1. INTRODUCTION

Under simple viewing conditions the spectral properties of a reflected light depend on the product of the spectral power distribution of the light source and a surface's reflectance function. Variation in either light-source properties or surface composition induces variation in reflected light. A remarkable fact of human color vision is that a surface's color appearance is stable under conditions of varying illumination. This phenomenon of color constancy has prompted the development of algorithms to estimate the spectral properties of a scene's illuminant and its surfaces

In two companion papers we analyzed how well two-stage linear recovery procedures can use bilinear models to determine surface and light-source chromatic properties from quantum-catch data. The applicability of these two-stage procedures, like those of Maloney and Wandell and D'Zmura, is restricted by the need for the number of photoreceptor types to equal or exceed either the dimension of the linear model for reflectance or the dimension of the linear model for illumination. A trichromatic visual system that uses a two-stage procedure is limited to recovering at most three spectral descriptors for either surfaces or light sources. We present here a general linear recovery procedure that uses bilinear models to recover spectral descriptions of reflectances and illuminants simultaneously. The procedure works in situations in which the dimensions of the linear models for reflectance and illumination each may exceed the number of photoreceptor types. A visual system that uses this new algorithm can recover, in principle, large numbers of spectral descriptors for both surfaces and light sources. The scope of the recovery procedure extends to all color constancy problems in which both (1) the number of quantum-catch data compares favorably with the number of unknown spectral descriptors to be recovered and (2) the number of surfaces providing quantum-catch data compares favorably with the number of spectral descriptors per surface to be recovered.

It is important to identify which bilinear models can be used by a color constancy algorithm to recover the spectral descriptors of surfaces and light sources uniquely. The necessary and sufficient conditions for unique recovery remain the same as those for the two-stage procedures: the bilinear model must provide a one-to-one relationship between quantum-catch data and sets of lit surfaces. We use this requirement to develop algorithms that check whether a given bilinear model, with the parameters of a particular color constancy problem, provides unique recovery.

We check the function of particular bilinear models in general linear recovery. These checks involve models for which the dimensions of both the reflectance and the illumination models exceed the number of photoreceptor types. These tests of model function extend the classification of linear methods for color constancy initiated in the two companion papers. We find that many views (provided by different light sources) of many surfaces are needed for recovering high-dimensional spectral descriptions of both surfaces and lights. We show that the general linear recovery algorithm lets p-chromatic systems, where p ≥ 2, determine spectral descriptions of arbitrarily high dimension.

In this paper we follow the format of the preceding companion papers. We first introduce the recovery algorithm (Section 2) and continue by discussing necessary conditions (criteria) for unique recovery to be possible (Section 3). The issue of whether a particular bilinear model does, in fact, provide unique recovery is not settled by these criteria. We are thus led, in Section 4, to provide a model check algorithm that possesses necessary and sufficient conditions. In Section 5 we check models and carry out further analysis in an effort to classify...
color constancy problems. This research was described elsewhere in preliminary form.21

2. GENERAL LINEAR RECOVERY PROCEDURE

We were led to develop the general linear recovery algorithm by noting that two-stage linear recovery procedures are restricted by the need to invert certain bilinear model matrices. In these two-stage procedures light-source descriptors are recovered in a first stage, and reflectance descriptors are recovered in a second stage,1 or vice versa.2 The matrix inverses are needed for carrying out these stages. In consequence, the dimensions of the recovered spectral descriptions for either lights or surfaces are restricted. However, such restrictions are not necessary. The new algorithm avoids the inversion of bilinear model matrices and so circumvents these restrictions. It recovers in a single stage both light-source and surface descriptors.

Each of the bilinear model matrices $B_j$ for $j = 1, \ldots, n$ is of matrix dimension $p \times m$, in which $p$ is the number of photoreceptor types, $m$ is the dimension of the linear model for illumination, and $n$ is the dimension of the linear model for reflectance (see Table 1 and Refs. 2–5). Their entries $(B_j)_{ki}$ are defined as follows [Eq. (8) of Ref. 1]:

$$(B_j)_{ki} = \int Q_k(\lambda)A_i(\lambda)R_j(\lambda)d\lambda, \quad j = 1, \ldots, n, \quad (1)$$

in which appear the photoreceptor spectral sensitivities $Q_k(\lambda)$, $k = 1, \ldots, p$; the illumination basis functions $A_i(\lambda)$, $i = 1, \ldots, m$; and the reflectance basis functions $R_j(\lambda)$, $j = 1, \ldots, n$. For a two-stage procedure to function, the bilinear model matrices must be of full rank, with $p \geq m$. The $p \times m$ model matrices $B_j$, in which the roles of surfaces and illuminants are interchanged, provide a transposed recovery procedure that is constrained by the requirement that $p \geq n$. In a companion paper we show typical basis functions for illumination and reflectance as well as a set of photoreceptoral spectral sensitivities (Fig. 1 of Ref. 3).

Let us now describe the bilinear model for quantum-catch data in terms of model matrices and descriptors for illumination and reflectance. Consider the quantum catches received from a Mondrian5 comprising $s$ surfaces with different (i.e., for $s \approx n$, linearly independent) reflectance functions that is lit, in turn, by $v$ illuminants. Each surface provides $p$ quantum catches, so that the total number of quantum-catch data from $v$ views of $s$ surfaces is $svp$. Following D’Zmura and Iverson,3 we introduce indices $t$ and $w$, which run over the number $s$ of surfaces and number $v$ of views, respectively. Then the quantum catch $q_{twk}$ of the $k$th photoreceptor type produced by the $t$th surface viewed under the $w$th light is related to the $n$ reflectance descriptors $r_{ij}$, for $j = 1, \ldots, n$, and the $m$ illuminant descriptors $a_{wi}$, for $i = 1, \ldots, m$, in the following way [Eq. (10) of Ref. 3]:

$$q_{twk} = \sum_{j=1}^{n} \sum_{i=1}^{m} r_{ij}(B_j)_{ki}a_{wi}. \quad (2)$$

This expression expresses the bilinear dependence of quantum-catch data on surface and illuminant descriptors.

