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Foreword to Special Issue on Mathematical Depth

1. INTRODUCTION

'Mathematical depth' is a notion that often turns up when mathematicians assess and evaluate the work of their fellows, but it has not been much studied by philosophers of mathematics. Though calling a piece of mathematics deep is clearly a term of high praise, it is apparently more specific than that, typically distinguished from other such honorifics (elegant, productive, difficult, and so on). Since there is no developed literature on the subject, it is an open question whether methodological inquiry into the notion would be productive.

Recognizing that no one observer commands an overview of modern pure mathematics, we invited a range of mathematicians, historians of mathematics, and philosophers of mathematics to the University of California, Irvine, in April of 2014 for a focused exploratory workshop on the topic of depth. Our hope was to get a preliminary sense of the terrain, to gauge the viability of methodological inquiry into what counts as deep mathematics and why.

Speakers were encouraged to present concrete examples of mathematics they took to be deep or not deep, along with some discussion of the mathematical facts that inspired those judgments. The workshop was structured to allow both the customary question-and-answer sessions after the individual talks and generous periods of more open-ended general discussion of all the examples presented so far. Particular attention was drawn to these four questions:

1. Is there agreement that the cited examples are deep or not deep?
2. Are there commonalities in the kinds of features cited in defense of depth and non-depth assessments in the various examples?
3. Is depth the same as or different from such notions as fruitfulness, surprisingness, importance, elegance, difficulty, fundamentalness, explanatoriness, beauty, *etc.*?
4. Is depth an objective feature or something essentially tied to our interests, abilities, and so on? Is the way depth is tied to those human traits more fundamental than the way our natural science is tied to those traits?

As it played out, the mathematics considered ranged from elementary to sophisticated; the discussions were co-operative and constructive; and the consensus was that 'depth' indeed deserves further study. We hope this special issue will serve as a first step on that path.

We begin this Foreword with a quick survey of (what we take to be) highlights of the discussions and of the talks not represented in written form below. (Those interested in a more complete picture are invited to peruse the video recordings.[1] Some references to relevant passages are included in the summaries and commentaries.) The papers to follow are then introduced. In the Afterword, we continue our survey of highlights, taking up those that refer more directly to the papers, and close with a few comments in our own voices.

2. THE DISCUSSIONS

The types of items offered as concrete examples of deep mathematics turned out to be quite varied: theorems (*e.g.*, Gödel's theorems, Dirichlet's theorem) and proofs (*e.g.*, the proof of Fermat's Last Theorem) were probably most commonly cited, but so were concepts or definitions (*e.g.*, 'Riemann surface', 'group'), assumptions or axioms (*e.g.*, large cardinal axioms), areas of mathematics (*e.g.*, Galois Theory), methods (*e.g.*, constructivism), even examples (*e.g.*, the Grigorchuk group[2]), and more. There were a few attempts at reducing one of these to another — *e.g.*, perhaps a deep theorem could be defined to be a theorem with a deep proof — but these often seemed problematic: Fermat's Last Theorem may not be particularly deep, despite the depth of Wiles's proof; conversely Jeremy Gray[3] indicates (in his paper, below) that a deep result (Gauss's theorem of quadratic reciprocity) can have a non-deep proof (a proof by cases), which might well inspire us to look for another, deeper proof; indeed, Mario Bonk[4] described how the achievement of hard-won deep concepts (*e.g.*, 'compactness', 'continuity') can render theorems and proofs shallow (in undergraduate point-set topology and one-variable real analysis) [Bonk, 6:40–7:45]. It was occasionally suggested that different criteria of depth are appropriate to different types of items so classified; however that may be, most appeared to agree that there is likely to be more than one way to be deep.[5]

[1] See https://www.youtube.com/playlist?list=PLQw7KTnzkpXfGo93vo3kQk7_jA_HgwbnC, which links to a series of YouTube videos. There is one for the welcoming remarks, one for each of the talks (which includes its Q&A), plus four more for the four separate general discussions — all arranged in the same order as the workshop itself. References to each talk and each talk's Q&A will be given by the speaker's name and the appropriate time stamp; points in the general discussions will be indicated by 'GD' followed by the appropriate number (1–4), plus the time stamp.

[2] This was proposed by Bonk [32:50]. The question of whether counterexamples can be deep was debated in the Q&A after the talk [Bonk, 48:20; 102:15].

[3] Professor Gray (Department of Mathematics and Statistics, Open University, and Mathematics Institute, University of Warwick) gave a talk entitled 'Depth', on which his contribution to this issue is based.

[4] Professor Bonk (Department of Mathematics, UCLA) gave a talk entitled 'Depth'.

[5] See, *e.g.*, the suggestions that 'depth' might be a 'splintered concept' (Arana) or a 'family resemblance concept' (Urquhart).

Another consideration related to different ways of being deep was raised by Robert Geroch [6:00],[6] who suggested that some mathematical items that may not seem deep on their own do appear deep in application. He cited four examples: existence and uniqueness results for systems of quasilinear first-order hyperbolic differential equations, which includes essentially all of the differential equations of classical physics; the classification of representations of SL(2,C), which correspond to possible particle types; Witten's proof of the positive-mass conjecture, which uses so-called spinor fields to prove a result that appeared to involve only tensor fields (and thus reflects a kind of physically salient impurity); and the CPT theorem, which shows that any relativistic quantum theory satisfying certain weak constraints must have certain symmetry properties. All of these results seem to acquire what depth they have only in the context of their physical interpretations, raising the question of whether depth has a pragmatic component that extends beyond purely mathematical virtues — or at least, whether our intuitions about depth may be colored by considerations of depth acquired in application.

Despite these uncertainties, there was little controversy of the sort envisioned in the first of the questions explicitly raised; for the most part, there appeared to be agreement on the depth or non-depth of the many examples on offer. Classification theorems were one exception: in his talk [Stillwell, 11:42] (and in the resulting paper below), John Stillwell[7] gave reasons for thinking the classification of finite simple groups is deep; Andrew Arana[8] wondered whether this might be an example of depth without fruitfulness [in the Q&A after Stillwell's talk, 52:23]; Gray went further, suggesting that there were just too many varied classes to be getting at something deep [*ibid.*, 55:30]; and Geroch [3:43] provocatively lumped all classification theorems under the heading of theorems about how mathematics relates to people, not about the mathematics itself. (This terminology — 'in us' *vs.* 'in the math' — was quickly adopted as evocative shorthand for 'subjective' *vs.* 'objective'.) Another disputed example was the proof of the irrationality of the square root of two: it was apparently regarded as deep by the ancients, but perhaps today it appears too simple to be deep. Some took this as evidence that depth is historically located or contextual; others that we can go wrong, that what we once thought deep could turn out not to be.[9] But the case was also disputed on the facts: a sizeable contingent felt that a deep theorem could very well have a simple proof.

