ON THE PHYSICAL POSSIBILITY OF ORDINAL COMPUTATION

WAYNE AITKEN JEFFREY A. BARRETT

Abstract. α-recursion lifts classical recursion theory from the first transfinite ordinal ω to an arbitrary admissible ordinal α [13]. Turing machine models for α-recursion and other types of transfinite computation have been proposed and studied [5] and [7] and are applicable in computational approaches to the foundations of logic and mathematics [11]. They also provide a natural setting for modeling extensions of the algorithmic logic described in [2] and [3]. Such α-Turing machines can complete a θ-step computation for any ordinal $\theta < \alpha$. Here we consider constraints on the physical realization of α-Turing machines that arise from the structure of physical spacetime. In particular, we show that an α-Turing machine is realizable in a spacetime constructed from $\mathbb{R}$ only if α is countable. Further, while there are spacetimes where uncountable computations are possible, there is good reason to suppose that such nonstandard spacetimes are nonphysical. We conclude with a suggestion for a revision of Church's thesis appropriate as an upper bound for physical computation.

1. Ordinal Recursion and Physical Computation

Church's thesis is that every computable function is recursive. Since Turing machines provide a computational model for the recursive functions, a function is Church computable only if it is Turing computable. But insofar as one takes what is actually computable to be a question of fact, one cannot simply stipulate the answer. More specifically, one should expect that what is in fact computable ultimately depends on the nature of the physical world. On this view, questions of what is computable must be answered relative to (i) a specified physical theory (and other assorted background physical assumptions) and (ii) a computational model described relative to these physical assumptions.[1]

Many of our best physical theories allow that what is physically computable may extend well beyond what is Turing computable. But how much more than Turing computable might be physically possible? And for that matter, how should one measure the strength of computations that extend beyond the Turing computable functions? α-recursion theory provides one context for answering such questions.

α-recursion lifts classical recursion theory from the first transfinite ordinal ω to an arbitrary admissible ordinal α [13]. If one thinks of Turing machines as providing natural computational models for ω-recursive functions, then one might wish to have similar computational models for α-recursive functions. Following earlier descriptions of infinite time Turing machines [5], Peter Koepke describes an

Date: October 23, 2009.

[1]Here we will assume a version of classical mechanics with no energy or velocity constraints and an α-recursive Turing model of computation. More specific auxiliary assumptions will be described as needed.

α-Turing machine that can complete θ computational steps on an α-length tape for $\theta < \alpha$ [7].[2]

Koepke characterizes the structure of an α-Turing machine by comparing it against a standard Turing machine. Standard Turing computations occur on an $\omega \times \omega$ tape, where the first parameter represents positions on the tape and the second represents computational time. These two parameters provide ordinal coordinates for a location on the tape at a time as indicated in Figure 1. Correspondingly, α-Turing machine computations occur on a much larger $\alpha \times \alpha$ tape as indicated in Figure 2. It will prove significant that this computational model requires that both space and time be partitionable into tape positions and computational clock ticks indexed by the ordinal α. We will suppose that this requires that positions on the tape are associated with disjoint spatial intervals that are ordered and indexed by α, and the same for each temporal stage of the computation. A spacetime will be any manifold that may be partitioned into disjoint spacetime regions.

An α-computation, for ordinal α, is described by characterizing a command C, a program P, and a computation of P.

Definition 1.1. A *command* $C = (s, c, c', m, s')$ indicates that if the local machine state is s and it reads c at the current head position, it writes c', then moves the head to the left one cell if $m = 0$ or to the right one cell if $m = 1$, then goes into local state s', where $s, s' \in \omega$ and $c, c', m \in \{0, 1\}$.

