X ... ' is true iff (by disquotational truth) compound X ... iff (by the grounding of chemistry) the molecular structures are so-and-so. See Maddy (2007, pp. 370–376), for an extended argument that the distinction between Robust and Thin Realism is independent of the distinction between correspondence and disquotational truth.

and an orange' to 'there's a thing, and another thing, and still another thing, and yet another thing, all fruits, all on the table.'[34] $2+2=4$, like disjunctive syllogism, is a generalization of these individual facts, again a generalization that doesn't depend on botanical facts, but only on the presence of logical structure.[35] In both cases, our real interest is in general features of any KF-structure.

So far so good. But there's a well-known limitation to this approach: if the world is finite, if there's an upper bound on the size of actual KF-structures, then simple logico-arithmetical inferences that reach above that bound will come out wrong. For example, if the most numerous logical structure that exists in our world has only ten objects, then the inferences corresponding to $6+7=1$ and $6+7=10$ will both be valid, that is, any situation with the $6+7$ structure will have both the 1 structure and the 10 structure, because there aren't any situations with the $6+7$ structure. So the worldly features that ground rudimentary logic aren't enough by themselves to sustain arithmetic.

This is the sort of thing that drives many philosophers to modality: we aren't just talking about all *actual* KF-structures, but about all *possible* KF-structures. In the spirit of Russell's dismissal of human limitations as 'merely medical,' we're tempted to think that the lack of large KF-structures is a 'merely physical' limitation, and to posit a sort of metaphysical possibility. Despite the confident judgments of many modal metaphysicians that flying pigs and talking donkeys are possible, I think it's reasonable to expect the Second Philosopher to be perplexed by this notion, at least initially, to be uncertain of its purport. Perhaps better to tackle the local problem on its own terms, simply to ask what *does* ground arithmetic, given that the ordinary physical ground of rudimentary logic is not enough. What supports our confident judgments about the large finite, the 'and so on', the '...' of an infinite sequence?

Before taking this on, let's pause a moment to note that the '...' also turns up early on in our study of logic. It's all very well to talk about individual logical truths and inferences, but as soon as we turn to precise discussion of even simple generalizations like disjunctive syllogism, we find ourselves reluctant to restrict, say, the length or complexity of the formulas allowed in the disjunctive premise—and this sort of thought puts us on the road to recursive definitions of 'term' and 'formula' and so on, to proofs by induction on length, and all the rest of formal logic. Just as every number has a successor, every formula has a negation; whatever it is that grounds the '...' of arithmetic might be expected to do the same for the '...' of logic. But what is that?!

Let me sneak up on this question by first asking another, about ordinary human psychology: what prompts us to think that for every number there's always a next number? This new question returns us for the moment to the developmental literature.[36] There the evidence indicates that we have two primitive proto-numerical systems. The first, which delivers information about small numbers (up to three for infants, four for adults), is the very mechanism that underlies our ability to detect and represent individual objects; its basic function is to keep track of these individuals as they appear, as in 'here's an object, here's another, here's still another.' This so-called 'object tracker' is obviously implicated

[34] The assumption that apples and oranges are fruits is left implicit.

[35] Though '$2+2=4$' doesn't look like a generalization over KF-structures—it looks like a claim about certain individuals, an operation and a relation—this can be viewed as a notational convenience, introduced because we operate more easily with singular terms, etc. This expedient fails when we reach quantified statements about numbers (see below).

[36] For summaries, see Maddy (2007, pp. 319–327), and the references cited there. For expert overviews, see Bloom (2000, chap. 9), Carey (2009, chap. 4, 7 and 8).

in our treatment of both rudimentary logic and elementary arithmetic. The other primitive system, called the 'analog system,' handles larger collections, but does so imprecisely: an infant can distinguish 8 dots from 16 and 16 dots from 32, but muddles 8 vs. 12; an adult's system is only slightly more accurate, up to a ratio of 2:3. Both these systems are present in animals, again suggesting an evolutionary origin, but neither is equipped for the '. . ..' So where does this element arise?

One conspicuous oddity here is that a young child might know what 'one,' 'two' and 'three' mean—that they apply to a single individual, a group of two individuals, a group of three individuals—and also know how to count, say, up to ten—that is, know how to generate the sequence 'one, two, three, four, five, . . . ten' in the correct order, and how to match number words with individuals in a group of objects in a one-to-one fashion—and at the same time, this same child might well not realize that the final number generated in such a counting procedure is the number of objects in the group counted. For example, asked to give the experimenter six objects, the child might give four; prompted to count, the child does so perfectly—'one, two, three, four'—and then confidently concludes that there are indeed six objects.[37] In fact, the course of development is fairly uniform: starting at about two years of age, the child understands that 'one' means one, but treats 'two' and higher roughly the same as 'some'; after several months, 'two' is mastered, then 'three'; somewhere around three-and-a-half years, somewhere between learning 'three' and learning 'five,' there's a sudden generalization, and in a flash, the child understands the meanings of all the number words in her counting range.

This breakthrough doesn't occur in nonhuman animals: they can be taught, laboriously, to associate groups of objects with numerical symbols of some kind, but this continues to be an arduous task at each stage, with 'four' or 'six' as difficult and time-consuming as 'three.'[38] The decisive factor in humans appears to be their command of number words:

> . . . human numerical cognition develops only as a result of children acquiring the linguistic counting system of their culture. (Bloom, 2000, p. 236)

The key insight of the toddler is that adding one object to a group corresponds to moving up one step in her store of number words; counting then becomes a way of detecting the number properties present in the world. The reason this is so difficult and takes so long may well be that the sequence of number words has a quite different cognitive basis from the object tracker or the analog system; it traces to the mechanisms underlying language, sometimes referred to, in black box terms, as the 'language-learning device.' This is why the young human eventually outshines his animal counterparts:

> . . . it is not a coincidence that nonhuman primates lack both generative numerical understanding and a generative language: they lack a generative numerical system just *because* they lack the capacity to develop a generative communication system. (Bloom, 2000, p. 236)

Numerical understanding apparently arises from one particular aspect of language, the sequence of number words,[39] but nonlinguistic animals lack even this much.

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37 See Bloom (2000, pp. 219–220), and the references cited there; also Carey (2009, pp. 297–302).