Some effort was made to distinguish depth from nearby notions. In his talk, James Tappenden[10] suggested, for example, that Smullyan's presentation of Gödel's first theorem is exceptionally beautiful or elegant without being deep [Tappenden,

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[6] Professor Geroch (Department of Physics, University of Chicago) gave a talk entitled 'Depth'.

[7] Professor Stillwell (Department of Mathematics, University of San Francisco), gave a talk entitled, 'What does "depth" mean in mathematics?', on which his contribution to this issue is based.

[8] Professor Arana (Department of Philosophy, University of Illinois, Urbana-Champaign) gave a talk entitled '*Grumeaux D'ordre Dans Une Pâte Informe*: On the depth of Szemerédi's theorem', on which his contribution to this issue is based.

[9] Similarly, the vagaries of fashion, or preference for one's own area of specialization, inevitably affect judgments of depth. Some took this to show that depth is changeable or subjective, others that these factors can distort our perceptions of objective depth.

[10] Professor Tappenden (Department of Philosophy, University of Michigan), gave a talk entitled, 'What depth and fruitfulness might teach us about methodology'.

26:50][11] and that finite injury arguments are deep without being beautiful or elegant [*ibid.*, 31:00].[12] Alasdair Urquhart[13] quoted Hardy as holding Euclid's theorem on the infinity of primes to be important without being deep ([Urquhart, 18:40]; see also the written version below). And it was hardly controversial that an item could be difficult without being deep. On the other hand, surprisingness (especially an elusive notion of 'objective surprisingness', introduced by Tappenden [in the Q&A after Bonk's talk, 51:50–56:05; see also GD#4, 5:03–9:00, 12:43–17:55]), and explanatoriness (see especially Marc Lange's talk and paper[14]) were often linked quite closely to depth.

The group as a whole made a concerted effort to identify and classify the various signs of depth touched on by the speakers and in discussion. Eventually some of these came to be tentatively understood as mere symptoms of depth, mere evidence of depth, not what depth consists in: for example, that a theorem or proof is (psychologically) surprising, or interesting, or took a long historical progression for us to reach ('standing on the shoulders of giants', as Stillwell puts it, echoing Newton), or reveals previously hidden connections, or transforms the community's thinking. As Lange pointed out [GD#1, 3:45–5:15], some of the remaining candidates appeared at least potentially circular: for example, though a theorem could be fruitful or productive, perhaps this only establishes its depth if the mathematics produced is itself deep. As noted above, there was outright disagreement on one point: some held that a deep theorem must have a long and/or difficult proof (or that a deep theorem must have a deep proof, and that a deep proof must be long and/or difficult); others thought a deep result all the more striking if its proof is short, simple, elegant.[15] Finally, there was widespread agreement that proofs by cases are not deep, but some divergence on why: some cited the lack of explanatory power; others (especially Urquhart [in the Q&A after his talk, 59:35]) emphasized the danger of mistakes lurking somewhere in a large number of separate cases.

Among these proposed criteria of depth, much attention focused on the widespread conviction that a theorem or proof is deep if it ties together far-flung areas of mathematics. This is often said, for example, of the proof of Fermat's Last Theorem (though the theorem itself is not necessarily regarded as deep, as noted above). If what qualifies areas of mathematics as 'far-flung' is just our impression that they are quite different, our pursuing them separately, this is a merely subjective criterion; if drawing these interconnections is to be an objective feature of a theorem or proof, then areas of mathematics must themselves be objectively distinguishable. One suggestion was that areas

[11] Tappenden had in mind the treatment in R.M. Smullyan, *Theory of Formal Systems* (Princeton: Princeton University Press, 1961).

[12] Strictly speaking, he spoke not of 'depth' *per se*, but of his preferred notion of 'fruitfulness for deep reasons' [Tappenden, 22:00].

[13] Professor Urquhart (Departments of Philosophy and of Computer Science, University of Toronto) gave a talk entitled 'Mathematical depth', on which his contribution to this issue is based.

[14] Professor Lange (Department of Philosophy, University of North Carolina, Chapel Hill) gave a talk entitled 'Depth', on which his contribution to this volume is based.

[15] Gauss's notion of depth, as expounded by Gray in his talk and his paper below, clearly sides with the deep-is-difficult side of the opposition.

of mathematics can be distinguished by their goals and methods [GD#4, 28:43–30:33, 33:50–40:00]. So, for example, number theory is out to investigate a potentially infinite sequence; algebra is in the business of identifying and studying structural features shared by many mathematical objects; analysis grew out of scientific applications and focuses on real number spaces and their generalizations; at least one aim of set theory is to provide a certain kind of foundation for classical mathematics. Each of these endeavors involves different methods, different ways of thinking, and these appear to be differences 'in the math'. Consider then, for example, theorems that connect the existence of large cardinals to analytical facts about sets of real numbers. If large cardinals are posited in pursuit of set theory's foundational goal (as a way of making set theory as generous as possible as the arena for classical mathematics) and questions of Lebesgue measurability, the Baire property, and so on arose in the ordinary pursuit of analysis, then, on the proposed interconnection criterion, these theorems are objectively deep.[16]

This line of thought might be extended to include concepts and assumptions as well as theorems and proofs: the concept of large cardinals or the assumption of their existence might be deep because they facilitate deep theorems and proofs. It seems unlikely, though, that depth for assumptions or concepts can be reduced to depth for theorems or proofs: for example, the Axiom of Choice, an assumption, might be thought to be deep because it occurs in different forms at the outset of many different fields; here it is the assumption itself that is doing the interconnecting, not the individual theorems or proofs in those disparate areas.[17]

Tappenden [GD#1, 8:55] put the general challenge of objectivity this way: if we were logically omniscient, if we could immediately perceive all the logical connections between all the concepts and assumptions, there would be no division of mathematics into fields, no distinctions of depth; these serve only to organize matters for creatures with our cognitive limitations. An analogous posture in the philosophy of science would be: if we were physically omniscient, if we immediately understood all the individual physical facts, there would be no need for higher-level theories; they serve only to organize matters for creatures with our cognitive limitations. We see this sort of contrast in Putnam's famous example of the square peg and the round hole: there is an account of why the former does not fit the latter in terms of the placement of individual atoms and forces,[18] but there is also the account in terms of

[16]The same might be said for a theorem or proof that opens up a new field. The fact that the field is new to us is 'in us', but the fact that it is new, that is, different from previously existing fields, would be 'in the math'. Similarly for organizational power, another suggested criterion of depth.

[17]Something similar could be said about the 'compactness theorem' in its various guises [Tappenden in GD#1, 11:18].