Definition 1.2. A *program* P is a finite set of commands such that:

(1) if $(s, c, c', m, s') \in P$, then there exists a $(s, d, d', m', s'') \in P$ where $c \neq d$, so in local state s the machine can handle either a 0 or a 1.
(2) if $(s, c, c', m, s') \in P$ and $(s, c, c'', m', s'') \in P$, then $c' = c''$, $m = m'$, and $s' = s''$, so the computation is fully determined by the program states and the initial cell contents, where $\{s | (s, c, c', m, s') \in P\}$ are the program states.

Definition 1.3. A *computation* is a triple $S : \theta \to \omega$, $H : \theta \to \alpha$, $T : \theta \to (\alpha \to \{0, 1\})$, where θ is the length of the computation, S is the program state function, H is the head position function, and T is the tape state function such that:

(1) the computational length θ representing the time t of the computation is a successor ordinal such that $\theta < \alpha$.
(2) at time $t = 0$ the machine starts with local machine state $S(0) = 0$ and head position $H(0) = 0$.

[2]An admissible ordinal α is a limit ordinal with properties that are useful in a theory of transfinite computation. If, for example, a subroutine halts in less than α steps and if a program calls the subroutine $\theta < \alpha$ times, then the program will also halt in less than α steps. There are several equivalent definitions of admissible ordinal. One is that an ordinal α is admissible if the α-stage L_α of the constructible universe is a model for Kripke-Platek set theory. While many admissible ordinals are countable, there are also uncountable admissible ordinals. Indeed, any regular uncountable cardinal is also an admissible ordinal.

While computations on Koepke's model might be any ordinal length, the computational closure properties of admissible ordinals make admissible-sized machines natural candidates for well-behaved ordinal computers. It is an added virtue that such machines also fit nicely with general recursion theory. There is, however, nothing in the cardinality considerations below that requires that the strongest physically realizable machine (in terms of what spacetime partitions are in fact physically possible) is characterized by an admissible ordinal—but if not, it would not be an entirely well-behaved computer.

(3) if $t < \theta$ and $S(t)$ is not a program state, then the machine stops at time $t+1$.

(4) if $t < \theta$ and $S(t)$ is a program state, then the unique command (s, c, c', m, s') where $S(t) = s$ and $T(t)_{H(t)} = c$ is executed as follows:

$$S(t+1) = s'$$

$$H(t+1) = \begin{cases} H(t)+1 & \text{if } m = 1 \\ H(t)-1 & \text{if } m = 0 \text{ and } H(t) \text{ a successor ordinal} \\ 0 & \text{otherwise} \end{cases}$$

$$T(t+1)_\beta = \begin{cases} c' & \text{if } \beta = H(t) \\ T(t) & \text{otherwise} \end{cases}$$

(5) if $t < \theta$ is a limit ordinal, then the full machine state at time t is

$$S(t) = \liminf_{r \to t} S(r)$$

$$H(t) = \liminf_{r \to t} H(r)$$

$$\forall \beta \leq \alpha \ T(t)_\beta = \liminf_{r \to t; S(s) = S(t)} T(r)_\beta$$

The α-computation is uniquely determined by the initial tape contents $T(0)$ and the program P. If the α-computation halts, $\theta = \beta + 1$ is a successor ordinal and $T(\beta)$ is the final tape content. The role played by inferior limits is to determine the full machine state at limit ordinals.

Note that while the tape is indexed by α, since the machine states in the commands are limited by ω and the programs are finite, there will be at most a countable number of possible programs. One consequence is that one might think of these programs very much as one thinks of classical Turing programs, they are just allowed to run longer and with a longer tape.

On this model an α-Turing machine has the resources to complete a θ-step computation of any standard Turing program for any ordinal $\theta < \alpha$ and track the results. Such a computational model allows for very strong extensions of the algorithmic logic described in [1], [2], and [3].

