Kant and Parallel Lines

Jeremy Heis
University of California, Irvine
HOPOS 2010

Kant and Euclidean Geometry

Kant's philosophy of geometry has been criticized as:

- refuted by the possibility of non-Euclidean physical spaces (e.g., Reichenbach).
- indissolubly linked with a diagrammatic -- and so pre-modern and unrigorous -- geometrical method (e.g., Russell).

Kant and Euclidean Geometry

Kant's philosophy of geometry has been criticized as:

- refuted by the possibility of non-Euclidean physical spaces (e.g., Reichenbach)
- indissolubly linked with a diagrammatic -- and so pre-modern and unrigorous -- geometrical method (e.g., Russell)

Others (e.g., Paul Rusnock) have argued that Kant should have known better, since some of his contemporaries clearly did.

Kant and Euclidean Geometry

Kant's philosophy of geometry has been criticized as:

- refuted by the possibility of non-Euclidean physical spaces (e.g., Reichenbach)
- indissolubly linked with a diagrammatic -- and so pre-modern and unrigorous -- geometrical method (e.g., Russell)

Others (e.g., Paul Rusnock) have argued that Kant should have known better, since some of his contemporaries clearly did.

- Kant: mathematics is rational cognition from the construction of concepts.
- Contemporary mathematicians (like Lagrange) were rejecting the mathematical use of diagrams.

Two interesting issues

1. Was Kant aware of the problems posed to geometry by parallel lines? Did he reflect on these problems?

Two interesting issues

1. Was Kant aware of the problems posed to geometry by parallel lines? Did he reflect on these problems?
2. Did these two "Euclidean" features of Kant's conception of geometry interact?

Two interesting issues

1. Was Kant aware of the problems posed to geometry by parallel lines? Did he reflect on these problems?
2. Did these two "Euclidean" features of Kant's conception of geometry interact?
   - These two features of Euclidean geometry are separable.
   - *Vide* Lobatschevsky and Hilbert.

Two interesting issues

1. Was Kant aware of the problems posed to geometry by parallel lines? Did he reflect on these problems?
2. Did these two "Euclidean" features of Kant's conception of geometry interact?
   - These two features of Euclidean geometry are separable.
   - *Vide* Lobatschevsky and Hilbert.

My answer to both questions will be Yes.

Kant's Reflexions 5-11 (Ak 14: 23-52)

- Given the significance of the Euclidean theory of parallels in discussion of Kant, it is surprising that little has been said of Kant's unpublished, scattered remarks on parallel lines.

Kant's Reflexions 5-11 (Ak 14: 23-52)

- Given the significance of the Euclidean theory of parallels in discussion of Kant, it is surprising that little has been said of Kant's unpublished, scattered remarks on parallel lines.
- Some facts:

Kant's Reflexions 5-11 (Ak 14: 23-52)

- Given the significance of the Euclidean theory of parallels in discussion of Kant, it is surprising that little has been said of Kant's unpublished, scattered remarks on parallel lines.
- Some facts:
  - There is no discussion of parallels in Kant's published work.

Kant's Reflexions 5-11 (Ak 14: 23-52)

- Given the significance of the Euclidean theory of parallels in discussion of Kant, it is surprising that little has been said of Kant's unpublished, scattered remarks on parallel lines.
- Some facts:
  - There is no discussion of parallels in Kant's published work.
  - Dating: Refl 5-10 (1778-1789); Refl 11 (1790)

Kant's Reflexions 5-11 (Ak 14: 23-52)

- Given the significance of the Euclidean theory of parallels in discussion of Kant, it is surprising that little has been said of Kant's unpublished, scattered remarks on parallel lines.
- Some facts:
  - There is no discussion of parallels in Kant's published work.
  - Dating: Refl 5-10 (1778-1789); Refl 11 (1790)
  - Adickes (editor of Ak 14) says the notes were written under a double impetus:

Kant’s Reflexions 5-11 (Ak 14: 23-52)

- Given the significance of the Euclidean theory of parallels in discussion of Kant, it is surprising that little has been said of Kant’s unpublished, scattered remarks on parallel lines.
- Some facts:
  - There is no discussion of parallels in Kant’s published work.
  - Dating: Refl 5-10 (1778-1789); Refl 11 (1790)
  - Adickes (editor of Ak 14) says the notes were written under a double impetus:
    - Kant was working on the section of the KrV on the mathematical method and mathematical definitions;
    - his student Schulz was putting together a work on the theory of parallel lines.