Note that the system expressed by Eq. (2) is a bilinear, viz., nonlinear system. We will now work toward formulating an equivalent homogeneous linear system. We introduce the $p$-dimensional data vectors $d_{tw} = [q_{tw1}, \ldots, q_{twp}]^T$ and $m$-dimensional vectors of illuminant descriptors $a_{tw} = [a_{tw1}, \ldots, a_{twm}]^T$ [Eq. (11) of Ref. 3]:

$$d_{tw} = \sum_{j=1}^{n} r_{jj}B_ja_{tw}, \quad (3)$$

for $t = 1, \ldots, s$ and $w = 1, \ldots, v$. We define the $pv$-dimensional vectors $d_t = [d_{t1}^T \ldots d_{tp}^T]^T$, the $mv$-dimensional vector $a = [a_1^T \ldots a_m^T]^T$, and the $pv \times mv$ block-diagonal matrices

$$C_j = \text{diag}[B_j, \ldots, B_j], \quad (4)$$

in which each of the blocks along the diagonal is $B_j$; Eq. (3) thus takes on the form [Eq. (13) of Ref. 3]

$$d_t = \sum_{j=1}^{n} r_{jt}C_ja, \quad t = 1, \ldots, s. \quad (5)$$

### Table 1. List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$</td>
<td>Number of photoreceptor types</td>
</tr>
<tr>
<td>$m$</td>
<td>Illuminant model dimension</td>
</tr>
<tr>
<td>$n$</td>
<td>Reflectance model dimension</td>
</tr>
<tr>
<td>$v$</td>
<td>Number of views</td>
</tr>
<tr>
<td>$s$</td>
<td>Number of surfaces</td>
</tr>
</tbody>
</table>

### Functions of wavelength

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(\lambda)$</td>
<td>An illuminant spectral power distribution</td>
</tr>
<tr>
<td>$A_i(\lambda)$</td>
<td>The $i$th illuminant model basis function, $i = 1, \ldots, m$</td>
</tr>
<tr>
<td>$L(\lambda)$</td>
<td>A reflected light</td>
</tr>
<tr>
<td>$Q_k(\lambda)$</td>
<td>The $k$th photoreceptor spectral sensitivity, $k = 1, \ldots, p$</td>
</tr>
<tr>
<td>$R_j(\lambda)$</td>
<td>A reflectance function</td>
</tr>
</tbody>
</table>

### Descriptors, vectors, and matrices

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_{tw}$</td>
<td>Illuminant descriptors</td>
</tr>
<tr>
<td>$B_j$</td>
<td>Bilinear model matrices</td>
</tr>
<tr>
<td>$\delta_j$</td>
<td>Block-diagonal bilinear model matrices diag[$B_j, \ldots, B_j$]</td>
</tr>
<tr>
<td>$d_t$</td>
<td>Quantum-catch data</td>
</tr>
<tr>
<td>$e_{ij}$</td>
<td>Variables relating two sets of reflectances</td>
</tr>
<tr>
<td>$F$</td>
<td>Recovery matrix</td>
</tr>
<tr>
<td>$G_{ij}$</td>
<td>Associated model matrices</td>
</tr>
<tr>
<td>$I$</td>
<td>Identity matrix</td>
</tr>
<tr>
<td>$L$</td>
<td>Model check matrix</td>
</tr>
<tr>
<td>$\rho_{ik}$, $\rho_j$, $\rho$, $P$</td>
<td>Inverse of the matrix of reflectance descriptors</td>
</tr>
<tr>
<td>$r_{ij}$, $r_t$, $r_j$, $R$, $\sigma_{ij}$</td>
<td>Reflectance descriptors</td>
</tr>
</tbody>
</table>

### Miscellany

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e$, $u$</td>
<td>Numbers of equations and monomial unknowns, respectively</td>
</tr>
<tr>
<td>$E$, $U$</td>
<td>Numbers of equations and monomial unknowns, respectively, provided by a model check algorithm</td>
</tr>
<tr>
<td>$Q$, $D$</td>
<td>Number sup of quantum-catch data and number $sm + om$ of unknown descriptors to be recovered</td>
</tr>
</tbody>
</table>
When \( s = n \), the descriptors \( r_{ij} \) can be regarded as the elements of a square \( n \times n \) matrix \( R \). If the \( n \) surfaces viewed are different, i.e., the vectors formed by each surface’s reflectance descriptors are linearly independent, then the reflectance matrix \( R \) is nonsingular, with inverse

\[
P = R^{-1}.
\]

When \( \rho_J \) is used to label the entries of the matrix \( P \), it follows that Eq. (5) can be rewritten in the form

\[
\sum_{t=1}^{n} \rho_{jt} d_t - C_j a = 0, \quad j = 1, \ldots, n.
\]

We have here the desired system of linear homogeneous equations. Note that the bilinear model matrices, represented by the matrices \( C_j, j = 1, \ldots, n \), have not been inverted. In consequence, the number \( p \) of photoreceptors need not match or exceed the dimension \( m \) of the illumination model, in contrast to the case with the earlier two-stage algorithms.1–5

It remains to write the system of Eq. (7) in a more convenient form, one in which the unknown descriptors can be recovered by determination of the one-dimensional kernel of a recovery matrix.2–4 Let us introduce the \( n \times 1 \) vectors

\[
\rho_j = [\rho_{j1} \ldots \rho_{jn}]^T, \quad j = 1, \ldots, n,
\]

which may be stacked, in order, to form a single \( n^2 \times 1 \) vector

\[
\rho = [\rho_1^T \ldots \rho_n^T]^T.
\]

The vector \( \rho \) and the \( vm \)-dimensional vector \( a = [a_{11} \ldots a_{vm}]^T \) of illuminant descriptors form a complete list of all unknown descriptors. It is convenient to form a single \((n^2 + vm)\)-dimensional vector \( \delta \) from these:

\[
\delta = \begin{bmatrix} \rho \\ a \end{bmatrix}.
\]

We use the \( pv \)-dimensional data vectors \( d_t, t = 1, \ldots, n \), introduced above in Eq. (5), to form the following \( pv \times n \) matrix \( D \), which records all \( pnv \) quantum catches:

\[
D = [d_1 \ldots d_n].
\]

From the matrices \( D \) and \( C_j, j = 1, \ldots, n \), can be formed the partitioned \( pnv \times (n^2 + vm) \) recovery matrix \( F \):

\[
F = \begin{bmatrix}
D & 0 & \cdots & 0 & -C_1 \\
0 & D & 0 & -C_2 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & D & -C_n
\end{bmatrix},
\]

in which all entries are zero other than the \( n \) blocks of data along the diagonal and the \( n \) blocks of bilinear model matrices for multiple views in the last block column. The homogeneous system of linear equations defined in Eq. (7) can now be written compactly as

\[
F \delta = 0.
\]

This is the final form of the linear homogeneous system. The unknown descriptors are the components of the vector \( \delta \), and the quantum-catch data and the bilinear model matrix entries determine the recovery matrix \( F \).

