Decision-Making on the Full Information Secretary Problem

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Abstract

The secretary problem is a recreational mathematics problem, suited to laboratory experimentation, that nevertheless is representative of a class of real world sequential decision-making tasks. In the 'full information' version, an observer is presented with a sequence of values from a known distribution, and is required to choose the maximum value. The difficulties are that a value can only be chosen at the time it is presented, that the last value in the sequence is a forced choice if none is chosen earlier, and that any value that is not the maximum is scored as completely wrong. We report a study of human performance on full information secretary problems with 10, 20 and 50 values in the sequence, and considers three different heuristics as models of human decision-making. It is found that some people achieve near-optimal levels of accuracy, but that there are individual differences in human performance. A quantitative evaluation of the three heuristics, using the Minimum Description Length criterion, shows inter-individual differences, but intra-individual consistency, in the use of the heuristics. In particular, people seem to use the heuristics that involve choosing a value when it exceeds an internal threshold, but differ in how they set thresholds. On the basis of these findings, a more general threshold-based family of heuristic models is developed.

Introduction

Many real world decision-making problems are sequential in nature. A series of choices is made available over time, and it is often efficient (and sometimes even necessary) to make a selection without waiting to be presented with all of the alternatives. On long cross-country drives, for example, people refill their cars at one of a sequence of towns on the route, without knowing the price of fuel at subsequent towns. This type of sequential decision has a continuous utility function. People aim to choose the cheapest price, and measure their success by how much their purchase exceeded this minimum.

Other sequential decision-making tasks have binary utility functions, where any incorrect decision is equally (and completely) incorrect. For example, consider being a witness for a police line-up, where, because of the circumstances of the case, the offender is known to be in the line-up. Police line-up policy demands that suspects are presented one at a time, may only be viewed once, and that a suspect must be identified at the time they are presented (e.g., Steblay, Deisert, Fulero, & Lindsay 2001). Suppose also (unrealistically, we hope) that the police insist that a suspect be identified, and indicate that they will force the identification of the last person in the line-up if none of the previous people are chosen. The aim is to choose the offender, and any misidentification has the equally bad outcome of selecting an innocent suspect.

This decision-making scenario has the same essential features as a recreational mathematics problem known as the 'secretary problem' (see Ferguson 1989 for a historical overview). In secretary problems, an observer is presented with a sequence of possible choices, and must decide whether to accept or reject each possibility in turn. The number of choices in the complete sequence is fixed and known, and only the rank of each possibility, relative to those already seen, is presented to the observer. If the observer chooses the best possibility in the sequence, their decision is correct, and any other choice is regarded as incorrect.

Variants of the secretary problem have been considered that change or relax different parts of the problem. In particular, the full information version of the secretary problem, sometimes known as the 'Cayley' problem, presents observers with a score from a known distribution for each possibility, and the goal is to choose the maximum score in the sequence. Rank information corresponds to the assumption that witnesses keep a relative ordering of people in line-ups, whereas value information corresponds to the assumption that witnesses evaluate some continuous measure of the probability that a person is the offender. In either case, the secretary problem has the important feature of using the same binary utility function as the line-up decision. The goal is to choose the actual offender, and any incorrect decision is equally wrong.

Problem Solving and Secretary Problems

Human performance on secretary problems is an interesting topic for cognitive science, for a number of reasons. It offers a well defined task, suited to laboratory experimentation, that nevertheless is ecologically representative of a class of real world situations. Because of their inherent complexity, secretary problems also provide an opportunity to study the relationship between rational analysis and heuristic strategies in human problem solving.