Overview of Conclusions (by section)

1. Kant's conception of the mathematical method requires that there be immediately certain practical propositions and requires that mathematical concepts have real, genetic definitions.

Overview of Conclusions (by section)

1. Kant's conception of the mathematical method requires that there be immediately certain practical propositions and requires that mathematical concepts have real, genetic definitions.
2. (*Pace* M Friedman), it is no objection to Kant that Euclid's parallel axiom does not satisfy his description of mathematical postulates, since Kant distinguished postulates from axioms and would have considered Euclid's principle to be an axiom.

Overview of Conclusions (by section)

1. Kant's conception of the mathematical method requires that there be immediately certain practical propositions and requires that mathematical concepts have real, genetic definitions.
2. (*Pace* M Friedman), it is no objection to Kant that Euclid's parallel axiom does not satisfy his description of mathematical postulates, since Kant distinguished postulates from axioms and would have considered Euclid's principle to be an axiom.
3. Kant recognized the mathematical gaps in Wolff's definition of parallel lines, and argued that his proofs, even if corrected, would only proceed philosophically, not mathematically.

Overview of Conclusions (by section)

1. Kant's conception of the mathematical method requires that there be immediately certain practical propositions and requires that mathematical concepts have real, genetic definitions.
2. (*Pace* M Friedman), it is no objection to Kant that Euclid's parallel axiom does not satisfy his description of mathematical postulates, since Kant distinguished postulates from axioms and would have considered Euclid's principle to be an axiom.
3. Kant recognized the mathematical gaps in Wolff's definition of parallel lines, and argued that his proofs, even if corrected, would only proceed philosophically, not mathematically.
4. Kant thought that Wolff's and Euclid's definitions of parallel lines were not properly mathematical definitions, since they do not contain in themselves their constructions.

Section I: Kant on Mathematical Definitions and Postulates

I.1. Kant's notion of the mathematical method

"[M]athematical cognition [is rational cognition] from the *construction* of concepts. But **to construct a concept means to exhibit** *a priori* **the intuition corresponding to it.** For the construction of a concept, therefore, a non-empirical intuition is required, which consequently, as **intuition, is an individual object, but that must nevertheless, as the construction of a concept (of a general representation), express in the representation universal validity for all possible intuitions that belong under the same concept.** [..] The individual drawn figure is empirical, and nevertheless serves to express the concept without damage to its universality, for in the case of this empirical intuition **we have taken account only of the action of constructing the concept …" (A713/B741)**

I.1. Kant's notion of the mathematical method

Two consequences:

1. Mathematical concepts are made, not given: viz., mathematical concepts and their definitions are grasped together.

I.1. Kant's notion of the mathematical method

Two consequences:

1. Mathematical concepts are made, not given: viz., mathematical concepts and their definitions are grasped together. (Otherwise, mathematical proofs couldn't be general.)

"In the case of arbitrarily thought concepts] I can always define my concept: for I must know what I wanted to think, since I deliberately made it up … [T]he object that [mathematics] thinks it also exhibits *a priori* in intuition, and this [object] can surely contain neither more nor less than the concept, since through the definition [Erklärung] of the concept the object is originally given." (A713/B741)

I.1. Kant's notion of the mathematical method

2. Possession of a mathematical concept is *sufficient* for representing an object falling under it and also *necessary*.

I.1. Kant's notion of the mathematical method

2. Possession of a mathematical concept is *sufficient* for representing an object falling under it and also *necessary*.

- "Through the definition of the concept the object is originally given" (A713/B741).

I.1. Kant's notion of the mathematical method

2. Possession of a mathematical concept is *sufficient* for representing an object falling under it and also *necessary*.

- "Through the definition of the concept the object is originally given" (A713/B741).
- We cannot think a circle without describing it from a given point (B154), and carrying out this procedure -- as construction *of a concept* -- requires possessing the concept <circle>.

I.2. Kant on Mathematical Definitions

Mathematical definitions are "real definitions."

- Real definitions "present the possibility of the object from inner marks" (*JL*, §106)
- Mathematical definitions are real since they "exhibit the object in accordance with the concept *in intuition.*" (A241-2)

I.2. Kant on Mathematical Definitions

Mathematical definitions are "real definitions."