[18]'The peg is . . . a rigid lattice of atoms . . . one could compute all possible trajectories . . . and . . . deduce from just the laws of particle mechanics . . . that the [peg] never passes through [the hole]' (Putnam, 'Philosophy and our mental life', reprinted in his *Mind, Language and Reality, Philosophical Papers*, Vol. 2 (Cambridge: Cambridge University Press, 1975), pp. 291–303. The quotation comes from p. 295.).

elementary geometry.[19] The subjectivist line here would see the geometric account as a crutch necessitated by our limitations, but an objectivist might well insist that geometric properties of the world are as real as atomic forces, that the geometric account traces patterns in the world no more dependent on us than the individual facts. An objectivist about depth could take an analogous line: even given all the logical facts, the concept of 'group' would still tie together disparate items in a mathematically fundamental way, quite independently of us; for that matter, she might continue, there are almost certainly veins of deep mathematics that we just are not equipped to notice.[20]

Tappenden also floated the notion [GD#2, 15:30], later picked up by others, that there might be an intermediate sort of objectivity: even if depth does depend on our interests and abilities, if we take these to be fixed, it will be an objective matter which items then count as deep. So, for example, even if the arrangement of mathematics into fields or what counts as fruits in fruitfulness do depend on our interests and abilities, it will still be an objective matter whether or not a certain mathematical item makes interconnections between these fields or yields such fruits. A subjectivist might count this as a victory, because fruitfulness is not 'in the math' (after all, even a Kantian idealist can say, 'once you fix the pure categories and the forms of intuition, the world is objectively spatiotemporal'); on the other hand, an objectivist might grant the point without embarrassment, noting that our scientific practice is shaped by which aspects of the world interest us, which aspects we can detect, no doubt leaving many aspects unnoticed, but that this in no way compromises the objectivity of those aspects we do notice.

3. THE PAPERS

In his paper, 'On the depth of Szemerédi's theorem', Andrew Arana investigates what depth is by looking at one recent theorem that is widely considered deep: the theorem that every sufficiently dense subset of N contains an arbitrarily long arithmetic progression. By looking at what various mathematicians have said in praise of Szemerédi's Theorem, Arana isolates and explores four different notions of depth, which he calls 'genetic', 'evidentialist', 'consequentialist', and 'cosmological'. According to these various accounts, a theorem is deep if (i) it was discovered by superlative mathematicians; (ii) its proof has some special property, such as requiring an exceptional amount of labor to find; (iii) it has important consequences; or (iv) it reveals unexpected structure. As Arana argues, each of these accounts has virtues and naturally fits some cases naturally described as deep. Nevertheless, each of these candidates has features that many philosophers would consider problematic: either by falling into vagueness,

[19] 'The [hole] is rigid, the peg is rigid, and as a matter of geometrical fact, the round hole is smaller than the peg . . . the peg . . . does not pass through the hole that is too small to take its cross-section' (*ibid.*, p. 296).

[20] Geroch was apparently so strongly inclined to see depth as 'in the math' that he tended to count an item as not-deep if it seemed to him to be 'in us' [in the exchange with Heis in the Q&A after his talk, 54:35].

by failing to pick out theorems that obviously count as deep, or by making depth subjective.

Though the use of 'deep' as a superlative honorific seems ubiquitous among contemporary mathematicians, Jeremy Gray ('Depth — a Gaussian tradition in mathematics') argues that the use of this word — and the values it expresses — began at a very specific time. Gray suggests that the word was first used in Gauss's *Disquisitiones Arithmeticae* to describe certain theorems in number theory, such as the theorem of quadratic reciprocity. After canvassing the writings of an impressively wide range of German mathematicians from Gauss to Hilbert, Gray concludes that a deep theorem in this Gaussian sense (i) is 'hidden', difficult to prove and discover, and (ii) has significant organizational power. Mathematicians before Gauss, such as Lagrange and Euler, did not speak of 'depth' because their mathematical work was primarily aimed at solving open mathematical problems. Mathematicians who share this approach, including the leading French mathematicians of the nineteenth century, valued a different virtue, complementary to depth. A 'virtuosic' mathematician has the ability to solve problems through absorbing a large number of examples and carrying out complicated and tricky calculations. Gauss and the German mathematicians who followed him valued depth over virtuosity because they had an opposed 'structural' view of mathematics.

Marc Lange ('Depth and explanation in mathematics') isolates and explores a comparative notion of explanatory depth. One proof of a theorem is explanatorily deeper than another proof if it answers why-questions that are left open by the less deep proof. A deeper proof thus explains an apparent mathematical coincidence. Further, one theorem is explanatorily deeper than another theorem if it is invoked in a proof that explains why the less deep theorem holds. For example, the theorem that an analogue of the binomial theorem holds in any commutative ring explains why the binomial theorem holds for exponentiation and why the general Leibniz rules hold for derivatives — two less deep theorems whose analogous form would otherwise seem coincidental. Since depth of proof and depth of theorems are distinct notions, Lange distinguishes between 'shallow depth', where an otherwise shallow theorem has a deeper proof, and 'deep depth', where we have a theorem that is itself deep. Paradigm cases of shallow depth are solutions of otherwise unimportant equations through clever substitutions: a proof that makes plain why that trick works then has 'shallow depth'. Throughout, Lange notes some analogies between explanatory depth in mathematics and explanatory depth in the natural sciences.

In his contribution ('What does "depth" mean in mathematics?'), John Stillwell considers a wealth of theorems to illustrate three different notions of depth: historical depth, foundational depth, and formal depth. Drawing on the intuitive idea that deep theorems are hidden and difficult to uncover, Stillwell considers measuring the depth of a theorem by the number of 'giants' whose shoulders were stood on to find its proof. Though many of mathematics' most famous theorems are in this way 'historically deep', like the Poincaré conjecture or Fermat's last theorem, Stillwell notes that many of these theorems lack what he calls 'foundational depth'. Foundationally deep theorems, like the fundamental theorem of algebra or the uncountability of the reals, are fruitful and fundamental for some area of mathematics, supporting and explaining a large group of facts. Last, Stillwell considers four formal ways of capturing depth: using Gödel incompleteness, Gödel speedup, unsolvable algorithmic problems, or reverse mathematics. Only the last way picks out theorems that are plausibly deep, though typically only for theorems concerning the reals.