While this is a perfectly coherent computation model, such α-Turing machines are not in general physically realizable. In particular, we will show that an $\alpha \times \alpha$ partition of spacetime into computational steps and tape positions is realizable in a spacetime constructed from $\mathbb{R}$ if and only if α is countable. But we will also show that an ω_1-Turing machine, where ω_1 is the first uncountable ordinal, is in principle realizable in a nonstandard spacetime constructed from the hyperreals $*\mathbb{R}$ (the line of nonstandard analysis) or from the long line $\omega_1 \times [0, 1)$ since both spaces allow for uncountable ordered partitions. An immediate consequence of this is that what computations are physically possible may depend on the precise topological structure of spacetime. On the other hand, there is good physical reason to take such nonstandard spacetimes to be poor candidates for faithful representations of physical spacetime. We will end by suggesting a revision of Church's thesis for physical computation.

2. The Injection Theorem

We begin by reviewing a few mathematical facts that are perhaps not as well-known as they deserve.

Lemma 2.1. *If there is an order-preserving injection of an ordinal α into $\mathbb{R}$, then there is an order-preserving injection of α into $\mathbb{Q}$.*

Proof. Let $f : \alpha \to \mathbb{R}$ be an order-preserving injection. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, there is for each $x \in \alpha$ a rational number r_x such that $f(x) < r_x < f(x+1)$. Consider the function $\alpha \to \mathbb{Q}$ given by $x \mapsto r_x$. This is an order-preserving injection since if $x < y$ then $r_x < f(x+1) \leq f(y) < r_y$. □

Corollary 2.2. *If there is an order-preserving injection of an ordinal α into $\mathbb{R}$, then α is countable.*

The converse holds as well:

Lemma 2.3. *If α is a countable ordinal, then there is an order-preserving injection from α into $\mathbb{Q} \cap (0, 1)$.*

Proof. We proceed by transfinite induction.

Zero: If $\alpha = 0$ then use the unique map from the empty set to $\mathbb{Q} \cap (0, 1)$.

Successor: Suppose there is an order-preserving map $f : \alpha \to \mathbb{Q} \cap (0, 1)$. Define $f' : \alpha + 1 \to \mathbb{Q} \cap (0, 1)$ as follows:

$$f'(x) = \begin{cases} f(x)/2 & x < \alpha \\ 3/4 & x = \alpha \end{cases}$$

Limit: Suppose λ is a countable limit ordinal and that there is an order-preserving map $f_\beta : \beta \to \mathbb{Q} \cap (0, 1)$ for each $\beta < \lambda$. Let $g : \omega \to \lambda$ be a bijection enumerating the countable set λ. For each $k \in \omega$, let β_k be the maximum of $g(0), \ldots, g(k)$. This gives a nondecreasing sequence of ordinals with limit λ. To see this, suppose otherwise that there is an upper bound $b < \lambda$. Let k be such that $g(k) = b + 1$. Then $\beta_k \geq g(k) > b$, a contradiction.

Define $h : \lambda \to \mathbb{Q} \cap (0, \infty)$ by the rule $x \mapsto n + f_{\beta_n}(x)$ where β_n is the first term of the sequence with $x < \beta_n$. Observe that h is an order-preserving injection.

Finally, observe that the rule $x \mapsto x/(x+1)$ defines an order-preserving injection $\mathbb{Q} \cap (0, \infty) \to \mathbb{Q} \cap (0, 1)$. Compose h with this injection. This yields the desired map. □

Theorem 2.4. *There is an order-preserving injection of an ordinal α into $\mathbb{R}$ if and only if α is countable.*

Proof. This follows from Corollary 2.2 and Lemma 2.3. □

An α-partition of time that represents the ordered ticks of a computation is possible if and only if there is an order-preserving injection from α to the parameter representing physical time. Similarly, an α-partition of space that represents an ordered tape is possible if and only if there is an order-preserving injection from α to the parameter representing physical tape locations. Hence, it follows from Theorem 2.4 that if spacetime is in fact faithfully represented by $\mathbb{R}$, then there can be at most a countable number of well-ordered computational steps and tape registers.[3]

While this represents a strong constraint on the physical realizability of α-computations, it still allows for much more computational power than Church's thesis. An $\omega + c$-computation (for a finite constant c) solves the standard halting problem, decides Goldbach's conjecture, and tests the proof-theoretic consistency of ZFC. The next admissible ordinal greater than ω, the first non-recursive ordinal, ω_1^{ck} is also countable. An ω_1^{ck}-computation could compute any function definable in the arithmetic or hyperarithmetic hierarchies.