- Real definitions "present the possibility of the object from inner marks" (*JL*, §106)
- Mathematical definitions are real since they "exhibit the object in accordance with the concept *in intuition.*" (A241-2)

"Genetic" definitions exhibit the object of the concept *a priori* and *in concreto* (*JL*, §106).

I.3. Mathematical Postulates

Mathematics is knowledge from construction of concepts.

Construction is an activity: the "(spontaneous) production of a corresponding intuition" (Ak8: 192).

I.3. Mathematical Postulates

Mathematics is knowledge from construction of concepts.

Construction is an activity: the "(spontaneous) production of a corresponding intuition" (Ak8: 192).

Mathematics therefore requires the possibility of certain spontaneous acts.

I.3. Mathematical Postulates

Mathematics is knowledge from construction of concepts.

Construction is an activity: the "(spontaneous) production of a corresponding intuition" (Ak8: 192).

Mathematics therefore requires the possibility of certain spontaneous acts.

This possibility is guaranteed by "postulates": "a practical, immediately certain proposition or a fundamental proposition which determines a possible action of which it is presupposed that the manner of executing it is immediately certain" (*JL*, §38).

I.3. Mathematical Postulates

- This postulate is immediately certain since to even think the postulate, one must of course possess the concepts employed in its formulation; but the procedure described in the postulate *is* itself the means by which the concepts in question are first generated.
- "Now in mathematics a postulate is the practical proposition that contains nothing except the synthesis through which we first give ourselves an object and generate its concept, e.g., to describe a circle with a given line from a given point on a plane; and a proposition of this sort cannot be proved, since the procedure that it demands is precisely that through which we first generate the concept of such a figure." (A234/ B287)

I.4. Genetic (basic) definitions are virtually interchangeable with postulates.

"The possibility of a circle is ... *given* in the definition of the circle, since the circle is actually constructed by means of the definition, that is, it is exhibited in intuition [...] For I may always draw a circle free hand on the board and put a point in it, and I can demonstrate all the properties of a circle just as well on it, presupposing the (so-called nominal) definition, which is in fact a real definition, even if this circle is not at all like one drawn by rotating a straight line attached to a point. I assume that the points of the circumference are equidistant from the center point. The proposition "to inscribe a circle" is *a practical corollary of the definition* (or so-called postulate), which could not be demanded at all if the possibility – yes, the very sort of possibility of the figure – were not already given in the definition." (Letter to Herz, 26 May 1789; Ak 11:53, emph. added)

Section II. Postulates and Axioms

II.1. Friedman on the Parallel "Postulate"

Michael Friedman thinks the Parallel Postulate does not fit Kant's picture (*Kant and the Exact Sciences*, p.88).

Geometry ... operates with an initial set of specifically geometrical functions ([viz, the operations of extending a line, connecting two points, and describing a circle from a given line segment]) ... To do geometry, therefore, ... [we] need to be "given" certain initial operations: that is, intuition assures us of the existence and uniqueness of the values of these operations for any given arguments. Thus the axioms of Euclidean geometry tell us, for example, "that between any two points there is only one straight line, the from a given point on a plane surface a circle can be described with a given straight line" (Ak 2:402).

II.1. Friedman on the Parallel Potulate

- Friedman continues: "Serious complications stand in the way of the full realization of this attractive picture... Euclid's Postulate 5, the Parallel Postulate, does not have the same status as the other Postulates: it does not simply 'present' us with an elementary constructive function which can then be iterated."

II.1. Friedman on the Parallel Potulate

- Friedman continues: "Serious complications stand in the way of the full realization of this attractive picture... Euclid's Postulate 5, the Parallel Postulate, does not have the same status as the other Postulates: it does not simply 'present' us with an elementary constructive function which can then be iterated."
- Euclid's Postulate 5: "if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles."

II.1. Friedman on the Parallel Potulate

- This makes Kant look unreflective: shouldn't he have known better?

II.1. Friedman on the Parallel Potulate

- This makes Kant look unreflective: shouldn't he have known better?
- This assumes that "axiom" and "postulate" are interchangeable.

II.1. Friedman on the Parallel Potulate

- This makes Kant look unreflective: shouldn't he have known better?
- This assumes that "axiom" and "postulate" are interchangeable.
    - Since Frege, they have been interchangeable.