Alasdair Urquhart's 'Mathematical depth' collects together various notions of depth, drawn from mathematicians, philosophers, and even writers on chess. Philosopher Penelope Maddy suggests that 'deep' is more or less equivalent to 'fruitful', 'important', and so on, while mathematicians G.H. Hardy and Timothy Gowers give very specific notions of depth: a theorem is deep if its proof requires conceptually intricate ideas. Instead of highlighting the length of chains of *definitions*, mathematician Doron Zeilberger effectively equates depth with length of *proof* and thus the computational power required to find a proof. This latter view suggests a formal proof-theoretic notion: the depth of a theorem is measured by the longest sequence of logical inferences in its proof, together with the number of auxiliary results and lemmas required. Interestingly, Urquhart observes, this proof-theoretic notion is analogous to combinations of chess moves that are often called 'deep'.

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Afterword to Special Issue on Mathematical Depth

1. THE DISCUSSIONS, REVISITED

The observation that depth is a matter of degree, emphasized by Lange, was widely accepted.[1] Tappenden noted [GD#4, beginning][2] that this twist provides one way of reconciling the objective and subjective notions: the various facts of relative depth might be objective, but the degree of depth we insist on before we are willing to call an item 'deep' could depend on our capacities, context, training and so on. Creatures with capacities far beyond ours, for example, might view all the items we call 'deep' as ultimately shallow, but this need not involve any relativity of the relations of comparative depth.

The notion of 'order from chaos' that Arana derives from his study of Szemerédi's theorem drew considerable interest. The previously maligned classification theorems were offered as examples of this sort of depth [in Q&A after Arana's talk, 50:55], Morley's miracle as another [*ibid.*, 54:00]. In contrast, it was suggested that Gödel's theorems [*ibid.*, 56:00] and degree theory [*ibid.*, 46:50] find disorder in order rather than the reverse. This reinforced the thought that there might be a number of ways of being deep and that they often come apart.

This notion of Arana's presents us with another important contrast. On one side, there is the idea of depth as an intrinsic property of a theorem or proof all on its own — 'order from chaos' being an example. On the other, there is the idea of depth as an extrinsic property — an item that draws interconnections between fields, that opens up new lines of inquiry, that facilitates proofs, that provides explanations, and so on —

[1] Indeed, it came out in discussion that the scale need not even be a linear ordering.

[2] See https://www.youtube.com/playlist?list=PLQw7KTnzkpXfGo93vo3kQk7_jA_HgwbnC, which links to a series of YouTube videos. There is one for the welcoming remarks, one for each of the talks (which includes its Q&A), plus four more for the four separate general discussionsŮall arranged in the same order as the workshop itself. References to each talk and each talk's Q&A will be given by the speaker's name and the appropriate time stamp; points in the general discussions will be indicated by 'GD' followed by the appropriate number (1–4), plus the time stamp.

the sort of thing that falls under Arana's general heading of 'consequentialism'. Arana worries that consequentialist views, views that locate depth somewhere in the kinds of consequences an item generates, may hinge on matters 'in us' rather than 'in the math', for example, on what we happen to be interested in, or how we happen to separate fields. We have seen that both subjectivists and objectivists have their own ways of understanding the separation of fields, and the same might be said of our interests and abilities: the subjectivist takes this to be a matter of fashion or psychology; the objectivist again sees the role of our interests and abilities as running parallel to our interests and abilities in science (they determine which parts of the world, which deep mathematical items, we focus on/are able to detect). But this leaves Lange's worry about consequentialist criteria untouched: there remains the danger of falling into circularity (what makes an item deep is having deep consequences).

Another notion that often surfaced is impurity — proofs are often praised for drawing on far-flung resources[3]— and we now see this as another potentially intrinsic criterion. Famous examples are analytic geometry, where algebraic notions are imported, and analytic number theory, where analysis is imported. It is perhaps no accident that Hardy, an analytic number theorist, traced depth to impurity! As Urquhart reports, Hardy focused on definitions or 'ideas', which are arranged into 'strata': the notion of an irrational number is deeper than that of an integer; the proof of the irrationality of the square root of two is deeper than the proof that there are infinitely many primes;[4] and the proof of the Prime Number Theorem, which uses the theory of functions, is deeper still. This suggests that depth corresponds roughly to the types of the items involved, that higher types are deeper: integers, reals, real functions [GD#4, 41:30]. A similar thought turns up in Stillwell's examples, where increasing depth tends to trace the gradual development of our understanding of the real numbers [Stillwell, 17:50], and the key move was admitting the completed infinite [*ibid.*, 24:00].

It is worth noting that Hardy is thinking of the previously given notions of 'real number' and 'function', seeing them arranged in strata and their degrees of depth as intrinsic. If we look instead at a notion in the process of being formed, a consequentialist view seems more natural; there we have a mathematical job to be done and what matters is getting the concept that will do it effectively, in other words, the concept's extrinsic virtues. Dedekind might be seen as giving a deep definition of 'real number', and Dyck a deep definition of 'group', though the relevant features are intrinsic in the first case and extrinsic in the second [GD#4, 46:30]. Bonk's discussion of 'Gromov's product' provides another example of a 'deep definition', so-called for extrinsic reasons [in his talk, 37:40–41:40]: it 'makes the theory work', 'captures the whole phenomenon of negative curvature in the right way', 'distills the essentials of the theory'.

One central focus of the group's efforts, at least for the philosophers, the search for candidate criteria that seem to stand a chance of being objective, of residing firmly 'in

[3]For example, as Arana points out, Szemerédi's own proof of his theorem is 'pure', in the sense that it involves no notions beyond those native to combinatorial number theory, but subsequent proofs were not. Indeed, referring to Arana's talk in his own presentation [Urquhart, 9:45], Urquhart quotes a remark that attributes the depth of Szemerédi's theorem to just this type of impurity in its proof.

[4]Thus Hardy's view, noted earlier, that the infinity of primes is important but not deep.

the math'.[5] This prompted the question: why do we want that? [GD#3, 15:20–23:10] If what we are doing is sociology, if we just want to know how the term is used, not how it might or should be used, then why this bias toward objectivity? Should we not just be exploring the usage and letting the results stand on their own? The response came that we are not only doing sociology, that we are wondering whether there is a notion of depth that might play a certain role, perhaps a role roughly comparable to the various theoretical virtues that help adjudicate between alternative scientific theories. But how can this be, the questioner continues, when the issue in science is which theory is more likely to be true, while the issue in mathematics is to draw some distinction between theorems that are all, presumably, true? Here the reply focused on assumptions: if we are to appeal to considerations of depth in our decision, for example, about whether to include the Axiom of Choice in our theory of sets, if we are to cite, for example, its welcome consequences or the way it appears in so many guises in so many disparate fields, then we might hope that this style of thinking is not just a way of tracking our psychology or sociology, that there are genuine mathematical virtues at stake.