The matrix \( F \) is the key to recovery. It depends in part on data and in part on the numerical structure of a bilinear model, represented by the matrices \( C_j, j = 1, \ldots, n \). From elementary linear algebra, we see at once that unique recovery, up to an arbitrary positive scalar, is possible if and only if \( F \) possesses a one-dimensional null space, i.e.,

\[
\text{dim}[\ker(F)] = 1.
\]

A necessary condition for this to be true is that the number of rows in the matrix \( F \) equal or exceed the number of its columns, minus one, a requirement that returns the feasibility condition of D’Zmura and Iverson [inequality (14) of Ref. 3] under the single restriction \( s = n \):

\[
\begin{equation}
\begin{aligned}
pnv &\geq n^2 + mv - 1.
\end{aligned}
\end{equation}
\]

The general linear recovery algorithm is thus as follows: (1) use quantum-catch data and the bilinear model to determine the matrix \( F \) [Eq. (12)]; (2) find the kernel of the matrix, where reside the illuminant and (inverse) reflectance descriptors.

The kernel of the recovery matrix \( F \) can be determined numerically with a singular value decomposition, which provides a basis for the null space of a singular matrix.23 In the event that (1) the matrix \( R \) of reflectance descriptors is invertible, (2) the \( v \) vectors of illuminant descriptors are linearly independent, (3) the feasibility condition [inequality (15)] is satisfied, and (4) conditions on the matrices \( B_j \) developed in Sections 3 and 4 below are met, then the singular value decomposition of \( F \) returns a single nonzero vector \( \delta \) that contains the descriptors \( \rho_{ij} \) and \( a_{ni} \) up to an arbitrary common scale; scaled reflectance descriptors \( r_{ij} \) are recovered from the \( \rho_{ij} \) by matrix inversion. Note that the algorithm recovers the descriptors up to an arbitrary scalar, as it cannot discriminate reflectances \( R \) lit by illuminants \( A \) from the scaled reflectances \( \kappa R \) lit by the reciprocally scaled illuminants \((1/\kappa)A\).

3. CRITERIA FOR GENERAL LINEAR RECOVERY

There are two natural classes of color constancy problems to consider. One involves two views of a set of surfaces. Such a problem is \((p m n v s) = (3 3 3 2 3)\), in which a trichromatic visual system attempts to recover three descriptors for each illuminant and surface when provided data from two views of the three surfaces.2–5 The cases that involve two views arise in situations in which the illumination of a set of surfaces changes. This change can occur in time, so that two different illuminants shine in succession on a set of surfaces. This type of change was
used by Land\(^8\) in his demonstrations of human color constancy. The change in illumination can also occur across space, as in the common situation in which outdoor surfaces, in partial shadow, are lit simultaneously by bluish skylight and by the yellowish light from the solar disk. The other class of natural color constancy problems involves a single view of a set of surfaces.\(^3\)

The general linear recovery procedure applies to almost every feasible problem in these two natural classes. Indeed, general linear recovery applies to all problems handled by the earlier two-stage linear recovery algorithms.\(^1\)\(^-\)\(^6\) The feasibility criterion for general linear recovery (inequality (15)), which was developed under the restriction that the number \(s\) of surfaces equal (or exceed) the dimension \(m\) of the reflectance model, subsumes the problems feasible for two-stage linear recovery, as these require the further restriction that the number of photoreceptors \(p\) equal or exceed the dimension \(m\) of the model for illumination and/or the dimension \(n\) of the model for reflectance.

However, our primary focus in this paper will be on problems in which two-stage recovery is impossible, namely, on those in which the number \(p\) of photoreceptors is less than the dimensions \(m\) and \(n\) for illumination and reflectance. In what follows we first examine necessary conditions (criteria) for unique recovery to be possible. In Subsection 3.A we discuss briefly the restriction \(s \geq n\). In Subsection 3.B we derive a criterion that rules out problems in which there exist illuminants that are invisible to bilinear models with the problem's parameters. We extend this criterion in Subsection 3.C to a more general test of whether there exist sets of surfaces that are invisible. This test of a particular bilinear model generalizes the necessary condition, met in our analysis of two-stage linear recovery,\(^3\) that bilinear model matrices be of full rank. In Subsection 3.D we derive a further inequality among problem parameters that extends the criterion \(pv > m\) for two-stage recovery to problems where \(p < m, n\). Finally, we consider the transposition of general linear recovery and resulting criteria in Subsection 3.E.

A. Restriction \(s \geq n\)

The first necessary condition is that the number \(s\) of surfaces equal or exceed the dimension \(n\) of the model for reflectance. To see this, note that for linear recovery to be possible the reflectance descriptors \(r_{ij}\) in Eq. (5) must form an \(s \times n\) matrix \(R\) of full rank: a unique left inverse for \(R\) with entries \(\rho_{p}\) is needed for unique recovery. No linear scheme can recover reflectance descriptors if \(s < n\), because the absence of unique inverses for (1) the \(s \times n\) matrix \(R\) of reflectance descriptors and (2) the \(p \times m\) model matrices \(B_j, j = 1, \ldots, n\), disallows manipulation that would provide a system of linear equations. The transposed criterion for illuminant descriptors is discussed in Subsection 3.E below.

B. Criterion \(p + v > m\)

A further necessary condition for unique recovery is expressed by the inequality

\[ p + v > m. \]  

(16)

In words, the sum of the number \(p\) of photoreceptor types and the number \(v\) of views must exceed the dimension \(m\) of the model for illumination. If, for some particular problem, this inequality does not hold true, then one can find invisible illuminants that cause recovery procedures with the problem's parameters to fail.

Note that in problems where \(p \geq m\), inequality (16) holds automatically. We argue here that the criterion is a necessary condition in problems where \(p < m\). The block-diagonal matrices \(C_j\) of Eqs. (4) and (5) are of matrix dimension \(pv \times mv\), and if \(p < m\), then \(\dim(\ker(C_j)) > v(m - p)\) for \(j = 1, \ldots, n\). If there exists a vector of illuminant descriptors \(a = [a_1^T \ldots a_r^T]^T\) with linearly independent constituents \(a_1, \ldots, a_r\), such that

\[ a \in \bigcap_{j=1}^{n} \ker(C_j), \]  

(17)

then recovery must fail: Eq. (5) shows that the quantum-catch data are identically zero for such illuminants.

The condition on the bilinear model matrix constituents \(B_j\) of the block-diagonal matrices \(C_j\) for such invisible illuminants to exist is that \(B_j a_n = 0\) for \(n = 1, \ldots, v\). This undesirable situation can occur if \(\dim(\ker(B_j)) \geq v\), and we must rule out this possibility. By hypothesis, \(\dim(\ker(B_j)) \geq m - p\), so we insist that \(m - p < v\), which, on rearrangement, produces the criterion inequality (16).