Postulates seem at first sight to be essentially different from axioms. In Euclid we have the postulate 'Let it be postulated that a straight line may be drawn from any point to any other'.

The postulates, so it seems, present the simplest procedures for making every construction, and postulate their, possibility. […] But what in actual fact is this drawing a line? It is not, at any rate, a line in the geometrical sense that we are creating when we make a stroke with a pencil. […] **Surely the truth of a theorem cannot really depend on something we do, when it holds quite independently of us.** So the only way of regarding the matter is that by drawing a straight line we merely become ourselves aware of what obtains independently of us. [...] So a postulate is a truth as is an axiom, its only peculiarity being that it asserts the existence of something with certain properties. From this it follows that there is no real need to distinguish axioms and postulates. (Frege, “Logic in Mathematics”)

II.1. Friedman on the Parallel Potulate

- This makes Kant look unreflective: shouldn't he have known better?
- This assumes that "axiom" and "postulate" are interchangeable.
    - Since Frege, they have been interchangeable.
    - But not for Kant.

II.B. Postulates are not the same as axioms

- Euclid has “common notions” and “postulates.”

II.B. Postulates are not the same as axioms

- Euclid has “common notions” and “postulates.”
- Early modern geometric texts (and editions of Euclid) had “axioms” and “postulates.” (Shabel; Dunlop 2009)

II.B. Postulates are not the same as axioms

- Euclid has "common notions" and "postulates."
- Early modern geometric texts (and editions of Euclid) had "axioms" and "postulates." (Shabel; Dunlop 2009)
    - Euclid's "common notions" were considered axioms.

II.B. Postulates are not the same as axioms

- Euclid has "common notions" and "postulates."
- Early modern geometric texts (and editions of Euclid) had "axioms" and "postulates." (Shabel; Dunlop 2009)
    - Euclid's "common notions" were considered axioms.
    - Euclid's list was expanded. E.g. "No two straight lines enclose a space."

II.B. Postulates are not the same as axioms

- Euclid has “common notions” and “postulates.”
- Early modern geometric texts (and editions of Euclid) had “axioms” and “postulates.” (Shabel; Dunlop 2009)
  - Euclid’s “common notions” were considered axioms.
  - Euclid’s list was expanded. E.g. “No two straight lines enclose a space.”
  - Euclid’s Parallel Postulate was relabeled an “axiom.”

II.B. Postulates are not the same as axioms

- Euclid has “common notions” and “postulates.”
- Early modern geometric texts (and editions of Euclid) had “axioms” and “postulates.” (Shabel 2003; Dunlop 2009)
  - Euclid’s “common notions” were considered axioms.
  - Euclid’s list was expanded. E.g. “No two straight lines enclose a space.”
  - Euclid’s Parallel Postulate was relabeled an “axiom.”
- Wolff: “postulata” are indemonstrable *practical* propositions; “axiomata” are indemonstrable *theoretical* propositions.

II.B. Postulates are not the same as axioms

- Kant departed from Wolff: an axiom is a "fundamental proposition *that can be exhibited in intuition*" (*JL*, §35).

II.B. Postulates are not the same as axioms

- Kant departed from Wolff: an axiom is a "fundamental proposition *that can be exhibited in intuition*" (*JL*, §35).
- But kept Wolff's view that postulates are *practical* fundamental propositions.

II.B. Postulates are not the same as axioms

- Kant departed from Wolff: an axiom is a "fundamental proposition *that can be exhibited in intuition*" (*JL*, §35).
- But kept Wolff's view that postulates are *practical* fundamental propositions.
- These definitions leave open that postulates are a species of, or even identical with, axioms.

II.B. Postulates are not the same as axioms

- Kant departed from Wolff: an axiom is a "fundamental proposition *that can be exhibited in intuition*" (*JL*, §35).
- But kept Wolff's view that postulates are *practical* fundamental propositions.
- These definitions leave open that postulates are a species of, or even identical with, axioms.
- But there is overwhelming circumstantial evidence against this possibility.

II.B. Postulates are not the same as axioms

- All of the texts Kant used in his lectures distinguished axioms as theoretical from postulates as practical.

II.B. Postulates are not the same as axioms

- All of the texts Kant used in his lectures distinguished axioms as theoretical from postulates as practical.
- Texts from Kant's students Schulz and Kiesewetter did also.