Most laboratory research on human problem solving has relied on artificial problems that are characterized by well-defined initial and terminating states that must be linked by a systematic, finite series of steps. Typically, these problems, like the 'Towers of Hanoi' or 'Cannibals and Missionaries', are deterministic, and have state spaces with combinatorially limited possibilities. A major focus of studying people's abilities to solve these tasks involves examining under what circumstances, if any, people make rational decisions. Violations of rationality are easy to measure, because the tasks permit a complete formal analysis. This approach to studying human problem solving assesses what Simon (1976) terms 'substantive' rationality: the ability of people to produce optimal final decisions. Typically, they do not address what Simon (1976) terms 'procedural' rationality—the efficiency of the processes required to make the decision—because of the limited combinatorial complexity of the problem.

More recently, however, some research has studied human performance on difficult combinatorial optimization problems, such as visually presented Traveling Salesperson Problems (TSPs), that have very large state spaces, and resist complete formal solution (e.g., MacGregor & Ormerod 1996; Vickers, Butavicius, Lee, & Medvedev 2001). In attempting to solve these problems, subjects are constrained both by the nature of the task (e.g., limited time), and by their cognitive capabilities (e.g., limited memory). In other words, their performance is constrained not only by the need to achieve a substantively rational outcome, but also by the need to use procedurally rational heuristic processes that are sufficiently fast and accurate, and are implementable with available cognitive resources. Procedural rationality offers an important additional constraint for understanding human problems solving processes, and for the development and evaluation of cognitive models of decision-making.

Secretary problems provide an opportunity to continue and extend this line of study. Because they are not inherently perceptual, secretary problems allow consideration of whether results obtained with problems like TSPs generalize to cognitively-based problem solving. Secretary problems also introduce uncertainty, and place demands on memory. While visual problems like TSPs are combinatorially large, the basic information about distances between points is always perceptually available in a complete and certain form to subjects. In contrast, the sequences of information in secretary problems are stochastic and presented only temporarily, requiring people to deal with uncertainty and rely on their memory.

Previous Research

Gilbert and Mosteller (1966) provide a thorough and useful overview of early mathematical analysis of several versions of the secretary problem. When only rank information is provided, the optimal decision rule takes the form of observing some fixed proportion of values in the sequence, remembering the maximum presented, and then choosing the first subsequent value that is greater, if one exists. Gilbert and Mosteller (1966, Table 2) detail the optimal 'cutoff' proportion for the initial sequence of observations, which depends upon the length of the sequence, but converges to the value $1/e \approx 0.368$. They also give the associated probability of making a correct decision using the optimal decision rule.

For the full information version, where values rather than ranks are presented, the optimal decision rule requires choosing the first value that exceeds a threshold level for its position in the sequence. Gilbert and Mosteller (1966, Tables 7 and 8) detail these optimal thresholds and the associated probabilities of making a correct decision. Since Gilbert and Mosteller's (1966) seminal work, a large literature has developed on mathematical analyses of a large number of variants on the secretary problem, often with a focus on the performance of heuristic decision rules (e.g., Freeman 1983).

Relatively less attention has been given to studying human performance solving secretary problems. Seale and Rapoport (1997) consider the rank information version of the problem with lengths of 40 and 80, and focus on the evaluation of plausible heuristic models of human decision-making. In an individual subject analysis, they found a parameterized version of the optimal cutoff rule provided the best fit. Kahan, Rapoport and Jones (1967) studied human performance on full-information versions of the problem with length 200, where the values were drawn from either a positively skewed, negatively skewed, or a uniform distribution. They found no evidence for the different distributions affecting the decisions made. They also compared individual and group decision-making, and found that decisions were made earlier in the sequence by individuals. Other empirical studies (e.g., Kogut 1990), make a large methodological departure by requiring subjects to sacrifice explicitly held resources to view additional presentations, usually because they are interested in applications of the problem to economic decision-making.