38 See Carey (2009, pp. 330–331).

39 See Bloom (2000, pp. 236–237): 'The claim here is not that numerical understanding emerges from language in general but that it emerges from learning the system of number words. Some

So, once again, back to our central question: whence the '. . .'? Even after the sudden realization that each of their counting words is linked to a certain worldly feature, human children will often agree to take up the challenge of naming the largest number—only to see each worthy candidate foiled by the sly expedient of adding 'plus one.'[40] This kind of exercise soon prompts the belief that there is always another entry in the sequence of number expressions, and this conviction is presumably based, not in the object tracker, not in the analog system—not even in reality, since there's actually a finite limit to the expressions we can produce—but in a fundamental feature of the language-learning device:

> All approaches agree that a core property of [the language-learning device] is recursion . . . [it] takes a finite set of elements and yields a potentially infinite array of discrete expressions. (Hauser *et al.*, 2002, p. 1571)

I'm not aware of any empirical work probing the child's understanding of numerical expressions too large to write or speak, but it seems fair to say that by the time we master something like the decimal system, we have come to think that despite the limits of paper, pencil and human breath, there is always, at least in principle, another numeral.

This, I submit, is the root of the '. . .': the 'in principle' structure of the sequence of numerical expressions, grounded in the language-learning device. Much as our primitive cognitive architecture, designed to detect KF-structure, produces our firm conviction in simple cases of rudimentary logic, our human language-learning device produces a comparably unwavering confidence in this potentially infinite pattern. And given the child's hard-won understanding that the number sequence measures the size of collections, this brings with it a corresponding conviction that the size of KF-structures is also, in principle, unlimited:

> It is not that somehow children know that there is an infinity of numbers and infer that you can always produce a larger number word. Instead, they learn that one can always produce a larger number word and infer that there must therefore be an infinity of numbers. (Bloom, 2000, p. 238)

Similarly, we

> might figure out that there is an infinity of possible musical compositions by noting the generative nature of musical notation. (Bloom, 2000, p. 238)

And we noted earlier that the same sort of conviction arises in formal logic.

At this point, the essential element is in place—our basic conception of the '. . .,' which eventually grows into our 'intuition' (as mathematicians and philosophers of mathematics like to call it) of an infinite sequence—but these primitive 'numbers' are still just convenient shorthand for the numerical properties of KF-structures: '$2 + 2 = 4$' stands in for

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people might have little knowledge of syntax, but so long as they have learned the system of number words (as is the case for numerical idiot savants), a rich understanding of number is possible. On the other hand, a person needs to have command of a counting system to have a precise understanding of larger numbers or any appreciation of the generative nature of number, regardless of how otherwise sophisticated his or her language is.'

[40] Thanks to Barbara Sarnecka and Teddy Sarnecki for this way of putting the point. Bloom's example is the realization that 'one can say "a trillion," "a trillion trillion," "a trillion trillion trillion," and so on' (Bloom, 2000, p. 238), not decimal notation or 'one,' 'one plus one,' 'one plus one plus one,' but the underlying idea is clear.

a more complicated fact of rudimentary logic.[41] Standard arithmetic is a first attempt to systematize these rudimentary logical facts with principles like the Peano Axioms, and it turns out to work well in that capacity, but it does so by quantifying over, by reifying, some things called 'numbers.' What does the Second Philosopher make of these new entities?

In §II, we confronted the same question for sets, and here the Second Philosopher follows a similar line of thought. First comes the move to reject Robust Realism: she has a mathematical theory that's doing just what it's designed to do—systematizing the individual facts of elementary arithmetic—and she sees no reason to doubt the propriety of those effective methods, so she rejects the Robust Realist's insistence that they be held to a standard imposed by some prior metaphysical theory.[42] Just as before, her understanding of whether standard arithmetic is true, whether numbers exist, rests on her assessment of how standard arithmetic does and doesn't resemble her familiar ways of investigating, and just as before, she concludes that both Thin Realist and Arealist ways of speaking are legitimate. But if the metaphysical status of numbers is comparable to that of sets, what grounds, what guides and constrains, the two practices is quite different: standard arithmetic is out to describe the structure present in the '...,' a fixed point of human cognition, probably genetically shaped; set theory employs whatever diverse and innovative methods it can find, in pursuit of explicit and implicit mathematical goals, guided and constrained solely by the hope of uncovering strains of deep mathematics.

For the record, notice that the Second Philosopher eventually sees the wisdom—that is, the mathematical advantages—of embedding standard arithmetic in a fully set-theoretic setting: she exchanges the vagaries of rudimentary logic for classical logic and the potential infinity of the '...' for the completed infinity of the set of natural numbers,[43] opening up the realm of analytic number theory and beyond.[44] Looking back, she now sees that the rudimentary logical facts of elementary arithmetic are more closely reflected in the

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41 Here the intuitionist might protest: why assume the '...' gives rise to a KF-structure?, why think rudimentary logic is correct? (Though both rudimentary and intuitionistic logic do without excluded middle, they disagree, e.g., about double negation elimination.) Assuming (as in §I) that KF-structuring and rudimentary logic are part of our most primitive cognitive machinery, it's not surprising that these are the Second Philosopher's first thought, but this psychological fact doesn't guarantee that these notions, developed for and grounded in a prevalent feature of the physical world, should be transferred to the '...,' to abstracta like numbers. To intervene at this stage of the Second Philosopher's arithmetic thinking, the intuitionist would presumably hold that the '...' or numbers inhabit some non-KF structure, but it isn't clear to me what arguments might be given for this so early on. Two other possible points of intervention are considered in footnotes 42 and 44.

42 The intuitionist might step in here. With the Robust Realist, he thinks we're first called upon to show that the Peano axioms are true of our intended subject matter. Noting the familiar troubles the Robust Realist encounters in the search for a reasonable, trustworthy epistemology, the intuitionist advocates some kind of idealistic metaphysics, some alternative to KF-structuring, and argues that intuitionistic logic is correct for these things. (The idea that the intuitionist's Creative Subject points the way toward a non-KF conception is explored in Maddy, 2007, pp. 231–233, 273–279 (sporadically), and 296, and in Maddy, Unpublished.) The Second Philosopher—who lets the method dictate the metaphysics, not the other way 'round—remains unmoved.