II.B. Postulates are not the same as axioms

- All of the texts Kant used in his lectures distinguished axioms as theoretical from postulates as practical.
- Texts from Kant's students Schulz and Kiesewetter did also.
- All of the examples of axioms that Kant gives in his writings are theoretical, all examples of postulates are practical.

II.B. Postulates are not the same as axioms

- All of the texts Kant used in his lectures distinguished axioms as theoretical from postulates as practical.
- Texts from Kant's students Schulz and Kiesewetter did also.
- All of the examples of axioms that Kant gives in his writings are theoretical, all examples of postulates are practical.
- Kant argued against Schulz that 7+5=12 is not an axiom, but it is a postulate. (25 Nov 1787 letter)

II.C. Kiesewetter on Postulates

- Kiesewetter, while composing his Kantian mathematics textbook, *Die ersten Anfangsgründe der reinen Mathematik* (1799), asked Kant for a definition of "Postulat" that would clearly distinguish postulates from "axioms" (Grundsätze). (12:267)

II.C. Kiesewetter on Postulates

- Kiesewetter, while composing his Kantian mathematics textbook, *Die ersten Anfangsgründe der reinen Mathematik* (1799), asked Kant for a definition of "Postulat" that would clearly distinguish postulates from "axioms" (Grundsätze). (12:267)

- Kiesewetter's text: "*Postulate*. A practical axiom is called a *postulate* or a *postulating proposition*. An axiom properly speaking [das eigentliche Axiom] and a postulate agree in that in neither case are they derived from a different proposition, and they only differ from one another in that an axiom relates to the knowledge of an object, augments the concept of it, (is theoretical), while a postulate adds nothing to the knowledge of the object, does not augment its concept, but rather concerns only the construction, the intuitive exhibition [Darstellung], of the object." (p.xxi)

II.C. Kiesewetter on Postulates

- Kiesewetter clearly saw (against Friedman) that axioms, but not postulates, "augment our knowledge of a concept."

II.C. Kiesewetter on Postulates

- Kiesewetter clearly saw (against Friedman) that axioms, but not postulates, "augment our knowledge of a concept."
  - to be certain of the postulate "to draw a straight line between any two points," it is sufficient to know the concept <straight line>.

II.C. Kiesewetter on Postulates

- Kiesewetter clearly saw (against Friedman) that axioms, but not postulates, “augment our knowledge of a concept.”
    - to be certain of the postulate “to draw a straight line between any two points,” it is sufficient to know the concept <straight line>.
    - Not so with “no two lines enclose a space.”

II.C. Kiesewetter on Postulates

- Kiesewetter clearly saw (against Friedman) that axioms, but not postulates, “augment our knowledge of a concept.”
    - to be certain of the postulate “to draw a straight line between any two points,” it is sufficient to know the concept <straight line>.
    - Not so with “no two lines enclose a space.”
- Kiesewetter clearly saw (against Frege) that postulates are not replaceable by theoretical axioms. “Now since all of mathematics rests on construction, postulates constitute a foundation stone of the structure of mathematics.” (p.xxi)

III. Kant’s Reflexions on the Definitions of <Parallel Line>

III.A. 3 Definitions of <circle>

III. Kant's Reflexions on the Definitions of <Parallel Line>

III.A. 3 Definitions of <circle>

1. "a line (on a plane) such that every possible line drawn on that plane from a determinate point is perpendicular to it." [Refl 5]

III. Kant’s Reflexions on the Definitions of <Parallel Line>

III.A. 3 Definitions of <circle>

1. “a line (on a plane) such that every possible line drawn on that plane from a determinate point is perpendicular to it." [Refl 5]
2. "a curved line, all of whose arcs are divided into equal parts by the perpendicular line that divides its cord into two equal parts." [Refl 6]

III. Kant's Reflexions on the Definitions of <Parallel Line>

III.A. 3 Definitions of <circle>

1. "a line (on a plane) such that every possible line drawn on that plane from a determinate point is perpendicular to it." [Refl 5]
2. "a curved line, all of whose arcs are divided into equal parts by the perpendicular line that divides its cord into two equal parts." [Refl 6]
3. "a line every point of which is the same distance from a single one" (A731/B760)

III. Kant’s Reflexions on the Definitions of <Parallel Line>

III.A. 3 Definitions of <circle>

1. “a line (on a plane) such that every possible line drawn on that plane from a determinate point is perpendicular to it." [Refl 5]

2. "a curved line, all of whose arcs are divided into equal parts by the perpendicular line that divides its cord into two equal parts." [Refl 6]

3. "a line every point of which is the same distance from a single one" (A731/B760)

   Kant argues for the third *on methodological grounds*.

