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Berkeley, Ancient Geometry and Infinite Divisibility

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**Abstract**

It has been argued that Berkeley's arguments against infinite divisibility rest upon his misunderstanding of convergent sequences; and that they conflict with the classical result that incommensurability implies infinite divisibility. Here I argue that there is no such misunderstanding or conflict by revisiting ancient geometrical practice. This practice had the metrical notion of a "part" as a unit of measuring magnitudes that is distinct from the cardinal notion of "number of points" in a magnitude. The conflict between Berkeley's denial of infinite divisibility and incommensurability, which is taken as the standard objection to Berkeley's view, is also shown to be apparent by distinguishing the Aristotelian conception of mathematics from the Pythagorean conception of mathematics supplemented by the theory of proportions due to Eudoxus and Theaetetus.

**Keywords**: Berkeley, infinite divisibility, incommensurables, Pythagoreans, Aristotle

## 1 Introduction

Writing in the fourth century BCE, Aristotle wrote in *Physics* 200b15 – 200b21, "What is infinitely divisible is continuous;" and in *Physics* 207b16 – 207b21 the converse, "What is continuous is divi[sible] ad infinitum."[1] The Aristotelian view that magnitudes are infinitely divisible was endorsed by Isaac Barrow in the mid seventeenth century. Barrow was the first Lucasian Professor of Mathematics at Cambridge, a post later held by his student Isaac Newton.[2] In Lecture IX of his *Mathematical Lectures* he said:

> There is no part in any kind of magnitude, which is absolutely the least. Whatever is divided into parts, is divided into parts which are again divisible...whatsoever is continued is always divisible into parts again divisible. I am not ignorant, how difficult this doctrine is admitted by some, and entirely rejected by others.

Berkeley, who had certainly read Barrow in the early eighteenth century, was one of the dissidents. He wrote in *A Treatise Concerning the Principles of Human Knowledge* (PHK, henceforth) §123 (W2: 99)[3]:

> The infinite divisibility of finite extension, though it is not expressly laid down, either as an axiom or theorem in the elements of that science, yet is throughout the same everywhere supposed, and

[1]All references to Aristotle in what follows are taken from the translations contained in Aristotle (1984).

[2]For Barrow's view I have used Barrow (1734) *Mathematical Lectures* published posthumously in 1683.

[3]I shall follow contemporary Berkeleyan scholarship abbreviations where, for example, 'W2:99' refers to volume 2 page 99 of Luce and Jessop (1948 – 1957) *The Works of George Berkeley, Bishop of Cloyne.*

> thought to have so inseparable and essential a connexion with the principles and demonstrations in geometry, that mathematicians never admit it into doubt, or make the least question of it.[4]

Here I want to consider this debate between Barrow and Berkeley because it raises an issue regarding ancient geometrical practice that deserves closer attention than it has received. Specifically, I will focus only on evaluating the claim from PHK §123 that infinite divisibility is neither an axiom nor a theorem of Euclidean geometry.[5] One reason for focusing on evaluating this claim is suggested by an entry Berkeley made in his notebooks — *The Philosophical Commentaries* (PC, henceforth) 263 (W1:33) — as a reminder to himself:

> To Enquire most diligently Concerning the Incommensurability of Diagonal & side. whether it Does not go on the supposition of unit being divisible ad infinitum, i.e of the extended thing spoken of being divisible ad infinitum (unit being nothing also V. Barrow Lect. Geom:). & so the infinite indivisibility deduc'd therefrom is a petitio principii.

This passage invites us to look more closely into ancient geometrical practice because it suggests that, in disagreeing with Barrow, Berkeley himself looked carefully at the geometry textbooks being used at the time for any proof connecting incommensurability (the view that there are pairs of magnitudes whose proportion cannot be expressed in terms of a common measure) and infinite divisibility (Aristotle's view that magnitudes are fundamentally non-atomic or not composed of points).

The main question I am asking is this: assuming Berkeley had access to a reliable edition of *Euclid: The Thirteen Books of The Elements* (*The Elements*, henceforth), is there evidence for his assertion that the infinite divisibility of finite lines is neither an axiom nor a theorem in *The Elements*? I answer this question affirmatively and in doing so I will argue for a more nuanced and plausible reading of Berkeley's denial of infinite divisibility. My main contribution is to show that Berkeley was not obviously wrong to deny that the infinite divisibility of finite lines is an axiom or theorem in ancient geometry.[6] By "not obviously wrong" I mean that given the issues involved in articulating the thesis of infinite divisibility, it is possible that Barrow and Berkeley were talking past each other. On the one hand, Barrow appears to endorse an Aristotelian conception of mathematics and continuity (in terms of infinite divisibility) but does not distinguish between geometry and number theory (or arithmetic, in their terms). On the other hand, Berkeley rejects these Aristotelian conceptions but firmly distinguishes between geometry and number theory. This means that the debate between Berkeley and Barrow is quite nuanced.

[4]It is noteworthy that in the same Lecture IX, Barrow (1734, pp. 153, 155) claimed that even though mathematicians rarely *openly* assume infinite divisibility, they *covertly* assume it. My paper is a historico-philosophical investigation into whether or not this assumption was necessarily made in ancient geometry.