C. Generalized Invertibility Criterion on Model Matrices When \(p < m\)

Although inequality (16) lets us eliminate color-constancy problems for which recovery must fail, we can derive stronger conditions by ruling out the possibility that there are invisible surfaces. If in Eq. (5) there is some illuminant vector \(a^0\) for which the vectors \(C_j a^0, \ldots, C_n a^0\) are not linearly independent, then there are surfaces that give rise to zero quantum catches under \(a^0\). Such invisible surfaces can be superimposed upon any solution \(r_{ij}\) of Eq. (5). That is, there exist reflectance descriptors \(r_{ij}^0, j = 1, \ldots, n, n\), not all zero, such that

\[ \sum_{j=1}^{n} r_{ij}^0 C_j a^0 = 0. \]  

(18)

In matrix form this equation reads as

\[ \sum_{j=1}^{n} r_{ij}^0 B_j A^0 = 0, \]  

(19)

where the \(v\) columns of \(A^0\) hold the vectors of illuminant descriptors \(a_{\wp}^0, w = 1, \ldots, v\), which are linearly independent, by hypothesis. We thus infer from Eq. (19) that the \(p\) rows of the \(p \times m\) matrix \(\sum_{j=1}^{n} r_{ij}^0 B_j\) lie in the subspace of dimension \(m - v\) that is orthogonal to the \(v\) columns of \(A^0\). In analogy to the derivation of the model-check algorithm for two-stage linear recovery,\(^3\) we see that all sub-determinants of \(\sum_{j=1}^{n} r_{ij}^0 B_j\) of size \(m - v + 1\) must vanish. This provides \(e\) homogeneous polynomial equations of degree \(m - v + 1\) in the \(n\) unknowns \(r_{ij}^0, j = 1, \ldots, n\), where \(e\) is the product of two binomial coefficients:

\[ e = \binom{p}{m - v + 1} \binom{m}{m - v + 1}. \]  

(20)

To rule out the possibility of invisible surfaces, it is nec-
ecessary and sufficient that these equations have only the trivial, zero solution:

\[ r_j^0 = 0, \quad j = 1, \ldots, n. \]  

(21)

Note that the number of equations must be positive, and examining the first binomial coefficient shows us that \( p \geq m - v + 1 \), which is equivalent to inequality (16). When \( m > v \), we can linearize the equations by expressing them in terms of the distinct monomials of degree \( m - v + 1 \), and the number \( v \) of these unknowns is

\[ u = \binom{n + m - v}{m - v + 1}. \]  

(22)

Thus, to rule out invisible surfaces, it is sufficient that this linearized system of homogeneous equations have no nontrivial solution, in which case the only invisible surface is, in fact, the null, zero surface. This criterion generalizes the requirement, posed earlier for two-stage linear recovery, \(^3\)\(^4\) that the bilinear model matrices be of full rank.

D. Criterion \( p v > n + 1 \) for Problems Where \( p < m \)

We showed \(^3\) that the inequality \( p v > n \) is a necessary condition for two-stage linear recovery. This inequality states that the product of the number \( p \) of photoreceptors and the number \( v \) of views must exceed the dimension \( n \) of the model for reflectance. It generalizes to multiple views the inequality \( p v > n \) for single views proposed by Maloney and Wandell. \(^1\) Yet the inequality \( p v > n \) depends on the assumption that \( p \geq m \), and it happens that we can strengthen the inequality in problems where \( p < m \).

Substituting \( p < m \) into inequality (15), we find that

\[ p n v > p v + n^2 - 1, \]  

(23)

and, subtracting \( p v \) from both sides, we find that

\[ p v (n - 1) > (n - 1)(n + 1). \]  

(24)

In all cases of interest, the quantity \( n - 1 \) is a positive integer, and inequality (24) simplifies to

\[ p v > n + 1. \]  

(25)

This is a necessary condition on color constancy problem parameters when the number \( p \) of photoreceptors is less than the dimension \( m \) of the model for illumination.

E. Transposition

As is the case for two-stage linear recovery, \(^3\) the roles of surfaces and light sources may be interchanged in the general recovery procedure. This interchange leads to a restatement of the inequalities on the problem parameters presented above. In particular, one needs a sufficient number of independent views: \( v \geq m \). Furthermore, the criterion \( p + v > m \) becomes \( p + s > n \) under transposition, and the criterion \( p v > n + 1 \) when \( p < m \) becomes \( p s > m + 1 \) when \( p < n \).

4. MODEL CHECK ALGORITHM

A recovery algorithm with parameters that meet the feasibility criteria presented in Section 3 need not work. As we argued in the companion papers, \(^3\)\(^4\) such criteria involve restrictions mainly on the parameters \( (p \ m \ n \ v \ s) \) of a color-constancy problem. Other than ensuring invertibility, the criteria ignore the structure of the bilinear model matrices [Eq. (1)]. Furthermore, the criteria do not take into account the structure of recovery matrices \( F \) [Eq. (12)], which are crucial to the recovery algorithm.

For recovery to be possible for all valid sets of quantum-catch data, it is necessary and sufficient for a bilinear model to provide a one-to-one relationship between sets of lit surfaces and quantum-catch data, up to an arbitrary positive scalar. \(^3\) In cases in which the number \( p \) of photoreceptor types equals or exceeds the dimension \( m \) of the model for illumination, formalizing the requirement for a one-to-one relationship leads to a model check algorithm \(^3\) for two-stage linear recovery. The model check algorithm is used to test whether a particular bilinear model provides a recovery algorithm that works perfectly with appropriate data. \(^4\)

The need for a one-to-one relationship between quantum-catch data and sets of lit surfaces also leads to model check algorithms for general linear recovery. The algorithms work in situations in which the number \( p \) of photoreceptors is less than both the dimension \( m \) of the model for illumination and the dimension \( n \) of the model for reflectance. In Subsection 4.A we present a linear model check algorithm that expresses necessary and sufficient conditions for perfect recovery in cases in which the number \( v \) of views equals (or exceeds) the dimension \( m \) for illumination. In Subsection 4.B we discuss the nonlinear problem met in situations in which the number of views is fewer (\( v < m \)).

A. Necessary and Sufficient Check for Problems Where \( p < m \) and \( v = m \)

We use the methods introduced in the companion papers \(^3\)\(^4\) to formulate necessary and sufficient conditions for unique recovery by bilinear models in cases in which the number \( v \) of views matches the dimension \( m \) of the model for illumination. The result is a model check algorithm that works by (1) using a bilinear model to create a model check matrix and (2) checking the dimension of the kernel of the model check matrix. If the dimension is one, then the bilinear model provides unique recovery.