III.A. 3 Definitions of <circle>

How much can be inferred from this definition of a circle?

I think, from a definition that does not at the same time contain in itself the construction of the concept, nothing can be inferred (which would be a synthetic predicate).

[On the flip side of this loose page:]

so that the proposition [viz., the definition] may be inverted and in this inversion be proved, which is necessary indeed for a definition. Euclid's definition of parallel lines is of this kind.

III.A. 3 Definitions of <circle>

- Kant rejects the earlier two definitions. The standard Euclidean definition "contains in itself the construction of the concept."
  - It is immediate from the definition that circles can be constructed.

III.A. 3 Definitions of <circle>

- Kant rejects the earlier two definitions. The standard Euclidean definition "contains in itself the construction of the concept."
  - It is immediate from the definition that circles can be constructed.
  - The other two definitions do not show immediately how to construct circles. The *definientes* are provable of circles, but the proofs require the ability to construct circles.

III.A. 3 Definitions of <circle>

- Kant rejects the earlier two definitions. The standard Euclidean definition "contains in itself the construction of the concept."
  - It is immediate from the definition that circles can be constructed.
  - The other two definitions do not show immediately how to construct circles. The *definientes* are provable of circles, but the proofs require the ability to construct circles.
- Kant then claims that Euclid's definition of parallels has the same flaw.

III.B. Euclid and Wolff's definitions of <circle>

Euclid: two lines that are extendible in both directions without ever intersecting.

Wolff: "If two lines AB and CD always have the same distance from each other, then they are parallel lines."

III.B. Euclid and Wolff's definitions of <circle>

Euclid: two lines that are extendible in both directions without ever intersecting.

Wolff: "If two lines AB and CD always have the same distance from each other. then they are parallel lines."

- One can prove the existence of E-parallels (but not W-parallels) without the Parallel Axiom.
- That there are two straight lines every equidistant is equivalent (given the axiom of Archimedes) to the Parallel Axiom.
- This is why Wolff is able to dispense with Euclid's axiom: he assumes, without argument, that there are parallels in his sense.

III.B. Euclid and Wolff's definitions of <circle>

- I.27. "If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another."
- I.28. "If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another."

III.B. Euclid and Wolff's definitions of <circle>

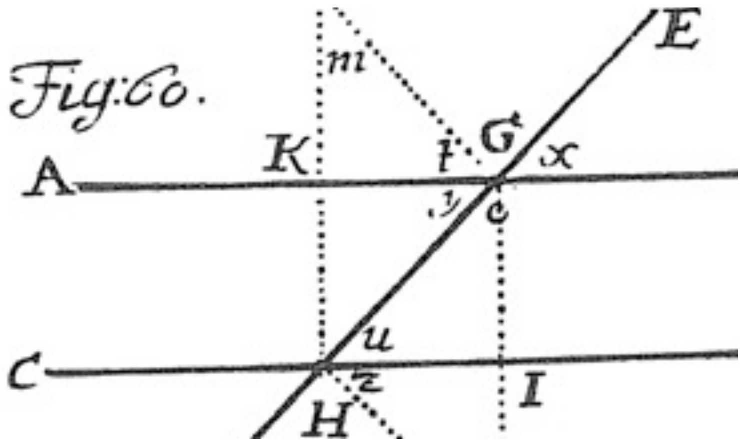
- I.27. "If a straight line falling on two straight lines makes the alternate angles equal to one another, then the straight lines are parallel to one another."
- I.28. "If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another."
- I.29. "A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles."

III.C. Wolff's Fallacious Proofs

- To infer I.29, Euclid needed his Parallel Axiom.
- Wolff realized that I.29 follows from his definition of <parallel lines> (but he did not recognize that problems show up elsewhere).