In this paper, we study human performance on the full information version of the secretary problem, where values are chosen from a uniform distribution. We consider problems of length 10, 20 and 50, under the binary utility function, but without any explicit

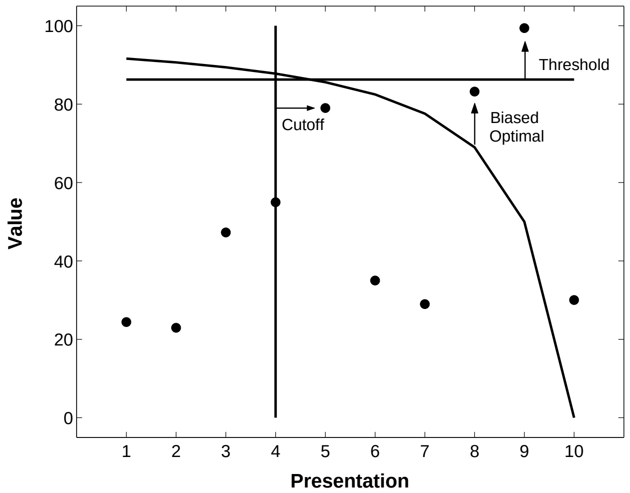

Figure 1: A sample secretary problem of length 10, with the sequence of values shown by filled circles, demonstrating the operation of the biased optimal (curved line), threshold (horizontal line) and cutoff (vertical line) heuristics.

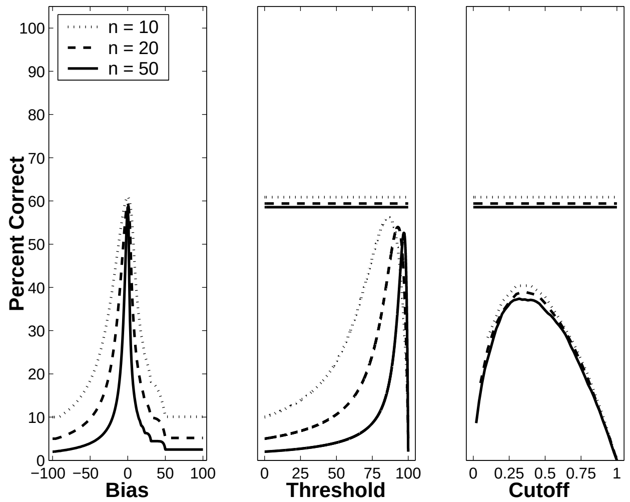

Figure 2: The accuracy of the heuristics, across their parameter spaces, for 10, 20 and 50 sequence length problems.

search cost. Our primary interest, like that of Seale and Rapoport (1997), is to develop and evaluate competing cognitive models of human decision-making.

Three Heuristics

We consider three possible heuristics as models of human decision-making. The first is a biased version of the optimal decision rule. This heuristic chooses the first value that exceeds a threshold level for its position in the sequence. The threshold levels correspond to the optimal values, for the given problem length, all shifted by the same constant. The second heuristic is inspired by Simon's (1956) notion of satisficing. It simply chooses the first value that exceeds a single fixed threshold. The third heuristic is inspired by the optimal decision rule for the rank information version of the secretary problem. It observes a fixed proportion of the values in the sequence, and remembers the maximum value up until this cutoff point. The first value that exceeds the maximum in the remainder of the sequence is chosen. For all three heuristics, if no value meets the decision criterion, the last value becomes the forced choice.

Figure 1 summarizes the functioning of the three heuristics on a problem of length 10. The sequence of values presented is shown by the filled circles. The threshold levels for the optimal heuristic (with no bias) follow the solid curve. The horizontal line shows the constant level used by the threshold heuristic. The vertical line shows the proportion used by the cutoff heuristic. Under these parameterizations, the biased optimal, threshold, and cutoff heuristics choose, respectively, the eighth, ninth, and fifth values presented.