Let us first rewrite Eq. (3) as

\[ D_t = \sum_{j=1}^{n} r_{ij} B_j A, \quad t = 1, \ldots, n, \]  

(26)

by taking the illuminant descriptors and quantum-catch data from each view to fill a column of the matrices \( A \) and \( D \), respectively. The matrix \( A \) of illuminant descriptors has dimension \( m \times v \), the bilinear model matrices are \( p \times m \), and the data matrices are \( p \times v \). Now suppose that a second set of surfaces lit by a second set of lights gives identical data. By Eq. (26),

\[ D_t = \sum_{i=1}^{n} s_{it} B_i Z = \sum_{j=1}^{n} r_{ij} B_j A, \quad t = 1, \ldots, s = n. \]  

(27)

If recovery is to be unique, then the only way for the two sets of surfaces under their respective light sources to give rise to identical data is for the reflectance descriptors \( r_{ij} \)
and $s_{ij}$ to be identical, up to some scale factor, and for the illuminant descriptor matrices $A$ and $Z$ to be identical, up to the reciprocal scale factor.

By hypothesis ($s = n$), the matrix $S$ of reflectance descriptors $s_{ij}$ is invertible, so leading from Eq. (27) to

$$B_i Z = \sum_{j=1}^{n} e_{ij} B_j A, \quad i = 1, \ldots, n,$$  \hspace{1cm} (28)

where

$$e_{ij} = \sum_{l=1}^{n} \sigma_{il} r_{lj},$$  \hspace{1cm} (29)

in terms of the elements $\sigma_{il}$ of the matrix $S^{-1}$ [Eq. (36) of Ref. 3]. Multiplying Eq. (28) on the right-hand side by $A^{-1}$, we find that

$$B_i H = \sum_{j=1}^{n} e_{ij} B_j, \quad i = 1, \ldots, n,$$  \hspace{1cm} (30)

where

$$H = ZA^{-1}.$$  \hspace{1cm} (31)

Subtracting from each of the $n$ Eqs. (30) its left-hand-side leads to the homogeneous system of equations

$$\sum_{j=1}^{n} e_{ij} B_j - B_i H = 0,$$

$$\vdots$$

$$\sum_{j=1}^{n} e_{nj} B_j - B_n H = 0.$$  \hspace{1cm} (32)

This system provides $n$ blocks of $pm$ linear equations in the $n^2 + m^2$ variables $e_{ij}$ and $h_{ij}$, where the latter $m^2$ variables $h_{ij}$ are the elements of the matrix $H$ of Eq. (31).

The system can be written in the compact form

$$L \omega = 0,$$  \hspace{1cm} (33)

where the matrix $L$ has dimensions $npm \times (n^2 + m^2)$ and the vector $\omega$ has dimension $n^2 + m^2$. The vector $\omega$ is defined to be

$$\omega = [e_{11} \ldots e_{1n} \ldots e_{n1} \ldots e_{nn}$$

$$h_{11} \ldots h_{1m} \ldots h_{m1} \ldots h_{mm}]^T.$$  \hspace{1cm} (34)

Continue by defining the matrix $X$, of dimension $pm \times n$, and the matrices $Y_i$, $i = 1, \ldots, n$, each of dimension $pm \times m^2$:

$$X = \begin{bmatrix} (B_1)_{11} & \cdots & (B_n)_{11} \\ \vdots & \ddots & \vdots \\ (B_1)_{1m} & \cdots & (B_n)_{1m} \\ \vdots & \vdots & \ddots \\ (B_1)_{p1} & \cdots & (B_n)_{p1} \\ \vdots & \vdots & \vdots \\ (B_1)_{pm} & \cdots & (B_n)_{pm} \end{bmatrix},$$  \hspace{1cm} (35)

$$Y_i = \begin{bmatrix} - (B_i)_{11} & \cdots & - (B_i)_{1m} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & - (B_i)_{1m} \\ \vdots & \vdots & \ddots \\ 0 & \cdots & - (B_i)_{pm} \\ \vdots & \vdots & \vdots \\ 0 & \cdots & - (B_i)_{pm} \end{bmatrix},$$  \hspace{1cm} (36)

Use the matrices $X$ and $Y_i$, $i = 1, \ldots, n$, to construct the model check matrix $L$ of Eq. (33) as follows:

$$L = \begin{bmatrix} X & 0 & Y_1 \\ \vdots & \ddots & \vdots \\ 0 & X & Y_n \end{bmatrix}.$$  \hspace{1cm} (37)

From Eq. (33) we see that the necessary and sufficient condition for a bilinear model to recover descriptors uniquely, in cases where $p < m$ and $m = v$, is that

$$\dim[\ker(L)] = 1.$$  \hspace{1cm} (38)

In the case in which the kernel of model check matrix $L$ is of dimension one, then the vector $\omega$ of Eq. (34) contains only the scaling solutions:

$$e_{ij} = 0 \quad \text{for } j = 2, \ldots, n,$$

$$h_{ij} = 0 \quad \text{for } i \neq j,$$

$$h_{i1} = 0 \quad \text{for } i = 2, \ldots, m,$$

$$e_{11} = h_{11}.$$  \hspace{1cm} (39)

The linear model check algorithm, then, tests whether the $npm \times (n^2 + m^2)$ matrix $L$ of Eq. (37) has a one-dimensional kernel. The algorithm is based on a singular value decomposition\(^4\) of the model check matrix $L$. Passing this test is both necessary and sufficient for a particular bilinear model to recover descriptors perfectly. In Section 5 below we present results of applying this algorithm to particular bilinear models.
B. Sufficient Checks for Problems

Where \( p < m \) and \( v < m \)

In cases in which the number \( v \) of views is less than the dimension \( m \) of the illumination model, the necessary and sufficient conditions for unique recovery lead to a set of nonlinear equations. This holds true in the situation of present concern, namely, general linear recovery in cases where \( p < m \), and also for situations in which two-stage linear recovery procedures work.\(^3\)\(^4\) However, we find that we can linearize the nonlinear equations to produce model check algorithms that express sufficient (but not necessary) conditions for unique recovery.