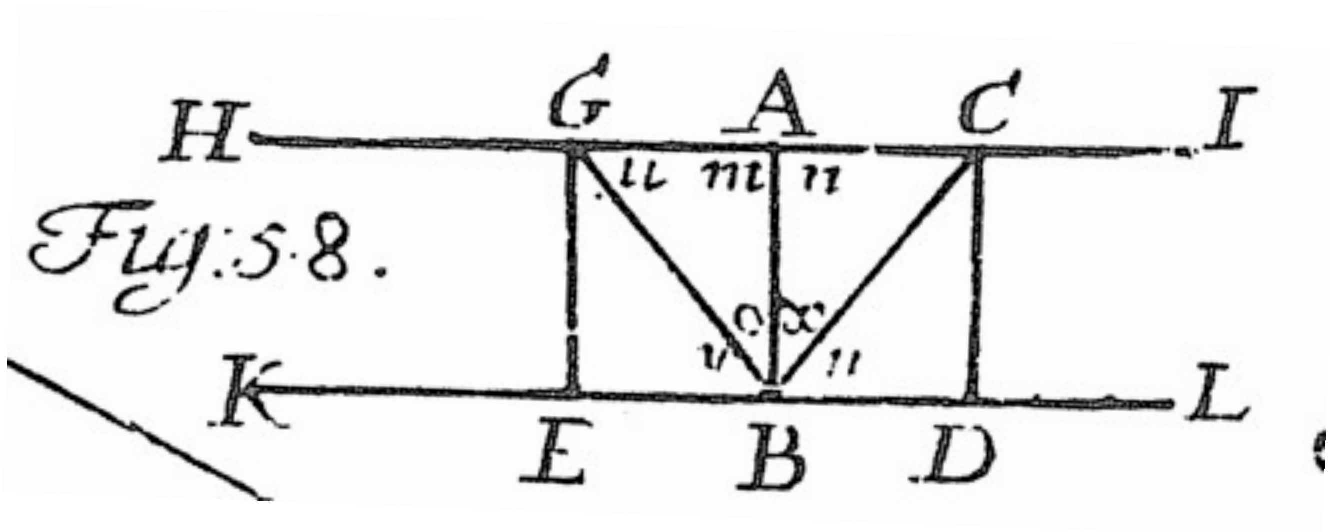
III.C. Wolff's Fallacious Proofs

- To infer I.29, Euclid needed his Parallel Axiom.
- Wolff realized that I.29 follows from his definition of <parallel lines> (but he did not recognize that problems show up elsewhere).
- In these Reflexions, Kant realizes both these facts about Wolff's approach.



Theorem 44 [=Euclid I.27-8]. If two lines AB and CD are cut by a transverse EF in G and H in such a way that either y=u, or x=u, or o + u is 180 degrees; then the lines will be parallel among themselves.

Demonstratio. 1. Send from H and G the perpendiculars HK and GI; then will K = I. truly also y=u per hypoth and HG=HG. Thus HK=GI, and so **since HK and GI are the distances of the lines AB and CD**, then the lines AB and CD are parallel to each other. QED.



<u>Wolff §230:</u> If HI is parallel and BA perpendicular to KL, then AB will also be perpendicular to HI. [This implies Euclid I.29]

Demonstratio. Let EB=BD and erect on E and D perpendiculars EG and DC; then GE = CD & E=D, consequently BG=BC & y=u [by SAS]. But since AB is perpendicular to KL *per hypoth*, then u+x = o+y. Therefore also x=o. Thus since obviously AB=AB, **then also will m=n** [by SAS], and thus BA is perpendicular to HI. QED.

III.C. Wolff's Fallacious Proofs

- The last application of SAS requires that the area enclosed by GAB and ACB be triangles and thus that GA and CA are straight lines.

III.C. Wolff's Fallacious Proofs

- The last application of SAS requires that the area enclosed by GAB and ACB be triangles and thus that GA and CA are straight lines.
- But nothing Wolff give us proves that the endpoints of 3 equal perpendiculars to a straight line can lie on a second straight line (in hyperbolic geometry, they can't).

III.C. Wolff's Fallacious Proofs

- The last application of SAS requires that the area enclosed by GAB and ACB be triangles and thus that GA and CA are straight lines.
- But nothing Wolff give us proves that the endpoints of 3 equal perpendiculars to a straight line can lie on a second straight line (in hyperbolic geometry, they can't).
- If we assume that every curve everywhere equidistant from a straight line is straight, then the proof goes through. But this is Euclid's parallel axiom.