Since Theorem 2.4 extends easily to any ordered set containing the rationals $\mathbb{Q}$ as a dense subset, any model of the spacetime continuum supporting uncountable α-recursive computation must fail to contain $\mathbb{Q}$ as a dense subset.

3. Hyperreal Spacetime

We now consider a contrasting result concerning the hyperreal or nonstandard line.

Theorem 3.1. *There is an order-preserving injection from the first uncountable ordinal ω_1 into the hyperreals (the nonstandard line) $*\mathbb{R}$.*

Proof. We will prove something stronger: we can inject ω_1 into the extended natural numbers $*\mathbb{N}$. To do this, define the set of initial embeddings P as the set of all injective order-preserving functions $\beta \to *\mathbb{N}$ where $\beta \leq \omega_1$.

We define a partial order on P by declaring that $f_1 : \beta_1 \to *\mathbb{N}$ to be less than or equal to $f_2 : \beta_2 \to *\mathbb{N}$ if and only if $\beta_1 \leq \beta_2$ and f_1 is the restriction of f_2 to the subset β_1. Observe that every totally ordered subset of P has a maximum: take the union of the graphs of the functions in such a subset. By Zorn's lemma there is a maximum element $f : \beta \to *\mathbb{N}$.

We claim that $\beta = \omega_1$. Suppose otherwise. Then β is countable. By the following lemma the image of this maximal f has an upper bound $n \in *\mathbb{N}$. We extend f to $\beta + 1$ by defining $f(\beta) = n + 1$. The existence of such an extension contradicts the maximality of f. $\square$

Lemma 3.2. *Every countable subset of $*\mathbb{N}$ has an upper bound in $*\mathbb{N}$.*

Proof. We appeal to the ultrafilter construction of $*\mathbb{N}$ where we view elements of $*\mathbb{N}$ as equivalence classes of sequences of natural numbers. Here we fix a nonprinciple ultrafilter of $\mathbb{N}$. The equivalence relation declares two sequences (a_i) and (b_i) to be equivalent if and only if $a_i = b_i$ for all i in a set of indices in the ultrafilter. Write $[a_i]$ for the class of a sequence. We also use the ultrafilter to define the order and the arithmetic operations on $\mathbb{N}$. All we need is the fact that if $a_i \geq b_i$ for all but a finite number of indices i then we have $[a_i] \geq [b_i]$.

---

[3]Philip Welch has a similar result in the context of a computational question in Malament-Hograth spacetimes [6]. Welch shows that the separability of the Riemannian spacetime manifold means that only countably many open spacetime intervals are available for computational steps [14] (See [15] for a broader survey article that also discusses this point, and [12] for recent results on the closely related strength of infinite time register machines.). More generally, one might note that if separability is a requirement for a physically plausible spacetime, then there can be only a countable number of open computational intervals. But settling the issue of what is physically computable by stipulating spacetime separability is presumably a bit too quick, especially in light of the main result of the next section.

Now let $n_1, n_2, \ldots$ be a countable number of elements of $*\mathbb{N}$. Choose sequences $(c_{i,j})_{j\in\mathbb{N}}$ so that $n_i = [c_{i,j}]$. Finally, for each $k \in \mathbb{N}$ let M_k be the maximum of $c_{i,j}$ for $i, j \in \{1, \ldots, k\}$.