The left panel of Figure 2 shows the accuracy of the biased optimal heuristic for bias values between -100 and 100 for problems of length 10, 20, and 50, calculated using the analytic method of Gilbert and Mosteller (1966, p. 55). At zero bias, the heuristic corresponds to the optimal decision rule, and so the maximum possible accuracy is obtained. The middle panel of Figure 2 shows the accuracy of the threshold heuristic for threshold values between 0 and 100 for problems of length 10, 20 and 50, calculated using the same analytic method. The maximum possible accuracy, corresponding to the use of the optimal decision rule, is shown for each problem length by the horizontal lines. Finally, the right panel of Figure 2 shows the accuracy of the cutoff heuristic for proportions between 0 and 1 for problems of length 10, 20 and 50, generated by simulation on a large sample of independently generated problems. Once again, the maximum possible accuracies are shown by the horizontal lines.

There are two observations worth making about the accuracy of the heuristics shown by Figure 2. It is clear that the threshold heuristic is capable of making better decisions than the cutoff heuristic. This is interesting, given that the cutoff heuristic is optimal for rank information secretary problems. It is also clear that the accuracy of both the biased optimal and threshold heuristics are very sensitive to their parameterizations, particularly for larger problem lengths.

Experiment

Participants Ten participants completed the experiment. There were 4 males and 6 females, with a mean age of 26.1 years.

Method Each participant completed the same three sets of problems. The first set contained 20 problems of length 10. The second contained 20 problems of length 20. The third set contained 20 problems of length 50. Participants always did the three sets in the same order—length 10, then 20, then 50—but the order of the 20 problems within each set was randomized across participants.

For each problem, the participants were told the length of sequence, and were instructed to choose the maximum value It was emphasized that (a) the values were uniformly and randomly distributed between 0.00 and 100.00, (b) a value could only be chosen at the time it was presented, (c) the goal was to select the maximum value, with any selection below the maximum being completely incorrect, and (d) if no choice had been made when the last value was presented, they would be forced to choose this value. As each value was presented, its position in the sequence was shown, together with 'yes' and 'no' response buttons. When a value was chosen, subjects rated their confidence in the decision on a nine point scale ranging from "completely incorrect" to "completely correct".

Results Table 1 summarizes the accuracy of the decisions made by all of the subjects for all of the problems. The average accuracy for the 20 problems in each set is given, together with averages across all problems for each subject, and across all subjects for each problem length. There are three observations worth making about these results. First, some subjects achieve levels of accuracy competitive with the optimal decision rule. Secondly, there appear to be individual differences between the subjects, with a range in average accuracy from 33% to more than 60%. Thirdly, there is some suggestion that human performance worsens as the problem length increases, even after accounting for the slightly decreased accuracy of the optimal decision rule.

Table 1: Accuracy of human decisions, showing the percentage of correct answers for each participant on each set of problems. Average accuracy for each participant, and for each problem length are also shown.

Participant	$n = 10$	$n = 20$	$n = 50$	Mean
1	65	65	55	61.37
2	45	45	20	36.67
3	55	45	50	50.00
4	40	35	25	33.33
5	55	35	55	48.33
6	65	45	20	43.33
7	45	60	50	51.67
8	55	50	45	50.00
9	70	55	55	60.00
10	50	35	55	46.67
Mean	54.50	47.00	43.00	

Model Evaluation One compelling aspect of the model evaluation undertaken by Seale and Rapoport (1997) is that it was done at the level of individual subjects, rather than by averaging decisions across subjects. As noted by Estes (1956), averaging non-linear decision processes in the presence of noise, and with significant individual differences, acts to corrupt the form of the empirical data being modeled. Because these criteria are likely met in the current problem, we also undertook individual subject evaluation of the biased optimal, threshold and cutoff heuristics.

A potential criticism of Seale and Rapoport (1997) is that the quantitative component of their model evaluation relied solely on the ability of a heuristic, at one or more parameterizations, to match the decisions made by a subject. As argued by Roberts and Pashler (2000), measures of goodness-of-fit fail to account for important quantifiable components in model selection. In particular, it is important also to assess the complexity of parameterized models, to ensure that good fit to empirical data does not merely arise because a model is so complicated that it can fit any data, including data that are never observed.