III.C. Wolff's Fallacious Proofs

- The last application of SAS requires that the area enclosed by GAB and ACB be triangles and thus that GA and CA are straight lines.
- But nothing Wolff give us proves that the endpoints of 3 equal perpendiculars to a straight line can lie on a second straight line (in hyperbolic geometry, they can't).
- If we assume that every curve everywhere equidistant from a straight line is straight, then the proof goes through. But this is Euclid's parallel axiom.
- Without the axiom, we cannot prove that there are Wolffian parallels -- that is, that <parallel line> has a real definition.

IV. Kant's Criticism of Wolff and Euclid

IV.A. Kant's Diagnosis of Wolff's Error

- Kant realizes that Wolff's proof of I.29 goes through if we build into the definition that parallel lines are everywhere equidistant *straight* lines -- but at the cost of making the proof merely conceptual or "philosophical".

"This proposition cannot be exhibited mathematically, but rather follows merely from concepts: that namely parallel lines alone have a determinate distance from one another, that the distance should be measured by the perpendicular lines, from a point A which connects the one to the other, that, because the distance must be reciprocally equal, the distance of the point B to the other line, therefore the perpendicular to the this one, must stand at the same time as the measure of the distance of this line to the other other, and so at the same time perpendicular to it. And [so] … both these perpendiculars are one and the same." [Refl 10]

IV.A. Kant's Diagnosis of Wolff's Error

Kant summarizes his criticism:

"If the equality of the distance of two lines constitutes the definition of parallelism, then the definitum and the definition must be invertible."

[[That is, we should be able to give the right kind of proofs of I. 27-8 and I.29.]]

"Although the proposition is invertible, it cannot be proven, since we can infer from the complete concept indeed to the concept of the equality of the angles, but the complete concept does not lead to its construction."

IV.B. Kant’s criticism of Euclid’s Definition

- Adickes thinks that Kant's criticisms of Wolff are confused, and, in any case, they do not touch Euclid's definition. I don't agree.

IV.B. Kant’s criticism of Euclid’s Definition

- Adickes thinks that Kant's criticisms of Wolff are confused, and, in any case, they do not touch Euclid's definition. I don't agree.
- Euclid's definition: "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."

IV.B. Kant’s criticism of Euclid’s Definition

- Adickes thinks that Kant's criticisms of Wolff are confused, and, in any case, they do not touch Euclid's definition. I don't agree.
- Euclid's definition: "Parallel straight lines are straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction."
- Like Wolff’s definition, Euclid’s definition does not "contain in itself its construction."

IV.B. Kant's criticism of Euclid's Definition

- Euclid's definition is not a real definition:

IV.B. Kant's criticism of Euclid's Definition

- Euclid's definition is not a real definition:
  - It includes a negative mark, which no real definition has.
  - Kiesewetter 1799, §68: "the <u>infinite</u> extending of lines, without them intersecting, does not admit of construction (intuitive exhibition)"

IV.B. Kant’s criticism of Euclid’s Definition

- Euclid’s definition is not a real definition:
  - It includes a negative mark, which no real definition has.
  - Kiesewetter 1799, §68: “the infinite extending of lines, without them intersecting, does not admit of construction (intuitive exhibition)”
- Euclid’s proof that parallel lines are constructible [I.31] *comes too late:* that parallel lines are constructible is not a "practical corollary" of the definition whose "possibility is already contained in the definition."

IV.B. Kant’s criticism of Euclid’s Definition

- A properly mathematical definition, on Kant's view, is built up from component marks in a way that parallels the step by step procedure for exhibiting the concept in intuition.

IV.B. Kant’s criticism of Euclid’s Definition

- A properly mathematical definition, on Kant's view, is built up from component marks in a way that parallels the step by step procedure for exhibiting the concept in intuition.
    - This step by step procedure would be built up from basic constructions (given by the postulates) (see Ak 20:410-1),
    - just as defined concepts are built up from component simple marks.

IV.B. Kant’s criticism of Euclid’s Definition

- A properly mathematical definition, on Kant's view, is built up from component marks in a way that parallels the step by step procedure for exhibiting the concept in intuition.
  - This step by step procedure would be built up from basic constructions (given by the postulates) (see Ak 20:410-1),
  - just as defined concepts are built up from component simple marks.
- Given this parallelism, it should be impossible to understand a definition without seeing that it can be constructed.