Fix i_0. We identify the set of indices j where $M_j > c_{i_0,j}$. Observe that if $j \geq i_0$ then $M_j \geq c_{i_0,j}$ by definition of M_j. Thus the set of j where $M_j \geq c_{i_0,j}$ includes all but a finite number of values of j. Hence $[M_j] \geq n_{i_0}$. This holds for arbitrary n_{i_0} so $M = [M_j]$ is an upper bound of $n_1, n_2, \ldots$ □

A consequence is that if spacetime were faithfully represented by $*\mathbb{R}$, then there could in principle be an uncountable number of well-ordered computational steps. It would consequently be possible to carry out computations that could not be carried out if time is faithfully represented by $\mathbb{R}$; and hence, there would at least in principle be empirical consequences to the difference.

4. Nonphysical Properties of Nonstandard Spacetimes

If spacetime were faithfully represented by $*\mathbb{R}$ there would be more computations possible than if it were faithfully represented by $\mathbb{R}$. But the empirical differences that larger continuous spaces might allow are perhaps clearer in the context of the long line $\omega_1 \times [0, 1)$.

Clearly the sort of spacetime partition required to characterize an ω_1-Turing machine is realizable in a spacetime constructed from $\omega_1 \times [0, 1)$. And it follows that there might be, as in $*\mathbb{R}$, computations realizable in this nonstandard spacetime that would not be possible in a spacetime constructed in the standard way from $\mathbb{R}$. But there may be more direct ways to distinguish $\omega_1 \times [0, 1)$ from $\mathbb{R}$.

Consider the following experiment. Build a clock that ticks each $1/2^n$ second for $n = 1, 2, 3 \ldots$ and stops ticking at one second after completing an ω-sequence of ticks, and move one meter to the right at each clock tick. If spacetime is faithfully represented by $\mathbb{R}$, then one would complete an ω-sequence of one-meter steps in one second, move further than any finite distance, and hence no longer be a spatial location.[4]

If, however, spacetime were faithfully represented by the long line $\omega_1 \times [0, 1)$, then the limit l of the ω-sequence of one-meter steps exists, and one would end up in spatial location l after one second. One might then check that one was still at a spatial location by noting that one could still move to the right one meter further. If possible, such an experiment might empirically distinguish between $\mathbb{R}$ and $\omega_1 \times [0, 1)$. On the other hand, both $\omega_1 \times [0, 1)$ and $*\mathbb{R}$ exhibit properties that suggest that such nonstandard spacetimes are poor candidates for representations of physical spacetime.

Consider the step experiment again in the context of the long line. After one second, one might move one meter further to the right; but if one moves one meter to the left, one must pass an infinite number of one's earlier one-meter steps and end up at an indeterminate location in that it cannot be expressed relative to earlier

[4]This supertask is often referred to as a reverse space-invader. Earman (1986) discusses such supertasks in the context of assessing what it might mean for a physical theory to be deterministic. Note that we here have a finer-grained way of understanding stepwise supertasks by using a transfinite ordinal to index the steps of the process.

steps.[5] Indeed, since l is the limit of the sequence of steps any move to the left, no matter how small, must pass an infinite number of terms of the sequence. Further, if one then moved one meter again to the right, one would not return to location l. That is distance measurements as performed by the local observer must typically fail to commute. The point is that in trying to tell a story where we suppose that one can make sense of an ω-sequence of distance measurements, we find that these operations cannot exhibit the properties that one might take to be essential to the very notion of a distance measurement.

There is a trivial sense in which the long line is not metrizable since there is no $\mathbb{R}$-valued distance function satisfying the usual properties. The problem here, however, is more serious. Since measuring in one direction one meter then measuring back one meter cannot typically return the observer to where she started in the long line, one can arguably make no sense whatsoever of the general notion of a distance measurement. In any case, insofar as one requires that commuting distance measurements be possible for a space to serve as a stage for the dynamical description of physical processes, the long line is manifestly unsuitable.