In model theoretic terms, there are clear differences in the complexity of the three heuristics being considered. For the set of 20 length 10 problems given to subjects, there are 10^{20} possible combinations of decisions. The biased optimal, threshold, and cutoff heuristics can predict, respectively, 78, 60, and 9 of these possibilities by varying their parameters. Similar differences in complexity hold for the longer problem lengths, with 88, 70 and 17 data distributions being indexed by the parameters for the length 20 problems, and 121, 90 and 30 for the length 50 problems. Accordingly, any superiority in the ability of the biased optimal heuristic over its competitors, or in the threshold over the cutoff heuristic, could possibly be due to greater complexity, rather than fundamentally capturing regularities in the empirical data.

These concerns are best addressed using advanced model selection methods (e.g., Pitt, Myung, & Zhang 2002), which provide criteria for choosing between models in ways that consider both goodness-of-fit and complexity. One interesting challenge in doing this is for the current models is that they are deterministic, and do not specify an error theory. This means that various probabilistic model selection criteria, such as Bayes Factors (e.g., Kass & Raftery 1995), Minimum Description Length (MDL: e.g., Grünwald 2000)

Table 2: Minimum Description Length (MDL) criteria values for the Biased Optimal (BO), Threshold (Th) and Cutoff (Cu) models, measured against the decision made by the ten participants on each problem length. Bold entries highlight strong evidence in favor of the preferred model.

	$n = 10$			$n = 20$			$n = 50$		
	BO	Th	Cu	BO	Th	Cu	BO	Th	Cu
1	32.7	**26.3**	40.1	34.4	**25.6**	52.6	34.6	**23.2**	60.6
2	**19.4**	32.4	33.2	38.0	40.4	45.0	34.6	32.8	60.6
3	35.4	**26.3**	35.7	38.0	29.7	35.2	29.7	32.8	47.5
4	**15.3**	35.1	42.0	26.3	25.6	47.8	**18.7**	32.8	44.1
5	32.7	32.4	40.1	30.4	29.7	50.3	52.0	48.9	60.6
6	**19.4**	29.5	38.0	**21.8**	29.7	42.1	29.7	28.1	43.8
7	26.6	22.9	40.1	26.3	21.2	41.5	18.7	17.8	35.8
8	32.7	26.3	40.1	41.5	**33.5**	50.3	**12.5**	23.2	47.8
9	26.6	32.4	38.0	26.3	25.6	52.6	24.4	23.2	36.0
10	**29.8**	37.6	40.1	**21.8**	37.0	52.6	34.6	**28.1**	43.8

or Normalized Maximum Likelihood (Rissanen 2001), are not immediately applicable. Grünwald (1999), however, develops a model selection methodology that overcomes these difficulties. He provides a principled technique for associating deterministic models with probability distributions, through a process called 'entropification', that allows MDL criteria for competing models to be calculated.

Table 2 shows the MDL values found by applying Grünwald's (1999) method to all three heuristics, taking each subject individually, and considering each problem length separately. Lower MDL values indicate better models, and differences between values can be interpreted on the log-odds scale. This means, for example, that the threshold heuristic (MDL value 26.3) provides an account that is about 600 times more likely than that provided by the biased optimal heuristic (MDL value 32.7) for the decisions made by the first subject for the length 10 problems, since $e^{32.7-26.3} \approx 600$. Kass and Raftery (1995) suggest, without being prescriptive, that a difference of six or more in the log-odds given by MDL values corresponds to 'strong' evidence in favor of the preferred model. Adopting the same standard, Table 2 highlights in bold those instances where the MDL for one of the heuristics provides strong evidence in its favor compared to both of the others.

