Answer Key
ECON 105C. Problem Set 5.

Question 1.

a) Write up the firms profit maximization problem taking into account that
\[ K_t = K_{t-1} + I_t \]

\[
\max_{l_t, K_t} \frac{1}{r} \sum_{i=0}^{T} [\pi(K)K_i - P_t I_i - C(I_i)]
\]

s.t. \( K_t = K_{t-1} + I_t \); \( \forall t = 0, \ldots, T \)

b) Set up the Lagrangian (Hint: remember that there are \( t \) constraints)

\[
L = \sum_{t=0}^{T} \frac{1}{(1+r)^t} [\pi(K)K_i - P_t I_i - C(I_i)] - \sum_{t=0}^{T} \lambda_t [K_t - K_{t-1} + I_t]
\]

c) What is the interpretation of the Lagrangian multiplier in this model? The Lagrangian multiplier represents the marginal value (=impact on firm’s profits) of an additional unit of capital. It is the shadow value of relaxing the constraint. 

d) Write up the Lagrangian expressing the multipliers in time \( t \) value.

Define \( q_t \) as the marginal value of capital in year \( t \) value (\( \lambda_t \) is the present discounted value of \( q_t \)), then:

\[ q_t = (1+r)^t \lambda_t \rightarrow \lambda_t = \frac{q_t}{(1+r)^t} \]

Plugging in in the Lagrangian we obtain:

\[
L = \sum_{t=0}^{T} \frac{1}{(1+r)^t} [\pi(K)K_i - P_t I_i - C(I_i)] - q_t [K_t - K_{t-1} + I_t]
\]

Note that now there is only one summation.

e) What is the interpretation of \( q_t \)?

Is the value of to the firm of an additional unit of capital in time \( t+1 \) in time \( t \) dollars.

f) Write the F.O.C. w.r.t \( I_t \) and \( K_t \) . (Hint: for the F.O.C. w.r.t. \( K_t \) remember that \( K_t \) appears in two constraints)
\[ I_t : \frac{1}{(1 + r)^t} \left[ -P_t - C'(I_t) \right] + q_t = 0 \]
\[ K_t : \frac{1}{(1 + r)^t} \left[ \pi(K_t) + q_t \right] - \frac{1}{(1 + r)^{t-1}} q_{t-1} = 0 \]

**g)** Focus on the F.O.C. w.r.t. \( I_t \). Interpret this condition. (Hint: before interpreting multiply both sides by \( (1+r)^t \) and solve for \( q_t \))

\[ \frac{1}{(1 + r)^t} \left[ -P_t - C'(I_t) \right] + q_t = 0 \quad \Rightarrow \quad q_t = P_t + C'(I_t) \]

At the optimum, the marginal cost of acquiring capital (the purchase price plus the adjustment cost) equals the marginal value of capital. The firm invests up to the point where the cost of acquiring capital equals the value of capital.

**h)** Consider the case when \( C'(0) = 0 \) (e.g. there is no investment in the period). What happens to capital? Does it increase, or decrease? Plot the condition derived in **g**) in the \((K_t, q_t/P_t)\) space (graph relating \( q_t/P_t \) to \( K_t \))

If there is no investment, the marginal cost of adjusting capital is also zero so \( C'(0) = 0 \). When this happens, the F.O.C. becomes: \( q_t = P_t \) or \( \frac{q_t}{P_t} = 1 \). If there is no investment capital does not change so this is the condition when \( \Delta K_t = 0 \). Note that \( K_t \) does not appear in this condition so it can be represented in the \((K_t, q_t/P_t)\) space as a horizontal line fixed at \( \frac{q_t}{P_t} = 1 \):
i) Explain the dynamics of $K_t$ in the graph. In order to do it consider what is the sign of $C'(I_t)$ and $I_t$ when $q_t > 1$ and when $q_t < 1$.

In the parts of the line above the line, that is when $q_t > 1$, capital is increasing. In the parts below the line capital is decreasing. To see why, look again at the F.O.C.:

$$q_t = P_t + C'(I_t)$$

Remember that if $I_t$ is positive, so is the marginal cost of adjusting capital, $C'(I_t)$, and capital is therefore increasing. If $C'(I_t)$ is positive the right hand side of the condition is greater than when $C'(0) = 0$ so $q_t$ greater than 1 and we are above the line.

j) Now focus on the F.O.C w.r.t. $K_t$. Multiply both sides of this equation by $(1+r)^t$ and rearrange so that $\pi(\bar{K})$ appears as a function of $q_t$ and $\Delta q_t$. Interpret this condition.

$$[\pi(\bar{K}_t) + q_t] - (1+r)q_{t-1} = 0 \Rightarrow \pi(\bar{K}_t) = (1+r)q_{t-1} - q_t$$

Now define $\Delta q_t = q_t - q_{t-1}$, so $q_{t-1} = q_t - \Delta q_t$ and the condition becomes:

$$\pi(\bar{K}_t) = rq_t - \Delta q_t - r\Delta q_t$$

At the optimum, firms marginal revenue product of capital ($\pi(\bar{K}_t)$) equals the opportunity cost of a unit of capital (left hand side). The intuition is that owing an
additional unit of capital requires forgoing of $rq_t$ and involves offseting capital gains of $\Delta q_t$.

k) Assume now that $\Delta q_t = 0$. Draw this condition in a graph relating $q_t$ to $\bar{K}_t$.

If you solve for $q_t$, the condition becomes: 

$$q_t = \frac{\pi(\bar{K}_t)}{r},$$

remember that 

$$\frac{d\pi(\bar{K}_t)}{d\bar{K}_t} < 0,$$

so when $\bar{K}_t$ increases, $\pi(\bar{K}_t)$ decreases and, therefore, so does $q_t$ so that the line is downward sloping:

l) Explain the dynamics of $q_t$ in the same graph.

In the points of the graph to the right of the curve, $q_t$ is increasing and to the left of the curve $q_t$ is decreasing. To see why, look at the condition again but solve for $\Delta q_t$:

$$\Delta q_t = \frac{rq_t - \pi(\bar{K}_t)}{(1 + r)},$$

given that 

$$\frac{d\pi(\bar{K}_t)}{d\bar{K}_t} < 0,$$

this implies that when capital increases so does $\Delta q_t$. Therefore, to the right of the line in the graph $\Delta q_t$ is positive and it is negative to the left of the line.

m) Combine the graphs derived in h) and k). Explain the dynamics of $K_t$ and $q_t$ starting from each of the four quadrants determined by the graph.
n) Use the graph in m) to explain the effects (and the dynamics) of a permanent tax credit. What happens if the tax credit is temporary?
The permanent tax credit can be interpreted as a permanent increase in the marginal value of capital so the relevant F.O.C. becomes: 
\[ q_t + \theta_1 = P_t + C'(I_t), \] 
where \( \theta_1 \) is positive represents the value of the tax credit. Given that the right hand side of the equation is the same, this implies that for the equality to hold, \( q_t \) has to be smaller than in the case with no tax credit. That is why the curve shifts down leading to a new long run equilibrium in which capital is greater than before. If the shift is temporary eventually we go back to the original situation.

\( \theta_1 \) is positive represents the value of the tax credit.

The increase in the interest rate affects only the second F.O.C: 
\[ q_t = \frac{\pi(K_t)}{r}, \] 
an increase in \( r \) leads to a lower \( q_t \) for the condition to hold, so the curve shifts down. In the new long run equilibrium, capital is smaller. If the shift is temporary eventually we go back to the original situation.