We have found two such algorithms for the class of problems in which \( p < m \). The first follows from developments like those in the companion papers.\(^3\)\(^4\) Unfortunately, the homogeneous systems of linear equations that are produced by this model check algorithm are very large. This fact prompted us to apply the analysis of Iverson and D'Zmura\(^5\) to the present problem. In cases in which the dimension \( n \) of the model for reflectance equals or exceeds the dimension \( m \) of the model for illumination, a second model check algorithm results that produces more compact systems of equations. The second algorithm also has limited applicability, as discussed below. We make available both of these algorithms through a technical report.\(^23\)

5. RESULTS

Here we present results of classifying color constancy problems that are met when (1) the number \( p \) of photoreceptors is less than both the dimensions \( m \) and \( n \) of the models for illumination and reflectance and (2) the number \( s \) of surfaces equals or exceeds the dimension \( n \) of the reflectance model. The difficulty in performing model checks in cases in which the number \( v \) of views is less than the dimension \( m \) of the illumination model leads us to limit our scope to those problems where \( v = m \), for which we have a necessary and sufficient model check algorithm (Subsection 4.A). For these problems we exhibit bilinear models with the problem parameters that can be used to recover spectral descriptions uniquely, when possible. We follow Section 2 (Methods) of the companion paper,\(^4\) to which the reader is referred.

We consider, in turn, dichromatic problems (Subsection 5.A), trichromatic problems (Subsections 5.B and 5.C), and, more generally, \( p \)-chromatic problems, for \( p \geq 2 \) (Subsection 5.D). We show the results of checks of particular bilinear models (Subsection 2.B of Ref. 4) with components listed in Table 2, together with the spectra of singular values of exemplary model-check matrices (Subsection 2.C of Ref. 4).

### A. Dichromacy

Figure 1 shows results for general linear recovery of spectral descriptions by dichromatic systems. The format of the diagrams follows that of similar figures in the companion paper (e.g., Fig. 1 of Ref. 4) and is detailed in the caption.

Results for problems with a three-dimensional illumination model are shown in Fig. 1A. The positive result for the problem with parameters \((p \ m \ n \ v \ s) = (2 \ 2 \ 3 \ 2 \ 3)\), indicated by the circle, follows from transposition (entailment (d) of Ref. 4) of the positive result\(^3\) for \((2 \ 3 \ 3 \ 2)\). The problem with parameters \((2 \ 3 \ 3 \ 3)\) fails totally, as indicated by the \(X\). This total failure is suggested by the failure of the model-check algorithm for the 36 models with these parameters that can be formed from the appropriate components of Table 2. The total failure of \((2 \ 3 \ 3 \ 3)\) can be shown analytically: the proof\(^3\) of the failure of models of the form \((2 \ 2 \ 2 \ 2)\) is readily extended to show that all models of form \((2 \ c \ c \ c \ c)\), \(c \geq 2\), also fail totally. We leave this proof to the reader. Figure 1A indicates, finally, that there are

### Table 2. Tested Bilinear Model Components

<table>
<thead>
<tr>
<th>Photoreceptors</th>
<th>Illuminants</th>
<th>Reflectances</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two dimensions</td>
<td>Judd et al.(^25)</td>
<td>Cohen(^26)</td>
</tr>
<tr>
<td>Smith–Pokorny(^24) proctanope</td>
<td>Dixson(^27)</td>
<td>Parkkinen et al.(^28)</td>
</tr>
<tr>
<td>Smith–Pokorny deuteranope</td>
<td>Fourier (1, sin)</td>
<td>Fourier (1, sin)</td>
</tr>
<tr>
<td>Smith–Pokorny tritanope</td>
<td>Fourier (1, cos)</td>
<td></td>
</tr>
<tr>
<td>Three dimensions</td>
<td>Judd et al.(^25)</td>
<td>Cohen(^26)</td>
</tr>
<tr>
<td>Smith and Pokorny(^24)</td>
<td>Dixson(^27)</td>
<td>Parkkinen et al.(^28)</td>
</tr>
<tr>
<td>Hurvich and Jameson(^29)</td>
<td>Fourier</td>
<td>Fourier</td>
</tr>
<tr>
<td>CIE 10° observer(^30)</td>
<td>Indoor (D65, A, P(_2))(^30)</td>
<td></td>
</tr>
<tr>
<td>Stockman et al.(^2)–10° observer(^31)</td>
<td>Fourier</td>
<td></td>
</tr>
<tr>
<td>Sony XC-007 CCD RGB</td>
<td>Fourier</td>
<td>Fourier</td>
</tr>
<tr>
<td>Sony XC-711 CCD RGB</td>
<td>Fourier</td>
<td>Fourier</td>
</tr>
<tr>
<td>Four dimensions</td>
<td>Judd et al.(^25)</td>
<td>Cohen(^26)</td>
</tr>
<tr>
<td>Dixson(^27)</td>
<td>Parkkinen et al.(^28)</td>
<td>Fourier</td>
</tr>
<tr>
<td>Fourier</td>
<td>Fourier</td>
<td></td>
</tr>
<tr>
<td>Five dimensions</td>
<td>Fourier</td>
<td>Parkkinen et al.(^28)</td>
</tr>
<tr>
<td>Six+ dimensions</td>
<td>Fourier</td>
<td>Fourier</td>
</tr>
</tbody>
</table>
bilinear models with parameters (2 3 4 3 4) that function perfectly.

In Fig. 2B are shown results for dichromatic problems with a four-dimensional model of illumination. The positive result for the problem (2 4 3 4 5) follows from transposition of the positive result shown in Fig. 1A for the problem (2 3 4 3 4). The total failure for (2 4 4 4 4) is again suggested by the failure of model checks and is an instance of the failure of (2 c c c c), c ≥ 2. Finally, the model check algorithm shows that there are bilinear models with parameters (2 4 5 4 5) that work perfectly.

In Fig. 2 are shown spectra of ordered singular values of model check matrices for exemplary bilinear models. As described in Section 2 of Ref. 4, one determines such a spectrum by computing a singular value decomposition of a model check matrix and ordering the singular values from greatest to least. We then check whether there is a plunge between adjacent values in the spectrum of ordered singular values of five log units or greater. We take such a plunge to indicate the presence of a nontrivial kernel, and the number of small singular values indicates the rank of the kernel.