43 It seems to me that these two moves are separable, but I won't try to sort this out here.

44 This presents another point of entry for the intuitionist. Here the mathematical considerations emphasized in McCarty (2005) come into play: as I would put it, there are good reasons to think that intuitionistic analysis also tracks important strains of mathematical depth. There's no reason the Second Philosopher can't embrace and pursue both; though intuitionistic logic differs from rudimentary logic, her commitment to that logic applies only to various physical situations (see footnote 41); for abstracta she's open to other possibilities (though she would like an alternative

language of finite sets,[45] but also, for that very reason, that number talk made a better shorthand. She now sees that the number sequence is really just a tool for measuring the sizes of KF-structures or finite sets, a role that can be played by various set-theoretic sequences, most conveniently by the von Neumann ordinals.[46] The Thin Realist might say she's discovered that she has no good evidence for the existence of two kinds of abstracta, of numbers in addition to sets; the Arealist that there's no call to clutter up our theory with both; both come to appreciate that a standard theory of finite sets and their number properties can be grounded in KF-structures and our intuition of the '. . ..' All that said, let's return our attention here to the simpler context of standard arithmetic.

The contrast, again, is between this intuitive grounding and the depth-based grounding of higher set theory. Now some would claim that higher set theory, too, is grounded in an image or picture or intuition, namely in the iterative conception, the cumulative hierarchy, the structure of V. It's worth observing that in fact this image of the universe of sets came along rather late in the history of the subject, around 1930;[47] whatever Cantor and Dedekind were doing in the 1880s, it doesn't appear to have involved this intuitive picture. Still, perhaps contemporary set theory *is* so grounded; perhaps this picture is now what guides and constrains the practice. As it happens, the punch-line of *Defending the Axioms* is that this isn't so. Many hold that intrinsic justifications for decisions in the practice of set theory—that is, justifications based on intuitive evidence like our picture of the cumulative hierarchy—are primary, and that extrinsic justifications—based on fruitfulness, productivity, etc.—are only secondary, at best, but I argue, on the contrary, that intrinsic justifications have force only insofar as they play an instrumental role in allowing us to track depth;[48] if a line of deep mathematics conflicts with our current concept of set, it's the concept that should give way.[49] So while in set theory, the intuition of V plays a merely heuristic role, in arithmetic, the intuition of an infinite sequence isn't secondary at all: it's what grounds the subject!

Notice also that much of the intuitive force behind the iterative conception would seem to derive from the same source as that of an infinite sequence: the recursive aspect of the language-learning device. The other key elements in the set-theoretic case are the completed infinite, starting with ω, and the combinatorial notion of the power set operation—additions that may or may not enjoy any fundamental cognitive backing. Either way, our intuitive picture of V, generated by an amalgam of these three elements, can't help but be considerably foggier that of an infinite sequence.

to KF-structuring to guide her thinking). She disagrees with the intuitionist only when he insists that classical mathematics be rejected.

45 In Maddy (1990), I would have counted finite sets of physical objects, rather than their KF-structures, as robustly real; the preface to Maddy (2011) gives a quick summary of my reasons for abandoning this position.

46 The view of natural numbers as measuring devices for cardinality properties of finite sets descends from Maddy (1990, chap. 3).

47 See Zermelo (1930).

48 For example, the iterative conception may suggest large cardinal axioms, but they must then prove their worth by their extrinsic merits.

49 This is one way of characterizing the debate over the axiom of choice: opponents argued that it was inconsistent with the idea of a set as the extension of a predicate, or as generated by a rule or construction; proponents were impressed by the mathematics it generated; in the end, the iterative conception, with its combinatorial notion of subset, replaced the earlier conception and preserved the vital mathematics.

Still it's important to recognize that the comparative clarity of the intuition of the '...' provides no guarantee of the other virtues we typically take for granted: that we all share the same notion of a infinite sequence, that this notion is coherent and fully determinate. William Tait expresses a related caution when he writes that the basis of our notion of number in our understanding of finite iteration (the '...') ...

> ... does not rule out the possibility that the idea of Number is incoherent ... Can it be that we could *correctly* construct [a proof of $0 = 1$] in primitive recursive arithmetic? On the basis of our understanding of Number, we would say not. But this understanding is our final court of appeal in the matter ... we cannot hope to prove absolutely that such a construction is impossible. Thus there is a sense in which security must always elude us. (Tait, 1981, p. 41).

The Second Philosopher would cite the large-scale similarity of brain structure from one person to the next as some reason to hope that we're all talking about roughly the same pattern, and to centuries of work in number theory as evidence of coherence and determinacy, but of course this is all merely empirical support and could at some point turn out to have been mis-leading.[50]

In sum, then, standard arithmetic differs from set theory, not in the metaphysics of its purported objects, but in the type of thing that grounds its practice: our concept of the '...' vs. the facts of mathematical depth. This places arithmetic at some distance from one end of our pair of extremes, the set-theoretic end, but this is only half of the story. On the other extreme, we've seen that claims of elementary arithmetic, like '$2 + 2 = 4$,' correspond to rudimentary logical facts, grounded in the KF-structuring of the ordinary physical world (where it occurs), but that standard arithmetic goes beyond rudimentary logic into the recursive conceptual element drawn from the human language-learning device.[51] So here we have the advertised intermediate point between the two extremes: a conceptual grounding distinct from the robust worldly features of KF-structure, on the one hand, and from the purely mathematical facts of depth, on the other.

But notice that arithmetic now appears to serve two distinct masters, to be answerable to two distinct grounds: the logical facts of elementary arithmetic and conceptual '...' of standard arithmetic.[52] This raises a new, pressing concern: how do arithmetic's two distinct

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[50] Cf. Woodin (1998, RI). Koellner (CRI) describes one upshot of this work: 'there are people who point out that there is always the possibility of an inconsistency in the transfinite. This points out that the situation is no different for the large finite'. See also Hamkins (QMO).

[51] Just to be clear: this needn't involve a rejection of arithmetic logicism. I use the term 'rudimentary logic' for various claims most people would classify as Logic and the term 'classical logic' for rudimentary logic with no indeterminate properties or relations and a material conditional, but these are merely convenient labels; no principled distinction between Logic and nonLogic is intended or presupposed. So nothing here precludes counting arithmetic as Logic for one reason or another. What matters for present purposes isn't a contrast between Logic and nonLogic, but between what grounds rudimentary logic and what grounds the theory of the '....'