This prompted Lange to propose a new topic to the group, to suggest that we move beyond our examination of the candidate criteria of depth, and also consider the various roles the notion of depth might be intended to play, the various employments to which it might be put [GD#3, 23:10–25:15]. It could be that some particular criterion fits better with a particular employment, rather than another. At that point, the candidate employments identified were serving as (i) a mere honorific, (ii) a way of adjudicating between assumptions or axioms, perhaps analogous to theoretical virtues in science, and (iii) a way of identifying the 'right' concepts or definitions. As noted above, broadly consequentialist criteria seem most appropriate in concept-formation instances of (iii). Maddy has suggested that consequentialist criteria, rather than intrinsic criteria like 'intuitive' or 'part of the concept of set', are fundamental for (ii), as well [GD#4, 47:10]. In any case, the line of thought described in the previous paragraph now comes to this: objective criteria seem preferable to subjective criteria for employment (ii), and perhaps also for employment (iii).

In the end, the five candidate criteria for depth 'in the math' that gained the widest support were these:

1. ties together apparently disparate fields;
2. involves impurity (definitions that reach into higher types, proofs that appeal to concepts other than those in the statement proved);
3. finds order in chaos;
4. exhibits organizational or explanatory power;
5. transforms a field or opens a new one.

[5] Even philosophers whose ultimate interest is in objective criteria would need to begin, as we did in the workshop, with a more open-ended, purely descriptive exercise of surveying, cataloging, and examining as many actual uses of the term as possible — by historical figures, contemporary mathematicians, workshop participants, *etc.* — much as a sociologist or an ordinary language philosopher might do. Only this sort of field work can provide the data or evidence for more normative methodological or philosophical questions.

Two questions were then posed: which of these are illustrated by which of our examples? And, are there any clear examples that fit none of them? To answer 'yes' to the second question would be to give an example that satisfies some important subjective criterion but none on the list of potentially objective criteria. It might be that there are no such cases, that all our subjective criteria are good indicators for objective criteria. But if there are such examples, then unless we have missed some candidates for criteria 'in the math', there would appear to be at least some instances of depth that only reflect features essentially 'in us' — and depth of this purely subjective sort might be a good candidate for employment (i), but perhaps not for (ii) or (iii).

However that may be, the notion that there are different ways of being deep was further ratified in the course of these considerations: examples were offered of (3) without (1) or (2) (Szemérendi's proof of his theorem), of (5) without (1)–(4) (Cantor's diagonal argument), and perhaps of (4) without (5) (Hilbert's basis theorem [GD#4, 55:30]). Gödel's theorem was discussed at some length as a clear case of (5), and perhaps also (2), as number theory is being called upon to prove facts of meta-logic. This much is clearly only the beginning of the story. No examples of depth that failed on all five criteria were proposed.

2. A FEW THOUGHTS FROM THE INDIVIDUAL EDITORS

2.1. Penelope Maddy

As Urquhart has pointed out (Urquhart, [12:50] and in his paper), in *Defending the Axioms*,[6] I lumped a number of different notions together under a broad umbrella of 'depth'. Most of these — fruitfulness, effectiveness, productivity — are transparently consequentialist. The odd man out is 'importance', which Hardy explicitly distinguishes from 'depth', but I was taking that in a roughly consequentialist spirit as well.[7] My method was to describe particular examples — ranging from Cantor's and Dedekind's introduction of sets into their practices, through Zermelo's defense of the Axiom of Choice, to contemporary thinking about determinacy and large cardinals — describing in each case the mathematical context, the jobs that needed doing, and how sets themselves or strong axioms about them yielded a theory capable of doing those jobs. There were probably examples of all the above consequentialist criteria (1, 4, and 5) at work here, but I did not attempt any further delineations. Instead, the focus was on identifying particular goals and values in the particular line of inquiry, and assessing the efficacy of the entities or assumptions directly in those concrete terms. This is much the way many of the detailed examples in the workshop have been presented: here is what was going on in the math; here is what needed doing; here is how it got done; and, often enough, here is the further mathematics that was opened up as a result.

At one point in the book I suggest that trying to give a general account of depth would most likely be unproductive — a remark Urquhart jokingly took to be 'discouraging as far as our workshop is concerned!' [14:45]. He goes on to give a charitable

[6]Oxford University Press, 2011.

[7]As noted in the discussions above, this is not surprising, given my focus on concept formation and axiom choice.

interpretation, as the widely-shared thought that there is not any single kind of depth, but I think in fact I had something a bit more discouraging in mind. Some mathematical notions — like 'continuity' (Cauchy, Weierstrass) or 'completeness' (Dedekind) or 'computability' (Turing) — have been subjected to revealing conceptual analyses, but it does not seem that 'depth' is like these. Terms ripe for conceptual analysis are often contrasted with natural-kind terms, but that characterization does not appear to fit, either: there does not seem to be an underlying essence that we are out to locate (like discovering the chemical composition of water). Some notions are formal — for example, whether or not one statement logically implies another depends only on the logical form, not on further details — or structural — for example, whether or not an account is explanatory depends (for the Kitcherian) only on its instantiating certain unifying patterns of argument, not on the details of how it does this — but it is not clear that depth fits here, either.

So, this line of thought continues, perhaps the features that make any individual mathematical item deep are irreducibly specific to the context and texture of that particular item, not to any identifiable subclass of that context and texture. We might notice, for example, that analytic number theory ties elementary number theory with analysis, that large cardinals tie analysis with set theory, but the general fact of 'tying together' would not be what makes either of these deep; what makes them deep would lie in the details of how the tying is accomplished and what it achieves. There are, after all, many ways of tying things together that are superficial or mere trickery.

Still, even if my skeptical forebodings should prove accurate, this would not be cause for despair (as Urquhart's tone of doom may suggest). It is not that there is nothing to say; it is just that what there is to say is relentlessly specific, the very sort of thing we saw in the workshop (and I tried to do in the book). We can and (as methodologists) should examine the wide range of uncontroversial cases to see what makes each of them tick. At least for objectivists, there are strains of mathematical virtue lurking among the vast array of logical relations, and it behooves us to learn as much as we can about where they are (the mathematician's job) and how they work (the methodologist's job). For that matter, if there are in fact formal or structural generalizations to be found, contrary to this skeptical line, then close attention to examples is the way to find them.