Consider the step experiment again, but this time in the context of the hyperreal (nonstandard) line. Here the problem is not one of making sense of distance measurements since one can provide a (nonstandard) metric for the space. Rather, the problem is that while the ω-sequence of steps is bounded, just as with the natural numbers, there is no limit of the sequence of steps in the nonstandard line. Consequently, one does not end up at a determinate spatial location. On the other hand, with the nonstandard line, one at least has the property that distance measurements commute: one can always move to the left one meter, then one meter to the right and be sure to end up where one started. The problem here is that the least-upper-bound property fails and one can have a bounded monotonically increasing series with no limit. Once again, we encounter a striking sort of indeterminism in describing even a process involving only a simple ω-sequence of steps.

Insofar as one takes such sorts of indeterminism to pose problems for physical prediction and explanation, one may take both types of nonstandard spacetime, spaces that lack a coherent notion of distance measurement and spaces for which the least-upper-bound property fails, to be nonphysical in the sense that they are inappropriate as a stage for physical events. The problem is that in neither case would one be able to provide general dynamical laws for the supertasking processes that would go beyond the limitations of the reals, which is why one might have wanted the nonstandard spacetimes in the first place.

Since it is entirely unclear how to characterize dynamical processes in nonstandard spacetimes like those constructed from the long or hyperreal line, one might take spaces that fail to support a coherent notion of distance measurement or fail to satisfy the least-upper-bound property to be physically implausible. Insofar as one takes such nonstandard spacetimes to be nonphysical, one might favor the standard spacetimes represented in our best physical theories, and thus take ordinal computations to be limited to the countable ordinals.

[5] One might imagine placing physical meter markers after each measurement as one approaches the limit position l. When one turns to the left, there can be no first meter marker one would encounter. For some, this fact alone may be enough to suggest that such a space is nonphysical.

5. Morals

What is physically computable may be more than what is Turing computable, but how much more depends in part on the structure of spacetime. On the other hand, while it is entirely possible that our world is faithfully represented by a nonstandard spacetime and while there may even be empirical consequence to which we choose, there is good reason to rule out representations like $*\mathbb{R}$ and $\omega_1 \times [0, 1)$ as nonphysical.

Insofar as (i) an α-partition of spacetime is required for an α-computation and (ii) our best physical theories are correct in taking $\mathbb{R}$ as faithfully representing the structure of spacetime, then Church's Thesis should be replaced by the following: *A function is physically computable only if it is α-computable for a* countable *ordinal α.* This is a canonical upper bound on the strength of physical computations insofar as one is committed to conditions (i) and (ii).

Condition (i) represents a commitment to a stepwise model of computation. If one has significantly different computational model, then the results described here may or may not hold.[6] Sacrificing condition (ii) would involve replacing our current best physical theories with new, as yet unspecified, physical theories. There is nothing inherently incoherent in supposing that spacetime has a nonstandard structure; but the results described here hold for any ordered set containing the rationals $\mathbb{Q}$ as a dense subset, and, as suggested by the thought experiments above, it is unclear that spacetimes where these results fail to hold can be made physically plausible.[7]

[6]There are other computational models one might adopt. One might, for example, take any dynamical evolution of a physical system to represent a computation. If so, one is only limited in computational power by the functions that might be realized in the trajectories of the physical states allowed by our best physical theories. But whether one wants to take something like this to be a *computation* depends on one's use of the notion.

[7]We would like to thank Philip Welch, Brian Skyrms, David Malament, Louis Narens, and John Manchak, for helpful discussions. We would also like to thank the two referees for their comments.

References

[1] W. Aitken and J. A. Barrett (2004) "Computer Implication and the Curry Parado", *Journal of Philosophical Logic* 33: 631-637.

[2] W. Aitken and J. A. Barrett (2007) "Stability and Paradox in Algorithmic Logic", *Journal of Philosophical Logic* 36: 61–95.