Figure 2 shows the spectra for exemplary models with parameters (2 3 3 3 3), (2 3 4 3 4), (2 4 4 4 4), and (2 4 5 4 5), from bottom to top. The kernels of the matrices are dimensions three, one, four, and one, respectively. A model passes the necessary and sufficient test posed by the model check algorithm (Subsection 4.A) if its matrix has a kernel of dimension one, so that these spectra indicate failure for (2 3 3 3) and (2 3 4 4) and success for (2 3 4 3 4) and (2 4 5 4 5). Although all problems of form (2 c c c c) must fail, the numerical results suggest that there may be perfect recovery procedures for all problems of the form (2 c + 1 c c + 1 c) or, by transposition, of the form (2 c + 1 c c + 1 c). We investigate this possibility more fully in Subsection 5.D below.
The positive results for the problems \((3 4 2 4 2)\) and model check algorithm of Subsection 4.B is applicable. The positive results for the problems \((3 4 2 4 2)\) and \((3 4 3 4 3)\) follow from transposition of the positive results\(^4\) for \((3 2 4 2 4)\) and \((3 4 3 4 3)\), respectively. The model check algorithm shows that there are bilinear models with parameters \((3 4 4 4 4), (3 4 5 4 5),\) and \((3 4 6 4 6)\) that work perfectly. Exemplary spectra for these three problems are shown in Fig. 4. The bottom curves show that the kernels of model check matrices of exemplary bilinear models with parameters \((3 4 4 4 4), (3 4 5 4 5),\) and \((3 4 6 4 6)\) have dimension one, so that the models pass the check.

In Figure 3B are shown results for trichromatic problems with a five-dimensional model of illumination. The positive results for the problems with parameters \((p m n v s) = (3 5 2 5 3)\) and \((3 5 4 5 3)\) follow from transposition of the positive results\(^5\) for \((3 2 5 2 5)\) and \((3 5 5 3 5)\), found in Ref. 4, and for \((3 4 5 4 5)\), shown in Fig. 3A. Continuing toward the right of Fig. 3B, the model check algorithm shows that there are bilinear models with parameters \((3 5 5 5 5)\) and \((3 5 6 5 6)\) that work perfectly. The middle two curves in Fig. 4 present exemplary spectra for these two problems.

Results for trichromatic problems with a six-
model checks for (3 6 6 6 6) are successful; the top curve in Fig. 3A, and for (3 5 6 5 6), shown in Fig. 3B. The for (3 3 6 3 6), found in Ref. 4, for (3 4 6 4 6), shown (3 6 5 6 5) follow from transposition of positive results. The positive results for (3 6 3 6 3), (3 6 4 6 4), and (3 6 5 6 5), (3 5 6 5 6), and (3 6 6 6 6). The spectra show that the corresponding model check matrices had one-dimensional kernels, so passing the check. See the text for further discussion.

The CIE daylight fundamentals and the Fourier models for reflectance and illumination; the latter are of dimension c and have their highest frequency components at frequency (c − 1)/2. The upper limit 31 stems from our choice to approximate functions of wavelength using the 31-dimensional vectors that arise in sampling the interval 400–700 nm at 10-nm intervals. The number of views is identical to the dimension m for illumination, viz., v = m = c, so that the linear model check algorithm can be applied. Figure 5 shows that each of the exemplary models gives rise to a model check matrix with a kernel of dimension one: each model passes the check.

The numerical results lead to the following conjecture: a trichromatic visual system can recover arbitrarily high numbers of descriptors per illuminant and reflectance when provided adequate information. We now prove this claim for problems with parameters of the form (3 c c c c), to which we apply the analysis based on Schur's lemma22 that was introduced by Iverson and D'Zmura.5

We define the three c × c bilinear model matrices \( \beta_k \), \( k = 1, 2, 3 \), with entries \( (\beta_k)_{ij} = (B_j)_{ii} \) [see Eq. (1)], so providing one bilinear model matrix per photoreceptoral spectral sensitivity.5 We use these matrices to rewrite Eq. (26) as follows [Eq. (4) of Ref. 5]:

\[
\Delta_k = R \beta_k A, \quad k = 1, 2, 3,
\]

in which the data matrices \( \Delta_k \) have entries \( q_{ijk} \) [see Eq. (2)] and dimension \( n \times v \) and the matrix of reflectances \( R \) has entries \( r_{ij} \) and dimension \( n \times n \).

Suppose now that surfaces with reflectance descriptor matrix \( R \) lit by sources with descriptor matrix \( A \) give rise to quantum catches that are identical to those produced.

Figure 5 provides some numerical evidence. The figure shows the spectra of singular values for the model check matrices of exemplary trichromatic bilinear models with parameters \((3 c c c c)\) for \(3 \leq c \leq 31\), c odd. Plotted on a log axis are the ordered singular values of the model check matrices for these models. We scaled the spectra to stagger the maximal singular values along the vertical axis at half-log-unit intervals. The parameters of the models whose spectra are shown are, from bottom to top, \((3 3 3 3 3), (3 5 5 5 5), (3 7 7 7 7), (3 9 9 9 9), (3 11 11 11 11), (3 13 13 13 13), (3 15 15 15 15), (3 17 17 17 17), (3 19 19 19 19), (3 21 21 21 21), (3 23 23 23 23), (3 25 25 25 25), (3 27 27 27 27), (3 29 29 29 29), and (3 31 31 31 31). Each has a kernel of dimension one and so passes the necessary and sufficient model check. See the text for further discussion.
by surfaces with reflectance descriptor matrix \( S \) lit by sources with descriptor matrix \( Z \):

\[
R \beta_k A = S \beta_k Z, \quad k = 1, 2, 3. \tag{41}
\]

We want to identify bilinear models for which the only possible solution to Eq. (41) is that where \( S = \kappa R \) and \( A = (1/\kappa)Z \) for some constant \( \kappa \). Note that the \( c \times c \) matrices \( S \) and \( A \) are invertible, by hypothesis, so leading to

\[
E \beta_k = \beta_k H, \quad k = 1, 2, 3, \tag{42}
\]

where \( E = S^{-1}R \) and \( H = ZA^{-1} \). For problems of the form \((3 \times c \times c)\) the \( c \times c \) matrices \( \beta_k, k = 1, 2, 3, \) are invertible, producing

\[
\beta_k^{\dagger} E \beta_k = H, \quad k = 1, 2, 3. \tag{43}
\]

It follows at once that

\[
\beta_k^{\dagger} E \beta_k = \beta_l^{\dagger} E \beta_l, \quad k, l = 1, 2, 3. \tag{44}
\]

Let us define the associated model matrices \( G_{k1} = \beta_k \beta_l^{-1} \), in terms of which Eq. (44) becomes

\[
E G_{k1} = G_{k1} E, \quad k, l = 1, 2, 3, \tag{45}
\]

and we see that \( E \) commutes with each of the associated model matrices. Following Iverson and D’Zmura,\(^5\) it suffices to consider two of these matrices, say, \( G_{21} \) and \( G_{31} \). By Eq. (45),

\[
[E, G_{31}] = 0 = [E, G_{31}], \tag{46}
\]

where the symbol \([A, B]\) denotes the commutator \(AB - BA\).