[52] Analogous dualities appear in the grounding of formal logic, and the formal theory of musical compositions, if there is such a thing: facts about physically realized formulas and musical notations on the one hand, the concept of the recursive '...' on the other. (As indicated in the previous note, I'm not interested here in separating Logic from nonLogic, but with all the ingredients arrayed before me on the table, I can't resist one passing observation. Presumably one motivation for logicism is the thought that Logic, unlike Mathematics, enjoys some sort of epistemic transparency. If this is cashed out in terms of decidability, the truth table method does

grounds fit together? More pointedly: what justification is there for using the conceptually based theory of the '...' to codify and systematize various robust logical facts about the physical world? Why should standard arithmetic be reliable as an indicator of the facts of elementary arithmetic?[53]

Before addressing these questions, I'd like to follow up another line of thought suggested by this account of arithmetic. For the historically minded, any view according to which a bit of mathematics is the study of some aspect of our basic cognitive endowment can't help but ring Kantian bells. In addition, the robustly realistic portion of the proposed account also has its Kantian echoes. Partly as stage-setting for the issue of applications, I pause in the next section to explore these themes. Readers less than fascinated by historical interrelations of this sort are invited to skip directly to §V.

§4. A Kantian digression. There's a common misconception that Kant takes the intuition of time to ground arithmetic just as the intuition of space grounds geometry. That this isn't correct is noted by 'all careful writers on the subject' (Friedman, 1992, p. 105); as Michael Friedman puts it:

> In the Transcendental Aesthetic, §5 (The Transcendental Exposition of the Concept of Time) corresponds to §3 (The Transcendental Exposition of the Concept of Space), where the synthetic a priori knowledge of geometry is explained in terms of the pure intuition of space. In §5, however, arithmetic is not mentioned; instead, the synthetic a priori science whose possibility is explained by the pure intuition of time is identified as 'the general doctrine of motion' (B49). ... The science of time, for Kant, is therefore not arithmetic, but rather pure mechanics or the pure doctrine of motion. (op. cit.)[54]

So arithmetic isn't simply the study of our pure intuition of time. Still, pure intuition must be involved somehow or other, because for Kant arithmetic, like geometry, is synthetic a priori.

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the job for propositional logic, but alas, Church's Theorem ends the hope of an algorithm for validity in first-order predicate logic (more or less what I've been calling 'formal logic'). Now consider rudimentary logic. Assuming that the world is finite, that there's a maximum, N, to the number of objects in any worldly situation, then the validity of an inference φ to ψ *is* decidable; we just have to check through the finite number of situations of form φ to make sure they're all of form ψ. (This decidable set of validities will include items like the $N = N + 1$ inference; that there are at most N objects is a rudimentary logical fact about the world.) The same goes for the truths of what I've called 'classical logic.' Undecidability enters the picture with formal logic, when the '...' is injected into our understanding of the 'situations' that need to be considered. So, does Church's theorem tell us that Logic is undecidable, or that formal logic goes beyond Logic into arithmetic?)

53 Superficially, this may sound like Hilbert's question—why should an ideal, infinitary system be a reliable guide to contentful, finitary arithmetic?—but Hilbert's 'finitary arithmetic' isn't our 'elementary arithmetic': Tait (1981) is generally regarded as having shown that 'finitary' for Hilbert coincides with primitive recursive arithmetic, which, Tait argues, rests in turn on the primitive notion of finite iteration. So Hilbert's contentful mathematics already includes the '...'; what he rules out are infinite totalities. Though primitive recursive arithmetic uses only free variables, not quantifiers as in standard arithmetic, what matters for our purposes is that it encompasses the '....'

54 See also Parsons (1969, p. 62, 1983, p. 133).

In fact, the basis for arithmetic is quite different from that of geometry. In the Aesthetic, geometry is traced to the spatial form of intuition; since this form helps constitute the world of appearance, it follows that geometry applies to our world and that we can know this a priori. But arithmetic depends on the categories, so understanding its source and justification requires not just the Aesthetic, but the Analytic, as well.[55] Taking for granted the conclusion of the Aesthetic—that space and time are forms of intuition, and thus transcendentally ideal—in the case of arithmetic, we must also ask: what is the status of the categories?

The answer to this question begins with the claim that the table of the logical forms of judgment provides 'the clue to the discovery of all pure concepts of the understanding' (A70/B95). The forms of judgment themselves, in turn, originate in the structure of the discursive intellect—that is, any being who cognizes the world through concepts, through features several objects can share. Such an intellect employs two distinct faculties: a receptive sensibility passively affected by the world, and a spontaneous understanding actively applying concepts.[56] Our human discursive intellect's forms of sensibility are space and time, but the sensibility of another discursive intellect might take very different forms. The connection with logic now comes in a rush: Kant claims that any judgment, by any discursive intellect, whatever his forms of intuition, will take one of the twelve forms listed in the Table of Judgments (A70/B95), and thus, that any judgment, by any discursive intellect, is bound by the laws of logic.[57] And finally, the infamous Metaphysical Deduction follows up the promised 'clue' and concludes that for each entry in the Table of Judgments, there is a corresponding entry in the Table of Categories (A80/B106). So the categories arise from the structure of the discursive intellect.

One important point here is that logical truths—like our well-worn sample, if the book is red or green, and it's not red, then it's green—or logical validity—the reliability of the corresponding inference—apparently come out as analytic on this account. At least this follows if 'analytic' means 'not-intuitive': spatiotemporal intuition is not involved; the same logic holds for any discursive intellect, whatever his forms of intuition. On the other hand, if 'analytic' means 'predicate concept contained in the subject concept,' the case is obviously more problematic; indeed, our example isn't even of subject-predicate form. Still, most commentators allow Kant some leniency on this definition, figuring 'true by virtue of the concepts involved' should be good enough. Even then, the forms of judgment aren't literally concepts, but they are rules for synthesis—perhaps the very same rules

55 Of course, geometric cognition, like all cognition, also depends on the categories and the work of the Analytic (see the Axioms of Intuition, where the principle that 'all intuitions are extensive magnitudes ... makes pure mathematics in its complete precision applicable to the objects of experience' (B202, A165/B206)). But as Kant puts it at the beginning of the Analytic: 'We have above [in the Aesthetic] traced the concepts of space and time to their sources by means of a transcendental deduction, and explained and determined their a priori objective validity' (A87/B119-120); the job that's left is a transcendental deduction for the pure categories and eventually the schematism, which is where number comes in (see below).

56 The contrast cases are the intuitive intellect, whose intuiting actually creates the object (God), and the empirical intellect, whose sensibility is directly stamped with the features of the object (eventually shown to be impossible in the *Critique*); both get by without an understanding that applies concepts. For more on the subject of this paragraph and the next, see Maddy (2007, §III.2).