In any case, the book does point to roughly consequentialist features of the set-theoretic examples it treats. Arana worries that consequentialist accounts are either subjective — depending on what happens to be of interest to us — or problematically objectivist, platonistic.[8] Sticking to the second horn of the proposed dilemma, the objectivist does hold that the vast net of logical relations between the various concepts and assumptions is itself objective (what Tappenden's logically omniscient being would perceive) and that the outstanding mathematical virtues of some strains of these (the ones connected with 'group' or the Axiom of Choice) are objective features of that net, but I do not see that any platonistic *abstracta* are needed to fill this in. Still, there remains Lange's concern about circularity: if an item is praised for its consequences, do not those consequences themselves have to be deep? Yes, I think they do. We value

[8]Arana also mentions Kant, but his empirical objectivity comes linked with transcendental idealism.

those bits of mathematics that allow us to meet our mathematical goals, but in the end, the legitimacy of those goals depends on their actually uncovering depth. [9] This exacerbates my concern that depth cannot be given a non-circular formal or structural analysis.[10]

Finally, in the discussions recounted above, it was suggested that if depth is used to adjudicate between axioms in the ways I have suggested, then we might well imagine that it had better be 'in the math', rather than 'in us'; we had better side with the objectivist. In the book, I do claim that depth is objective, that the Axiom of Choice, determinacy and large cardinals, and sets themselves would have the mathematical advantages they do even if we mistakenly thought they did not or failed to think of them at all.[11] This is asserted there with perhaps more bluster than argument, and it was my uneasiness about these and other matters surrounding depth that prompted me to propose it to my colleagues as the topic for a rather unorthodox workshop like this one.[12] I am not sure I am any more confident now than I was before about the objectivity of depth, but I do think I have come to understand what hinges on this, at least for the particular philosophical account of set theory outlined in the book. Let me close with a gesture in that direction.

I describe there two accounts of the metaphysics and epistemology of set theory: *thin realism* and *arealism*. Thin realism differs from more familiar versions of realism in the philosophy of mathematics by positing a minimalist metaphysics that renders the epistemology trivial;[13] arealism, in contrast, does without ontology entirely.[14] While these seem diametrically opposed, the considerations that reveal the facts about sets to the thin realist and the considerations that guide the arealist's shaping of her theory of sets are precisely the same: the strains of mathematical depth. I conclude from this that set theory is indeed answerable to objective constraints, but that this process can be described with equal legitimacy as the thin realist does or as the arealist does.

[9] See *Defending the Axioms*, pp. 81–82.

[10] Recall, *e.g.*, that Tappenden's 'fruitfulness' is actually 'fruitfulness for deep reasons' [Tappenden, 22:00].

[11] The kind of objectivism I have in mind recognizes the points touched on above: what areas of deep mathematics we uncover (Foreword, pp. 6) and what degree of depth we demand before we bestow the term (Afterword, p. 1) might well depend on our interests and abilities; the claim is just that the underlying relations of comparative depth are objective.

[12] I am grateful to them, to the speakers, and to all the participants for signing on for a project that started out as something of a hobby horse of mine. (I hope it has now grown into something more than that.)

[13] Slightly less cryptically: the thin realist begins with an assessment of the effectiveness of set-theoretic methods for achieving goals of the practice (ultimately, as noted in the following sentence, the discovery of deep mathematics); given the means-ends rationality of these methods, she draws the metaphysical conclusion that sets must be just the kind of thing that can be investigated in these ways; the question 'why do these methods track the truth about sets?' then becomes trivial ('because that's what sets are'). For more, see *Defending the Axioms*, Part III.

[14] Continuing from the previous note: the arealist makes the same assessment of set-theoretic methods as the thin realist, and draws the same conclusion about their means-ends rationality. But, unlike the thin realist, he sees no grounds to think that set theory is a body of truths, that sets exist, or that an 'epistemology', *per se*, is called for. See *Defending the Axioms*, Part IV.

So far, so good. But what if depth should turn out not to be 'in the math', but only 'in us'? I say this without pretending to a defense, but my inclination is to think that in the imagined eventuality, the ground beneath thin realism would collapse. We would be left with a version of arealism as the only viable option remaining, and a version that grasps the first horn of Arana's dilemma: the strains we choose to follow among the vast array of logical interconnections reflect not any independent mathematical virtues, but the goals and values our interests and abilities prompt us to pursue. The vibrant practice of pure mathematics would remain, of course, as we actively search out mathematical items that can achieve these idiosyncratic goals we've set for ourselves — and perhaps that is just what there is, in the end. In any case, this is what seems to me to be at stake.

2.2. Michael Bennett McNulty

During the workshop, Tappenden introduced the notion of 'intermediate objectivity', which is briefly discussed above (Foreword, p. 6). I am particularly interested in the sort of intermediate objectivity that relates to the *organization* of mathematical practice [GD#2, 3:00] (see above, pp. 1–2). In some cases, a proof's connections to multiple mathematical fields warrant the attribution of depth to the proven theorem. Although the division of mathematics into fields is arguably rooted 'in us' (as opposed to 'in the math'),[15] Tappenden suggested that such depth attributions are intermediately objective in the following sense. Once one fixes the divisions of mathematical disciplines, that a theorem's proof utilizes mathematics from diverse areas does not depend on a subject's choice. I find this notion of intermediate objectivity to be promising: a wide variety of examples of depth attribution appear to rest on the organization of mathematics. Hence, in the following, I clarify the intermediate objectivity of organizational attributions of depth by cataloguing different senses of subject-dependence.

The worry is that the division of mathematics into subdisciplines is merely subjective: due, perhaps, to our predilections or cognitive imperfections. If this were the case, then the depth of pieces of mathematics based on these criteria too would be merely subjective. However, I contend that there is a clear sense in which organizational attributions of depth are distinct from other more subjective attributions. Here are four ways in which depth attributions may be dependent on 'us', broadly speaking.

1. Depends upon personal choice.
2. Depends upon personal abilities or limitations.
3. Depends upon communal choice.
4. Depends upon communal abilities or limitations.[16]

[15] Above, we consider the possibility that the differences between mathematical disciplines are 'in the math' (Foreword, p. 3).

[16] 'Communal' in sorts 3 and 4 refers to the mathematical community relevant to the attribution of depth at issue.

Each sort of subject-dependence in this list is, I claim, more objective than those that precede it in a sense that is made clear by the consideration of examples of the sorts.

1. A mathematician praises a theorem as deep insofar as it is a fundamental theorem in her area of research and as she wants to support this field. Other mathematicians, perhaps those with other research specialties, may disagree with her.
2. A mathematician claims that a theorem is deep because its proof is beyond her ken. Other more skilled or differently trained mathematicians may disagree with her. The subject cannot simply choose what she finds to be deep, in this sense. Proofs are simply given to her as understandable or not (although, of course, she may gain the ability to understand the proof in time).
3. A mathematician writes that Fermat's Last Theorem (FLT) is deep because its proof connects diverse areas of mathematics. Other mathematicians must agree with the assessment: given the current divisions of mathematics, it is undeniable that the proof of FLT utilizes mathematics from disparate areas. So the ground of FLT's depth is not up to personal choice, abilities, or limitations. However, we can recognize that, were the community to change — that is, if the mathematical community consented to a different division of mathematical disciplines — the proof of FLT may no longer connect so many areas, in which case it would be less deep.
4. A mathematician claims that the Four Color Theorem (4CT) is deep because no solitary person in the relevant mathematical community has completed its proof.[17] Other mathematicians must agree with the claim: there is no non-computer-aided proof of 4CT. Nevertheless, this fact is not independent of us: were human cognitive capacities different, we might be able to produce a proof along the same lines as the current computer-aided proofs. Alternatively, in the future, we may discover a distinct humanly executable proof of 4CT.