By Schur’s lemma,\(^6\) if \( G_{21} \) and \( G_{31} \) share no common nontrivial invariant subspace, then the matrix \( E \) is a multiple of the identity matrix; i.e., the two sets of surface reflectances are related by a single positive scalar, and the two sets of light sources are related by the reciprocal of that scalar. It thus suffices to show that one can always find matrices \( G_{21} \) and \( G_{31} \), for arbitrarily high \( c \), that have no common nontrivial invariant subspace.

Let us choose \( G_{21} \) to possess distinct eigenvalues. It follows that \( G_{21} \) can be diagonalized, and without loss of generality we take

\[
G_{21} = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_c \end{bmatrix}. \tag{47}
\]

It now follows from Eq. (46) that matrix \( E \) is diagonal, because the components \([E, G_{21}]_{ij}\) of the commutator are \( e_{ij}(\mu_j - \mu_i) \); and, because the eigenvalues of \( G_{31} \) are distinct, the off-diagonal components of \( E \), namely \( e_{ij} \) for \( i \neq j \), must vanish.

Consider now the components \([E, G_{31}]_{ij}\) of the commutator \([E, G_{31}]\); these are \((e_{ii} - e_{jj})g_{ij}\), where \( g_{ij} \) are the components of \( G_{31} \). Since these expressions vanish for all \( i, j \) by Eq. (46), the simplest way to ensure that \( e_{ii} = e_{jj} \) for all \( i, j = 1, 2, \ldots, c \), is to construct \( G_{31} \) so that \( g_{ij} \neq 0 \) for some \( i \) and all \( j \). This can always be done.

For color-constancy problems with parameters \((3 \times c \times c)\), for arbitrarily large \( c \), we can thus always construct bilinear models with modified gamma matrices that share no common nontrivial invariant subspace. By Schur’s lemma, the matrix \( E \) is a scalar multiple of the identity, and so the bilinear model provides a one-to-one relationship between lit surfaces and quantum-catch data, up to an arbitrary positive scalar.

Note that the ability to specify associated model matrices \( G_{21} \) and \( G_{31} \) that force the scaling solution is tantamount to specifying bilinear model matrices \( \beta_k, k = 1, 2, 3 \), that provide perfect recovery. Perhaps the simplest of many ways to construct model matrices \( \beta_k, k = 1, 2, 3 \), given suitable \( G_{21} \) and \( G_{31} \), is to pick an arbitrary invertible \( \beta_1 \) and set \( \beta_2 = G_{21} \beta_1 \) and \( \beta_3 = G_{31} \beta_1 \).

### D. Capacity of a Polychromatic Bilinear System

The result of Subsection 5.C shows that there are trichromatic systems that can recover \( c \) descriptors, \( c \geq 2 \). It must be the case that there are systems with greater numbers of photoreceptor types that can also recover \( c \) descriptors, \( c \geq 2 \). We can easily reduce such a system to a trichromatic system, for which we know recovery is possible, simply by ignoring the responses of the excess photoreceptor types. This argument is equivalent to one that applies entailment (a) of Ref. 4: \((3 \times c \times c) \Rightarrow (p \times c \times c), p > 3\). We conclude that there are \( p \)-chromatic visual systems that can recover \( c \) descriptors for both illuminants and reflectances, for \( p \geq 3 \) and \( c \geq 2 \).

In contrast, all bilinear models with parameters of the form \((2 \times c \times c)\) produce recovery algorithms that fail totally, which follows by extending the proof\(^4\) of the total failure of the special case \((2 \times 2 \times 2)\).

We investigated further chains of problems of this sort. We performed successful model checks of all problems of the form \((2 \times c \times c + 1)\), for \(3 \leq c \leq 29\). The success of these model checks implies success, through entailment (a) of Ref. 4, for problems of form \((p \times c \times c + 1)\) for \(p \geq 2 \) and \(3 \leq c \leq 29\) (and their transposes). Although we have not proved the existence of perfect recovery algorithms for arbitrarily high \( c \) in these chains, we are confident that this is the case.

The successful results of the further model checks, performed on models with illuminant and surface parameters limited to a maximum value of \( 31 \), suggest that there are \( p \)-chromatic visual systems that can recover \( c \) descriptors, \( c \geq 2 \). It must be the case that there are systems with greater numbers of photoreceptor types that can also recover \( c \) descriptors, \( c \geq 2 \). We can easily reduce such a system to a trichromatic system, for which we know recovery is possible, simply by ignoring the responses of the excess photoreceptor types. This argument is equivalent to one that applies entailment (a) of Ref. 4: \((3 \times c \times c) \Rightarrow (p \times c \times c), p > 3\). We conclude that there are \( p \)-chromatic visual systems that can recover \( c \) descriptors for both illuminants and reflectances, for \( p \geq 3 \) and \( c \geq 2 \).

### 6. DISCUSSION

Our goal has been to describe and analyze a general algorithm that recovers spectral descriptions for lights and surfaces simultaneously, using linear methods. This simultaneous recovery is possible whenever the number of surfaces equals or exceeds the number of reflectance descriptors to be recovered per surface. The general linear recovery procedure can be used for all color constancy problems amenable to two-stage linear recovery.\(^3\)\(^4\) General linear recovery has a broader scope because the number of photoreceptor types need not equal or exceed the
dimension of the model for illumination. The problems that are not amenable to general linear recovery form a relatively small class that are inherently nonlinear.  

Like the linear model check algorithm for two-stage linear recovery,  
that for general linear recovery  
well to classify bilinear models. Yet the nonlinear model check algorithm for general linear recovery  
met in cases in which the number of views is less than the dimension  
for illumination,  
and linearized systems of equations that are usually too large for us to treat numerically.  

Analytic methods suffice to show that there are infinite chains  
perfectly. This result supports the intuition that poly-
chromatic systems can use reflected lights to determine  
spectral descriptions of arbitrarily high dimension when  
provided adequate information. Of course, this does not  
imply that an arbitrary polychromatic visual system has  
this property. Yet the successful results of our numerical  
tests through  
, with the empirically motivated trichromatic systems of Table 2, suggest that standard  
trichromatic systems have a very high capacity for recovering  
spectral descriptions.  

Numerical results suggest that there are a number of chains, including  
for which recovery  
works perfectly for a variety of bilinear models. Further chains are feasible if the base number of photoreceptor types is increased [for example, if  
the chain  
is feasible], and we conjecture that all such chains provide perfect recovery schemes that can work to arbitrarily high accuracy.

ACKNOWLEDGMENTS  

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