[3] W. Aitken and J. A. Barrett (2008) "Abstraction in Algorithmic Logic", *Journal of Philosophical Logic* 37: 23–43.

[4] J. Earman 1986, *A Primer on Determinism*, Dordrecht: Reidel.

[5] J. D. Hamkins and A. Lewis (2000) "Infinite Time Turing Machines", *Journal of Symbolic Logic* 65(2):567–604.

[6] M. Hogarth (2004) "Deciding Arithmetic Using SAD Computers", *British Journal for the Philosophy of Science* 55: 681–91.

[7] P. Koepke (2007) "α-Recursion Theory and Ordinal Computability", In *BIWOC - Bonn International Workshop on Ordinal Computability*, Ioanna Dimitriou, Joel Hamkins, and Peter Koepke, eds., Bonn Logic Reports (2007), 48–55.

[8] P. Koepke (2005) "Turing Computations on Ordinals", *Bulletin of Symbolic Logic* 11:377–397.

[9] P. Koepke and R. Miller (2008) "An Enhanced Theory of Infinite Time Register Machines", in *In Logic and Theory of Algorithms*, Lecture Notes in Computer Science 5028 (2008), 306-315.

[10] P. Koepke, R. Siders (2008) "Register Computations on Ordinals", Archive for Mathematical Logic 47: 529–548.

[11] P. Koepke, R. Siders (2006) "Computing the Recursive Truth Predicate on Ordinal Register Machines", in A. Bechmannn, et. al. (eds.), *Logical Approaches to Computational Barriers*, Computer Science Report Series 7: 160–169.

[12] P. Koepke and P. Welch (2009) "The Strength of Infinite Time Register Machines", forthcoming.

[13] G. E. Sacks (1990) *Higher Recursion Theory*, Perspective in Mathematical Logic. Springer-Verlag: Berlin-Heidelberg (1990).

[14] P. D. Welch (2008) "The Extent of Computation in Malament-Hogarth Spacetimes", *British Journal for the Philosophy of Science* 59: 659–74.

[15] P.D. Welch (2009) "Discrete transfinite computation models" in B. Cooper and A. Sorbi (eds.) *Unconventional Computation*, forthcoming.

FIGURES

TIME	SPACE										
	0	1	2	3	4	5	6	7	$\cdots$	n	$\cdots$
0	1	0	0	1	1	1	0	0		0	
1	0	0	0	1	1	1	0	0		0	
2	0	0	0	1	1	1	0	0		0	
3	0	0	1	1	1	1	0	0		0	
4	0	0	1	1	1	1	0	0		0	
5	0	0	0	1	1	1	0	0		0	
6	0	0	0	1	1	1	0	0		0	
7	0	0	0	1	1	1	0	0		0	
$\vdots$											
t	0	0	0	1	0	1	0	0		0	
$t+1$	0	0	0	1	1	1	0	0		0	
$\vdots$											

Figure 1: A standard Turing machine with head position indicated by shaded regions

α TIME	α SPACE										
	0	1	2	3	4	5	$\cdots$	ω	$\cdots$	θ	$\cdots$
0	1	0	0	1	1	1		0		0	
1	0	0	0	1	1	1		0		0	
2	0	0	0	1	1	1		0		0	
3	0	0	1	1	1	1		0		0	
$\vdots$											
t	0	0	0	1	0	1		0		0	
$t+1$	0	0	0	1	0	1		0		0	
$\vdots$											
ω	0	0	0	1	0	1		0		0	
$\omega+1$	0	0	0	1	0	1		0		0	
$\vdots$											
$\theta<\alpha$	0	0	0	0	0	0		1		0	
$\vdots$											

Figure 2: An α-Turing machine

Cal. State, San Marcos, CA 92096, USA
E-mail address: waitken@csusm.edu

UC Irvine, Irvine, CA 92697, USA
E-mail address: jabarret@uci.edu