57 'Thus' because logical laws are implicit in the forms of judgment themselves (see, e.g., Longuenesse, 1993, pp. 90–93.) Of course, for Kant, the laws in question are those of syllogistic (A303-305/B359-361).

as for the corresponding category, just applied in a different context[58]—so that Henry Allison, for example, is moved to speak of them as ‘judgmental concepts’ (Allison, 2004, p. 149). To avoid opening a gap between nonintuitive and analytic, it seems best to opt for this friendly amendment and to classify logic as analytic. Notice, however, that this isn’t the trivial analyticity of ‘all bachelors are unmarried,’ whose predicate, in Kant’s terms, is just one feature we explicitly included when we constructed the subject. Instead, the ‘judgmental concepts,’ like the corresponding pure categories, must be given concepts, not constructed by us, whose features are difficult to tease out, and about which we can easily go wrong.

Now that we’ve identified the source of the categories, the characteristically Kantian question arises: why should these features of our understanding, products of our cognitive endowment, apply to the world? In the case of the forms of intuition, the Aesthetic has answered this question with its Transcendental Idealism: since those forms help constitute the world of appearance, we can know a priori that they will apply to it. But the categories are more remote, emerging from the more rarified features shared by any discursive intellect whatsoever. Why should they be reliable in application to our world? This is the question addressed in the Transcendental Deduction.

If the case for the applicability of the categories in the Analytic is to run parallel to that for the forms of intuition in the Aesthetic, then Kant needs to establish that they are necessary conditions for experiencing the world of appearance.[59] To that end, he first observes that

> All experience contains in addition to the intuition of the senses, through which something is given, a **concept** of an object that is given in intuition, or appears. (A93/B126)

The plan, then, is to show that the categories underlie this fact:

> They . . . are related necessarily and *a priori* to objects of experience, since only by means of them can any object of experience be thought at all. (A93/B126)

In the B-deduction, this argument proceeds in two steps. First Kant argues that any thought of an object requires the unification or synthesis provided by the categories, but even if this is so, there remains the possibility that the deliverances of the sensibility might resist this thought-processing. In the second part, Kant observes that we sense objects in stretches of space and time, which requires that these stretches themselves be unified in cognition—and this synthesis again is provided by the categories. So in the end, the categories *are* necessary conditions for all experience of objects.[60]

Before turning to the place of number in all this, let’s pause a moment to reflect on the account of logic that emerges. Its laws are implicit in the forms of judgment, which are inseparable from the categories; since the world of any discursive intellect, whatever

58 This is one leading thought on what drives the Metaphysical Deduction (see Allison, 2004, pp. 152–156). My general understanding of Kant has been helped immeasurably by Allison’s book (2004) and Gardner’s introduction (1999). On Kant’s view of arithmetic, I’m particularly indebted to Parsons (1969, 1984) and also Rohloff (2007).

59 Transcendental Idealism is presupposed, having been established in the Aesthetic.

60 For a full analysis of the Deduction, see the introductory exposition of Gardner (1999, chap. 6), or the more thorough treatment of Allison (2004, chap. 7).

his forms of intuition,[61] is shaped by these a priori concepts, it will satisfy those logical laws. Consider, for example, the logical forms subject-predicate and if-then;[62] the corresponding categories are object-with-properties and ground-consequent.[63] Setting aside the remainder of the Tables, the Kantian idea could be put this way: the world of any discursive intellect consists of objects-with-properties standing in ground-consequent relations, and we can know a priori that the laws of logic apply in any such world. I trust it's obvious that the fundamental outlines of the Second Philosopher's position sketched in §I more or less result simply by removing the idealism from Kant's account (and updating the logic a bit by way of Frege, so that object-with-properties becomes objects-in-relations): many aspects of the world itself are so-structured, and for that reason, rudimentary logic applies to them.[64] Much as the applicability of logic for Kant depends, not on the forms of intuition, but only on the pure categories, we noted earlier that, for example, the reliability of the library inference depends, not on any physical details of the book, but only on the KF-structure of the situation.

Now back to Kant proper. If the Deduction has succeeded, he's established that the categories help constitute the world of appearance, that these a priori concepts must necessarily shape the inputs of sensibility. What hasn't been explained is how this takes place:

> Pure concepts of the understanding . . . [and] empirical (indeed in general sensible) intuitions, are entirely unhomogeneous . . .how is the **subsumption** of the latter under the former, thus the **application** of the category to the appearances possible . . . ? (A137/B176)

The answer to this question follows hard upon the Deduction, in the Schematism, where we're introduced to the notion of a 'transcendental schema':

> It is clear that there must be a third thing, which must stand in homogeneity with the category on the one hand and the appearance on the other . . . **intellectual** on the one hand and **sensible** on the other. Such a representation is the **transcendental schema**. (A138/B177)

Admittedly, this solution can sometimes seem little more than a restatement of the problem, but thinking of the schema for the a priori concept of a triangle might help. There's the

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61 I assume that whatever this discursive intellect's forms of intuition are, he will experience objects in their terms, which will require the general sort of synthesis Kant appeals to in the second part of the Transcendental Deduction. No peculiarities of space and time in particular appear to play a role there.

62 I use these slightly updated terms for what Kant calls 'categorical' and 'hypothetical' (A70/B95), the first two relational forms.

63 Kant calls these 'inherence and subsistence' and 'causality and dependence (cause and effect)'(A80/B106). As the pure categories involve no element of space or time, both of which seem closely linked to our current understanding of causation, I prefer the more neutral term 'ground-consequent,' which Kant uses in his gloss of 'hypothetical' judgments (A73/B98). 'Cause and effect' then emerges when 'ground-consequent' is schematized (see below). No connection to the notion of 'ground of a discourse' is assumed or intended.

64 This is only the metaphysical side of the story. The Second Philosopher's epistemological notion that our initial belief in the simpler truths of rudimentary logic arises from our most basic cognitive structures also recalls Kant's thinking; the Second Philosopher differs in trading his transcendental psychology for empirical psychology and taking the psychological structures to be produced by the world structures, not vice versa. In Maddy (2007, §§III.2 and III.3) and Maddy (TLM), the second-philosophical view of logic is explicitly derived from Kant in this way.

concept of a three-sided planar figure, and there are various particular representations of triangles in pure and empirical intuition; the schema is the ‘the representation of a general procedure of the imagination for providing a concept with its image’ (A140/B180-181). In other words, the schema is a rule or recipe, if you will, for generating a spatiotemporal representation answering to the concept, in this case, a triangular representation in intuition (perhaps something like this: trace a line segment, then another from one of its endpoints, connect the endpoint of the second segment to the other endpoint of the first).