I should emphasize that these categories are more markers on a spectrum than discrete sorts. In particular, there is middle ground between 3 and 4. The communal choices mathematicians make are more or less influenced by our human cognitive abilities and limitations: for example, with improved or modified cognitive abilities, we would, perhaps, divide mathematics into subdisciplines differently (as noted above, Foreword, p. 5). Nevertheless, these categories give a good sense of the different degrees of subject-dependence that may be present in attributions of depth. As one moves through the list, the grounds for depth attribution become less dependent on a single individual's choice and more binding for mathematicians generally. If a judgment is true based on communal choice, then in the case that the (relevant) mathematicians change that choice, the judgment may no longer be true. If a judgment, on the other hand, is true based on human cognitive abilities or limitations, it is not within

[17] Zeilberger endorses this conception of depth, as Urquhart explains in his paper.

our power of choice to change it. Now, again, there is some blurriness here — our choices are informed by our limitations, and we may make choices to try to address our limitations — but I contend that there is good reason to differentiate 3 and 4 and to rank 4 as more objective than 3.

Standards of mathematical depth attribution falling under sorts 3 or 4 are intermediately objective. Mathematical entities deep for these reasons are not deep *independently* of humans, but they are *also* not deep merely due to a single agent's whim or contingent limitations (hence their intermediately objective status). The standard of connecting diverse fields of mathematics lies in the spectrum between 3 and 4, as considerations of human limitations that play a role in the division of mathematical fields.[18] That said, our particular division of mathematical fields is not necessitated by our cognitive abilities and limitations; so I claim that this depth standard is closer to sort 3 than 4.[19] That a proof utilizes far-flung mathematics is not subjective in the sense of being due to personal choice or cognitive abilities. Rather, once the mathematical community have organized their practice in a particular way, facts about the connections amongst the areas of mathematics are *given* to us just as uncontroversially objective mathematical facts are.[20]

Finally, this account of intermediate objectivity also clarifies the objectivity of other kinds of depth attribution. Gray proposed the idea that some mathematics is deep insofar as it gives rise to entirely new fields of mathematics [GD #3, 7:54–11:10]. As new fields can be distinguished given the organization of mathematics, this criterion is intermediately objective. In addition, the account on hand entails the intermediate objectivity of depth attributions based on the impurity of a theorem's proof, raised above (p. 2). After all, the impurity of a proof — its marshaling of resources from diverse areas of mathematics — depends upon the organization of mathematics.

2.3. James Owen Weatherall

On two occasions during the workshop, mathematical depth was related to some notion of depth in science. The first occasion (Foreword, p. 3) was Geroch's discussion of examples from mathematics that he did not think were particularly deep *qua* mathematics, but which he believed were deep in light of their applications to physics. The second was Lange's suggestion [Lange, 15:40], emphasized in his contribution to

[18]As we raised the Kantian example above (Foreword, p. 6), I should note that, according to this categorization, the theorems of Euclidean geometry are intermediately objective for Kant. This intermediate objectivity is of sort 4, because the truth of Euclidean geometry is due to our human cognitive peculiarities: (Euclidean) space is the form of human outer experience.

[19]Of course, there are cases in which mathematicians dispute the division of mathematical fields or the precise borders amongst them. In my comments, I mean only to consider those cases in which there is widespread acceptance regarding the division of fields relevant to a specific attribution of depth. I here make no claims about the sort of subject-dependence associated with claims resting upon contested divisions of mathematics.

[20]That said, since organizational attributions of depth depend upon mathematical communities, they lack some of the characteristics of wholly objective judgments. For instance, dependence on the community means that the truth of intermediately objective judgments depends upon the time of utterance.

the current issue, that on his analysis, depth in mathematics is analogous to explanatory depth in science: in both cases, for Lange, greater depth corresponds to answering more why questions.

I propose to bring these two discussions together, to consider how depth in mathematics and depth in science intersect in cases where mathematics plays an essential role in scientific explanation. I will suggest that in two of Geroch's examples, the mathematics he cites appears deeper in light of its applications to physics because it is instrumental in comparatively deep explanations in physics, in Lange's sense of 'deeper than'. Since these examples are somewhat obscure, I will also describe a third, simpler, example that I think illustrates the same point. In another of Geroch's examples, a conjecture motivated by physics led to mathematics that (I will claim) is deep in its own right. Geroch's final example seems to me to combine features of both of these classes of examples. The upshot is that at least in the context of mathematical physics, our judgments of depth in mathematics and depth in scientific explanation may intertwine, in the sense that pursuing deep explanations in physics can be a route to deep mathematics, while contributing to deep explanations in physics can lead us to judge otherwise pedestrian mathematics as deeper.

The first example is the classification of irreducible representations of SL(2,C) [Geroch, 9:37].[21] I agree with Geroch that this is not especially deep mathematically; in effect, it is a matter of brute calculation and some clever organization of results.

I also agree that this example seems deep, or rather, *deeper*, in application. The irreducible representations of SL(2,C) play a crucial role in a classification of elementary particles due to Eugene Wigner, who showed that these representations correspond precisely to possible values that certain properties of elementary particles, namely their mass and spin, may take. There are several striking features of this classification. One is that it correctly captures the fact that although the mass of a particle may be any non-negative real number, the possible spins of a particle are discrete, represented as integral multiples of 1/2. A second striking feature is that, physically speaking, there is an important difference between the cases where a particle's mass is zero and where its mass is positive, a difference fully captured by Wigner's classification (it corresponds to differences in the structure of representations of SL(2,C) for the corresponding parameter choices). All of this works because SL(2,C) is the double cover of the so-called Lorentz group, SO(1,3). The Lorentz group is the symmetry group of a vector space with a Lorentz signature inner product on it, which is precisely the structure of the tangent space to each point of spacetime according to relativity theory. Wigner's classification shows a sense in which a certain variety of physical possibility corresponds to possible ways of being invariant under a certain class of symmetries associated with spacetime.