There are several conclusions that can be drawn from this analysis. First, despite its simplicity, the cutoff heuristic does not provide a good model of the human decisions. For almost every subject and every problem length, it has the greatest MDL value, and often is so much larger as to provide strong evidence against its suitability. Secondly, there is clear evidence of inter-individual differences in the use of the biased optimal or threshold heuristics. There are approximately as many instances, for each problem length, where the biased optimal or threshold heuristic is strongly favored as an account for an individual subject. Thirdly, there is also some evidence of intra-individual consistency in using the biased optimal or threshold heuristic. This is because, in most instances, strong preferences favor the same heuristic for the same subject on different problem lengths.

Once the MDL criteria have been used to control for effects of model complexity, it is sensible to examine the goodness-of-fit of the heuristics. This was done by considering the average percentage of correct predictors made by each heuristic, for just those participants with MDL values favoring the heuristic. The biased optimal heuristic correctly predicted an average of 81%, 78% and 88% of participant decisions for, respectively, the 10, 20 and 50 length problems. The threshold heuristic correctly predicted an average of 74%, 78% and 79% of decisions. These results suggest that, while the heuristics may not provide a complete account of human performance, they do capture important regularities in the decision-making data.

Where there is strong evidence for a participant using either the biased optimal or threshold heuristic, it is also worthwhile to examine the parameter values used. For participants using the biased optimal heuristic, the bias parameter was always negative, indicating they underestimated the optimal threshold value for each position in the sequence. As the problem length increased from 10 to 20 then 50, however, the average bias changed from -5.1 to -1.8 then -1.9. This suggests that, for the longer sequences, participants were better calibrated to the optimal curve. For participants using the threshold heuristic, the average best-fitting threshold increased from 88.1 to 93.2 then 94.6. These values compare to theoretically optimal thresholds of 86.4, 92.6 and 97.2, as shown in the middle panel of Figure 2. It is clear that participants are sensitive to the need to increase the threshold as the length of the sequence increases, and do not seem either to under- or over-estimate the optimal value. It should be acknowledged, however, that these parameter values analyses are based on limited data, and additional data are required to confirm the suggested trends in this experiment, as well as to ascertain whether there are significant individual differences that also need to be considered.

Discussion

This study constitutes a first attempt to understand human decision-making on the full information version of the secretary problem. A first contribution of the study is to reject the usefulness of the cutoff heuristic, on both theoretical and empirical grounds, as an account of human decision-making. This is a worthwhile finding, given that Seale and Rapoport (1997) found good evidence for people using this strategy on the rank information version of the secretary problem.

More importantly, it seems clear that both the biased optimal and threshold heuristics do capture something fundamental about human decision-making on the full information version. Both heuristics take the form of choosing the first value that exceeds a threshold, with the key difference being that the biased optimal heuristic uses thresholds that are sensitive to the position in the sequence, rather than being fixed.

Indeed, the biased optimal and threshold heuristics represent the two extremes of a continuum of threshold-based decision-making heuristics. Instead of using a constantly changing or a fixed threshold, it is possible for a decision process to use a small number of thresholds, and apply each to a sub-sequence of the presented values. For example, for a problem of length 10, a heuristic might apply one threshold for the first five values, decrease it for the next three values, and finally decrease it again for the penultimate value[1]. These sorts of heuristics seem likely to have complexity that lies somewhere between that of the biased optimal and threshold heuristics. It may well be the case that human performance is best explained by an account that is more sophisticated than the threshold heuristic, but does not have the full complexity of the biased optimal approach.

A final interesting problem for future research is whether the observed individual differences in accuracy are related to more traditional measures of problem solving ability and psychometric intelligence. In the everyday world, the ability to solve practical problems is generally regarded as an expression of intelligence. There is some evidence (e.g., Vickers *et al.* 2001) of a relationship between solution quality on TSPs and measures of IQ. Given that secretary problems are representative of a class of real world sequential decision-making tasks, they allow the possibility that there is a similar relationship for non-perceptual tasks to be examined.

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[1]Because the last value is always a forced choice, the threshold is always effectively zero for any heuristic.