Here, at last, is where number fits in. The Table of Judgments includes three forms of quantity—singular, particular, and universal—very roughly, ‘Socrates is wise,’ ‘Someone is wise,’ ‘Everyone is wise.’[65] The corresponding categories of quantity are unity, plurality, totality. Of these three,

> the third category . . . arises from the combination of the first two . . . totality . . . is nothing other than plurality considered as a unity. (B110-111)

And finally, number, it turns out, is the schema for these categories:

> The pure **schema of magnitude** . . . as a concept of the understanding, is **number**, which is a representation that summarizes the successive addition of one . . . unit to another. (A142/B182)

Number, then, is a rule or recipe for generating a spatiotemporal representation of a unit (unity), adding successive units (plurality), and collecting the units into a whole (totality). In other words, for counting.[66]

But this story presents us with a serious puzzle: why suppose that number enters the picture only at the point when the pure categories are schematized? Consider again a discursive intellect with forms of intuition different from our own. He still has the pure categories of unity, plurality and totality; why couldn’t he form units, generate a plurality, combine it into a totality? To perform actual counting of objects, he’d have to schematize these categories according to his own forms of sensibility, but why shouldn’t he be able to appreciate as well as we do that $2 + 2 = 4$, just as he does the reliability of disjunctive syllogism? Clearly our geometry is beyond him, but why shouldn’t he grasp our elementary arithmetic?

One observer who’s pondering mightily on a version of this question is Charles Parsons:

> The difficulty can be put this way: the synthetic and intuitive character of geometry gets a considerable plausibility from the fact that geometry can naturally be viewed as a theory about actual space and figures constructed in it. . . . The content of arithmetic does not immediately suggest such a special character or such a connection with sensibility. (Parsons, 1969, p. 58, 1983, p. 128)

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65 It’s actually difficult to understand what Kant has in mind for singular judgments (see Parsons, 1984, pp. 140–141). Also, I’ve adjusted the ordering of the three forms of judgment to line up properly with that of the corresponding categories (see Parsons, 1984, p. 141 and footnote 17; Longuenesse, 1993, pp. 248–249).

66 The categories of quantity are intended to cover continuous magnitudes as well, but I leave that aside here.

Furthermore, in the *Critique*,

> The status of the pure categorical notions, and in particular the relation of number to the pure categories, is obscured by Kant's characterizing number as the schema of quantity (A142/B182) and by the fact that most of Kant's explanation of notions of quantity comes in the Axioms, where he is principally concerned with the schematized categories. (Parsons, 1984, p. 146)

Faced with these obstacles, Parsons turns for illumination to Kantian writings both before and after the *Critique*.

The evidence from before the *Critique* (1781/1787) comes from the *Inaugural Dissertation* (1770), where Kant writes:

> It is one thing, given the parts, to conceive for oneself the *composition* of the whole, by means of an abstract notion of the intellect; and it is another thing to *follow up* this general notion . . . through the sensitive faculty of knowledge, that is to represent the same notion to oneself in the concrete by a distinct intuition. (Quoted and translated from §1 of the *Dissertation* in Parsons, 1984, p. 145)
>
> There is a certain concept which in itself indeed is intellectual, but whose activation in the concrete . . . requires the assisting notions of time and space (by successively adding a number of things and setting them simultaneously beside one another). This is the concept of *number*, which is the concept treated in ARITHMETIC. (Quoted and translated from §12 of the *Dissertation* in Parsons, 1984, p. 145. See also Parsons, 1969, p. 63, 1983, p. 134.)

Here we apparently find the very distinction needed to describe our nonhuman discursive intellect: he has the 'intellectual concept' of number, but has a different way of 'activating it in the concrete.' Even if neither he nor we can access the fact that $2 + 2 = 4$ without our respective 'activations in the concrete,' it would seem that the shared 'intellectual concept' alone serves to ground that fact.

The evidence from after the *Critique* is found primarily in a letter to Johann Schultz (1788). There Kant takes the subject matter of arithmetic to be 'merely *quantity* . . ., that is, the concept of a thing in general by determination of magnitude.' As for the role of sensibility,

> Time . . . has no influence on the properties of numbers (as pure determinations of magnitude), . . . the science of number, not considering the succession, which every construction of magnitude requires, is a pure intellectual synthesis which we represent to ourselves in our thoughts. (Quoted and translated in Parsons, 1984, pp. 149–150—also Parsons, 1969, p. 63, 1983, p. 134—as amended in Parsons, 2012, p. 112.)

Here, post-*Critique*, the pure categories of quantity (magnitude) are explicitly cited and the science of number based squarely on their intellectual synthesis. In Parsons' final analysis,

> The conclusion to be drawn from examining these texts, in my opinion, is that Kant did not reach a stable position on the place of the concept of number in relation to the categories and the forms of intuition. (Parsons, 1984, p. 152)

There may have been larger, more systematic motivations for Kant's position in the *Critique*,[67] but viewed simply in terms of the considerations reviewed here, the road appears open to a Kantian treatment of arithmetic dramatically different from that of geometry—as grounded in the pure, unschematized categories, independent of the forms of intuition.

If this is right, then there's at least a version of Kantianism that places arithmetic much closer to logic than to geometry:[68] both logic and arithmetic are grounded in the forms of judgment and the pure categories; geometry rests on the pure intuition of space. All three sciences could be known a priori, because both the forms of intuition and the pure categories are instrumental in constituting our world of appearance. It follows that all three would also be, in some sense, transcendentally ideal, but logic and arithmetic would hold in the world of appearance for any discursive intellect, while geometry is confined to our human world. And on this version of Kantianism, $2 + 2 = 4$, like disjunctive syllogism, would be classified as analytic—independent of intuition—but again not in the trivial sense that 'all triangles are three-sided' is analytic.[69]

We noted above the close resemblance between Kant and the Second Philosopher on logic. We now see that, for this version of Kantianism, their affinity carries over to the Second Philosopher's position on elementary arithmetic: for both, the status of $2 + 2 = 4$ is the same as the status of disjunctive syllogism. In this connection, it's worth noting Parsons' observation that in the *Critique*

> Kant focuses on singular propositions about numbers, so that the question how to interpret generalizations about them is not raised. (Parsons, 1984, p. 139. See also Friedman, 1992, p. 109)

This dovetails with Kant's insistence that arithmetic has no axioms (A163-165/B204-206). If Kant limits himself in this way, if he doesn't aspire to account for more than the likes of $2 + 2 = 4$, then his interest is confined to the Second Philosopher's elementary arithmetic, and the further question of what grounds the '...' is irrelevant to our comparison between the two.