Why is this deep? I think Lange's account is helpful here. There are a number of properties particles may have: mass, spin, charge of various sorts. But when one begins investigating the bestiary of known particles, it is immediately clear that these properties follow certain patterns. For instance, as just noted, spin can only take discrete values. This leads to a natural why question: why these particular values? Wigner's

[21] The CPT theorem [Geroch, 15:55], which I will not discuss, has a similar character.

classification answers this question, and it does so in full generality — it applies to *all* known particles — by appealing to a high-level structural feature of spacetime, namely the properties of the Lorentz group and its representations. (Recall Lange's remarks in section 4 of his essay, that appeal to symmetry groups is often the hallmark of deep explanations in physics.) I would suggest that the reason the classification of irreducible representations of SL(2,C) seems deeper in application is that it plays a central role in a deep explanation.

I think this example is more than sufficient to make the point that when mathematics plays an essential role in a deep explanation in science, the mathematics seems to gain some depth. But it is worth remarking that one need not delve into relativistic quantum physics to find examples of a similar character. Consider Newton's derivation of Kepler's first law, which states that the planets follow elliptical orbits, from an inverse square law of gravitation.[22] Here the mathematics involved is simple: it requires only basic calculus, and the derivation takes just a few lines. Still, in application to the solar system, this derivation shows how a broad empirical regularity, previously taken as a brute fact, follows from a much more general principle concerning how massive bodies affect one another. In this sense, the derivation plays a crucial role in a deep explanation. And, as with the representations of SL(2,C), it seems to me that the mathematics itself seems deeper in light of the application.

Another of Geroch's examples concerns the positive-mass conjecture in general relativity [Geroch, 11:27]. The positive-mass conjecture states that, for a certain class of physically interesting spacetimes (those that are 'asymptotically flat' and satisfy some background assumptions), a quantity known as the ADM mass is non-negative. The ADM mass is meant to represent the total mass-energy content of space at a time, and so, insofar as we do not think negative mass-energy is physically realistic, the positive-mass conjecture captures a sense in which this class of spacetimes satisfies a weak plausibility criterion. The result was first proved in 1979 by mathematicians Richard Schoen and Shing-Tung Yau, using variational methods. Two years later, in 1981, Edward Witten, a physicist, published a second proof, using so-called spinor fields on spacetime.

This result is of a character different from the other examples. While one might take it to play a role in a certain kind of (deep?) explanation — perhaps as part of an answer to the question, why is gravity always attractive on large scales? — it is more naturally understood as establishing that general relativity is compatible with our background understanding of mass and energy as non-negative. On the other hand, Geroch's suggestions notwithstanding, the positive-mass conjecture is widely considered a deep — or at least, celebrated and important — result in geometry: for instance, Schoen and Witten were both awarded MacArthur fellowships soon after their proofs appeared, and the positive-mass conjecture was explicitly cited by the ICM for both Yau's 1982 Fields medal and Witten's 1990 Fields medal.

Another consideration in this case is that Geroch specifically pointed to the second proof in his example, noting that Witten used spinors, which do not appear in the statement of the conjecture and play no role in classical general relativity, to prove a

[22] I am grateful to David Malament for suggesting this example.

result that does not appear to have anything to do with spinors. This indicates that, for Geroch, it is not the result itself that matters, as one might have expected if the result were deep in virtue of the role it plays in explanation, but rather the impurity of Witten's proof that makes a difference. Impurity, meanwhile, is a criterion for depth in pure mathematics that was identified by several contributors to the issue. All of this suggests to me that the application to physics is less important in this example, and it is better conceived as an instance in which a conjecture motivated by physics led to mathematics that is deep *qua* mathematics.[23] Of course, this is compatible with the possibility that the result seems still deeper because of the role it plays, or could play, in scientific explanation.

Geroch's final example, concerning existence and uniqueness results for hyperbolic systems, is a more complicated case. It seems to me that it lies somewhere on a spectrum between the examples already discussed. Results concerning solutions to the so-called Cauchy problem, the general name for initial-value and boundary-value problems in the theory of partial differential equations, have been a locus of considerable mathematical attention since the problem was first formulated by Augustin Cauchy. Cauchy proved a special case in 1842, but the general theorem that is usually cited in this regard was not proved until 1875, by Sofia Kovalevskaya. The Cauchy-Kovalevskaya theorem states that the initial-value problem for any partial differential equation whose coefficients are analytic in the unknown function and its derivatives has a unique solution (at least locally). It was not until 1957 that it was understood that this result does not generalize to smooth coefficients, following publication of a counter-example due to Hans Lewy. Given this long history, with partial contributions by many celebrated mathematicians, it is difficult to say that the theorems are not deep.

On the other hand, the methods by which such results are proved, at least in special cases, are now the topic of standard courses on partial differential equations, even at the undergraduate level, and so it is also difficult to think of these results as deep in the strongest sense. I am inclined to say that judgments of depth, here, may be context dependent, and results that appeared deep, or at least difficult, in the late nineteenth century may seem more accessible today.[24] Still, insofar as this example *is* deep, on a purely mathematical level, it is another case in which deep mathematics resulted from questions motivated by physics: namely, questions concerning when a certain physical configuration has a unique evolution. Meanwhile, the uniqueness of solutions of partial differential equations plays an important role in explanations of determinism in classical physics (and in the evolution of the Schrödinger equation in quantum mechanics). More, the particular explanations of determinism that make use of these results certainly seem deep, on Lange's account: the reason that essentially all classical

[23]Although I do not have space to develop it here, another example with this character concerns the Yang-Mills equation. In that case, attempts to find and classify solutions to a differential equation whose interest comes from high-energy particle physics led to an entirely novel approach to classifying smoothness structures on 4-manifolds, leading to a revolution in differential geometry and a Fields medal for Simon Donaldson.

[24]Similar points were made by several workshop participants, though a consensus was not reached on this point.

systems behave deterministically is that the rate of change of the properties of a system depend on the current properties of that system in such a way that the behavior of those properties may be represented by differential equations. And unique solutions of those equations for given initial conditions are secured by these theorems.

This last example seems most telling, since it strongly suggests that in cases where mathematics plays an essential role in scientific explanation, it can be difficult to disentangle judgments of depth of mathematics from depth of the scientific explanations in which that mathematics figures. We have seen both that a role in deep scientific explanations may make (trivial?) mathematics seem deeper than it otherwise would, and that questions from physics (or elsewhere in science) can lead to truly deep mathematics. But for a wide range of intermediate cases, such as the final example, where one wants to say the mathematics is deep, though perhaps not exceptionally so, it is not clear that one can fully distinguish the depth *qua* mathematics from its depth in application.

3. CLOSING REMARK

Our hope is that this issue will stimulate interest in the topic of mathematical depth. If the rough groundwork presented here inspires others to pursue the topic, it may be possible to convene a second workshop to build on the first, but however that may be, we would like to thank the speakers and participants in this workshop for their cheerful and cooperative spirit in pursing this difficult and uncertain topic.

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