Incidentally, Parsons (1984, pp. 139, 146) and others criticize Kant for failing to distinguish between the number n and an n-element set. A similar complaint might be lodged against the Second Philosopher's account of elementary arithmetic, with its focus on the logical structure of one object, another object, and nothing more,[70] rather than the number 2. In §III, '2' in the likes of '$2 + 2 = 4$' is regarded as a mere notational convenience;[71] the thin reification of numbers comes only in the transition to standard arithmetic. If the Second Philosopher is right in this, and if Kant's only interest is in elementary arithmetic, perhaps his neglect of the number 2 is less surprising. The fact remains that for a complete treatment of number theory, the question of the '...' must be addressed.

And here the Second Philosopher appeals to an idea that's roughly Kantian in another way, not in the tight parallel both draw between logic and elementary arithmetic, but in the

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67 See, for example, Parsons (1969, p. 61, 1983, pp. 131–132), Longuenesse (1993, p. 283).

68 This would appear to bring Kant surprisingly close to logicism.

69 As above, I'm taking the concept triangle to be 'three-sided planar figure.'

70 The position of Maddy (1990) comes closer to Kant, or Parsons' version of Kant, on this point, identifying the relevant worldly structure as a two-element set, understood (as I would now say) in the sense of Robust Realism; the number 2 is then taken to be a property of a two-element set. As mentioned in footnote 45, the preface to Maddy (2011) summarizes what I now take to be a persuasive case against this view.

71 See footnote 35.

notion that a mathematical theory—geometry and elementary arithmetic for Kant, standard arithmetic for the Second Philosopher—can be grounded in an aspect of our cognitive endowment. The pressing concern from the end of §III then arises for both: why should the deliverances of our cognitive mechanisms apply to the world? Kant's answer, as we've seen, depends essentially on his transcendental idealism—the relevant conceptual devices help to constitute the world—but the Second Philosopher is reluctant to follow that path. What is her alternative? The structure of the language-learning device inclines us think of the number properties we detect as embedded in an infinite sequence; why should we trust that this inclination won't lead us astray? How does standard arithmetic mesh with the robust worldly realities that ground elementary arithmetic?

§5. Arithmetic and the world. The general question of how and why mathematics so often works in application to the world is large and complex. Elsewhere,[72] I've suggested that contemporary pure mathematics is best regarded as providing a store of abstract models, that the mathematizing scientist should be understood as claiming that the worldly phenomenon in question resembles an abstract model in various ways—some of which we can specify and some of which, at least for now, we don't completely understand—and furthermore, that the much-discussed 'miracle of applied mathematics' is perhaps less miraculous than it first appears.[73] But standard arithmetic, as we're understanding it here, isn't some rarified deliverance of pure mathematics; it arose directly as a piece of applied mathematics, as a way of systematizing and codifying the individual facts of elementary arithmetic. The question is why a study that's grounded in conceptual features of the language-learning device should work so well as a systematization and codification of the worldly realm of KF-structures.

It may help to consider a few familiar examples of applied mathematics for comparison: we treat the ocean as infinitely deep when analyzing waves on its surface; we treat discrete items, like incomes or test scores, as continuous variables in statistical analyses; we treat fluids as continuous substances in fluid dynamics. In each of these cases and many more, a finite worldly phenomenon is embedded in an infinitary mathematical setting—much as we embed the worldly facts of elementary arithmetic in the infinitary structure of standard arithmetic. What does the applied mathematician require by way of legitimizing these mathematizations?

As noted in the brief discussion of idealization in §I, at least part of the answer must be that the mathematization should be beneficial and benign: it should have real advantages, and there should be persuasive evidence that it doesn't distort the target physical phenomena in misleading ways. In all three of our cases, the benefits, the advantages, are roughly the same: the treatment becomes more manageable, predictions more feasible. It would be extremely difficult to factor in the effects of waves rebounding from the bottom of the ocean; applying the calculus to continuous variables in statistics allows easy computation of values like optima; the use of continuum mechanics in fluid dynamics brings a wide range of engineering uses within workable reach. Furthermore, in each case there is evidence that the mathematizing is benign: given the depth of the ocean, the rebound effects are small; if the population in question is large enough, the approximations are close; as Tritton explains in his fluid dynamics text, the effective use of continuum

72 See Maddy (2008) or (2011), Part I, especially §2, 'how applied mathematics became pure.'

73 See Maddy (2007, §IV.2.iii).

mechanics depends on there being ‘a significant plateau’ (Tritton, 1988, p. 50) between volumes so small that properties like temperature or average velocity fluctuate wildly as individual molecules enter and depart, and volumes so large that these same properties vary dramatically from one portion to another. But in the end, despite these reasons for thinking the mathematization is benign, Tritton goes on to remark that the applicability of continuum mechanics in this context

> . . . is only a hypothesis. The above discussion [concerning the ‘significant plateau’] suggests that it is plausible, but nothing more. The real justification for it comes subsequently, through the experimental verification of predictions of the equations developed on the basis of the hypothesis. (Tritton, 1988, p. 51)

Often in the applied mathematician’s working life, the only way of knowing that neglected effects are small enough or approximations close enough or idealizations nondistorting enough is to check.

Now what about the case for embedding the facts of elementary arithmetic in the infinitary setting of standard arithmetic? If the physical world is finite, if there’s a upper bound on the size of actual KF-structures, then standard arithmetic, like our other examples of applied mathematics, will deliver some falsehoods, like ‘every number has a successor,’ or ‘for every prime there’s a larger prime,’ or even ‘$n \neq n+1$’ for n above that upper bound (as noted in §III). Nevertheless, it’s also clearly beneficial, again in much the same sense as our previous examples: most conspicuously, definition by recursion and proof by induction are extremely powerful tools for doing a lot of things at once. So, for example, we can prove by induction that addition is commutative for all pairs of numbers; first figuring out the size N of the largest logical structure in the world, then establishing commutativity piecemeal up to that large N and no further, would be onerous, if not impossible. We also have a strong conviction that this mathematization is benign, because, after all, we’re simply ignoring the ‘merely medical’ or ‘merely physical’ limitations on the extension of the number sequence.

But here there’s a glaring disanalogy with our comparison examples: no one defends the harmlessness of taking the ocean to the infinitely deep by claiming that the ocean *is* infinitely deep ‘in principle,’ or that the existence of the ocean floor is just an insignificant accident; likewise for test scores or fluids. In these cases, we consciously and deliberately falsify our description of the situation—for good reason, of course—and we fully recognize that this move needs defending—by the sorts of plausibility arguments and empirical evidence just cited. In contrast, when we move from elementary arithmetic to the ‘. . .’ of standard arithmetic we don’t feel that we’re falsifying, we’re just moving to the realm of ‘in principle.’ And we believe there’s no choice in the matter: Peano arithmetic *must* be correct, every number *must* have a successor, mathematical induction *must* be reliable. What must be right doesn’t stand in need of defense.

This deep psychological conviction is understandable, given that the infinitary structure in this case, unlike the others, springs directly from our basic cognitive machinery. This sentiment, in turn, motivates efforts to show that standard arithmetic is just logic or analytic or, for Kant, transcendentally ideal. But consider for a moment what these efforts are out to explain: the a priori truth of arithmetic, or for Kant, its a priori applicability. And aren’t these philosophical explananda just manifestations of that same underlying psychological conviction—that our application of standard arithmetic simply can’t go wrong?

The most the Second Philosopher allows in this direction is some claim to a priori knowledge of elementary arithmetic, and then only in the most extended externalist sense. Despite the undeniable attraction of our idea of the ‘. . .,’ she doesn’t see that it provides any guarantee of uniqueness or coherence or determinateness—as we saw in §III—nor, we might now add, of usefulness or applicability. These days, no one takes the applicability of geometry to be known a priori; the Second Philosopher is suggesting that the same should be true of standard arithmetic. In the end, whatever our convictions, the case for the harmlessness of standard arithmetic in application to the elementary logical structures of the world rests on Tritton-like evidence that our concept of the '...' is shared, coherent, sufficiently determinate to settle questions of elementary arithmetic, and to settle them correctly enough to be effective. And after all, this evidence—centuries of it!—is overwhelming.

§6. Conclusion. I've tried here to outline a philosophy of arithmetic that's plausible from a second-philosophical point of view. It begins from an account of elementary arithmetic—the arithmetic of simple claims like $2 + 2 = 4$—that places them among the robust objective truths of rudimentary logic. These logico-arithmetic validities hold in any worldly situation with the requisite KF-structuring—objects with properties, standing in relations, with dependencies—their truth is contingent, not on the physical details of the situation, but on the presence of that underlying form.

To reach the arithmetic of a potentially infinite number sequence—the kind of thing encoded in the Peano axioms—requires a further step to the '...,' to the notion of an infinite sequence. This idea isn't to be found among the basic number-detecting capacities that we humans share with various animals; rather, it turns up in our natural language number sequence, apparently a direct manifestation of the recursive element of the so-called 'language-learning device.' Here a conceptual endowment, not a worldly structure, provides the grounding, which raises a series of questions: how can I be sure that your notion of an infinite sequence is the same as mine?; even if we all share a single notion, how can we be sure that it's coherent or fully determinate in all respects?; how can we be sure that standard arithmetic won't lead us astray about the worldly truths of elementary arithmetic? Much philosophical theory-making has gone into efforts to answer these questions with strong assurances, but the Second Philosopher doesn't see how a conceptual grounding can provide any such guarantees, at least without a strong idealism about the physical world that she finds otherwise unmotivated.

From her point of view, embedding the individual facts of elementary arithmetic into the infinitary structure of a standard sequence is an instance of abstract modeling comparable to many others in applied mathematics—introducing falsehoods just as they do, and justified, as they are, when and only when these falsehoods are beneficial and benign. The case of standard arithmetic strikes us as different only because its source lies so deep in our conceptual mechanisms that we don't see ourselves as deliberately choosing to falsify, but this psychological fact doesn't alter the justificatory structure of the situation: the prudence of the move to standard arithmetic is still a matter of the advantages it confers and the unlikelihood of significant distortion, not its psychological force. And these factors can only be assessed Tritton-style, by ordinary experience.

So in the end arithmetic both resembles rudimentary logic—in the worldly grounding of the logico-arithmetical claims of elementary arithmetic—and differs from it—in the conceptual grounding of standard arithmetic. When it comes to the purported abstract ontology of numbers, arithmetic reveals its commonalities with set theory: in both cases, there's no fact of the matter to choose between a thin-realistic description that posits abstracta, but denies the need for a nontrivial epistemology, and an arealistic description that doesn't take 'truth,' 'existence,' and so on, to be at issue. But standard arithmetic differs starkly from set theory, too, in the nature of the forces that guide and constrain the practice: the intuitive picture of an infinite sequence is the object of arithmetical study, while set theory aims to uncover those productive contours of mathematical depth. The grounding of standard arithmetic lies in our conceptual endowment, fully distinct from both the simple worldly realm and the purely mathematical virtues.

In sum, then, we have an integrated second-philosophical take on the so-called 'a priori disciplines,' from rudimentary logic, through arithmetic and number theory, to the far reaches of pure mathematical set theory. Only a small foothold remains for a priori knowledge of an externalist variety in rudimentary logic, and for a weak shadow in the thin-realistic description of set theory, but firm groundings run throughout, from the world's KF-structures, to the recursive element of the language-learning devise, to the more esoteric facts of mathematical depth. For the naturalistically minded, I hope this account marks a congenial elaboration of the second-philosophical world view. For those less sympathetic, my question is this: we have here[74] at least the beginnings of answers, loosely empirical answers, to the traditional philosophical questions of what grounds logic, arithmetic, and set theory; are ordinary answers of this sort somehow unacceptable in principle?[75] Finally, if nothing else, perhaps we've established for all parties that the Second Philosopher has more resources at her employ than it might have seemed at first blush![76]

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[74] In combination with Maddy (2007, 2011) (also Maddy, 2014, Unpublished).

[75] Maddy (Unpublished) includes some meditations on this question.

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