[5]See Jesseph (1993, Chap. 2) and Jesseph (2005) for how Berkeley's thought on this issue evolved from the *Philosophical Commentaries* to the PHK.

[6]For an opposing view see Fogelin (1988), Jesseph (1993, 48 – 53), Franklin (1994), and Jesseph (2005, 278 – 284). My paper is not intended as a critical evaluation these views, although along the way I identify the ways in which I differ. There is a lot that I have learned from Jesseph (1993) and there is a lot in his view that I agree with. My intention is to open up the possibility for fruitful debate regarding these matters by adding more distinctions than have been made in the literature to the historico-philosophical background that could have influenced Berkeley's philosophy of mathematics.

One way in which I intend to explain or to show this nuance involves making certain distinctions. Here I go further than the existing literature in (1) distinguishing between *parts* of a finite line and *points* in a finite line; and (2) distinguishing between two ways of measuring the size of lines within a dense point-set conception of a line: the *metrical* approach (i.e., in terms of number of units of distance) and the *cardinality* approach (i.e., in terms of number of points). These are important distinctions to make because Berkeley prefaces his arguments against infinite divisibility in PHK §124 with, "If the terms extension, parts, and the like, are taken in any sense conceivable." An evaluation of Berkeley will have to begin by looking at what ancient geometrical practice meant by "parts of a magnitude" and whether ancient geometers were committed to any particular theory about the metaphysical composition of continua (e.g., that they cannot be composed of points) in their practice. Once these distinctions and clarifications are made (as I do in sections 3 and 4 below), Berkeley's denial of infinite divisibility can be given a plausible *metrical* reading because it presupposes that magnitudes are composed of actual points. Aristotle, as is well known, denies this presupposition. So, in saying that Berkeley is not obviously wrong, I intend to develop this plausible metrical reading by exposing and criticizing the Aristotelian assumptions that went into formulating the thesis of infinite divisibility.

Textual evidence that motivates making these distinctions can be found in a passage from *An Essay towards a New Theory of Vision* (NTV, henceforth) §112 (with my emphasis):

> For by the distance between any two points, nothing more is meant than the number of intermediate points: If the given points are visible, the distance between them is marked out by the number of the interjacent visible points: *If they are tangible, the distance between them is a line consisting of tangible points*; [...] This, perhaps, will not find an easy admission into all men's understanding: However, I should gladly be informed whether it be not true, by any one who will be at the pains to reflect a little, and apply it home to his thoughts.

Although Berkeley appears to slide here between distance as "number of points" and distance as "a line consisting of tangible points", I will use this passage to provide a plausible reading of Berkeley's denial of infinite divisibility in the following way. For any two points $x, y$ in a finite line such that $x < y$, the Euclidean distance $d(x, y)$ or parts between these two points is some finite *number* of *units of distance*, where a unit of distance is a line of a certain length. The order relation $<$ can be defined using Postulate 1 (To draw a straight line from any point to any point) as follows: $x < y$ if the straight line is drawn from $x$ to $y$. The distance function $d(x, y)$ is defined using Postulate 3 (To draw a circle with any centre and distance) as the length of diameter $\overline{xy}$ of the circle. A finite line, according to Berkeley, is not infinitely divisible because it has a finite number of *parts* or units of distance — its length; but it would be wrong to infer from this that a finite line has a finite number of *points*. This is why discussing in detail what 'parts' and 'points' mean is important for giving a plausible reading of Berkeley's claim in PHK §§123 – 124.

Another way in which I intend to explain or to show the nuance involved in Berkeley's view involves showing that when Aristotle says continuous magnitudes are infinitely divisible, he means two different things, which are not necessarily equivalent. He means, first, that magnitudes are not composed of *actual* mathematical atoms, geometrical minima or points. Call this the *philosophical thesis* of infinite divisibility. This is tied to the synonymy he thinks there is between continuity and infinite divisibility. But he also means, or at least this is how others have read him, that bisections of a given line can be done an indefinite (potentially infinite) number of times — this is taken to follow from *The Elements* Book I Proposition 10 (To bisect a given finite straight line). Call this the *mathematical thesis* of infinite divisibility. The problem is not only that these two formulations of infinite divisibility are not necessarily equivalent, but also that Aristotle uses the mathematical thesis of infinite divisibility to support the philosophical thesis.[7] But there is nothing about the mathematics or theorems in *The Elements* that suggests continua are fundamentally non-atomic or not composed of points with distance and order relations. Part of what I do in subsections 3.2 and 3.3 below is to show that when Berkeley denies the infinite divisibility of finite lines, he is rejecting the philosophical thesis and consequently Aristotle's mathematical argument for it.

There is another mathematical argument for infinite divisibility, which Berkeley was clearly aware of given the quote from the notebooks: incommensurability implies infinite divisibility. I meet this objection to the plausible reading I am offering by distinguishing Aristotle's conception of mathematics from the Pythagorean conception of mathematics in section 4. Here I argue that it is Aristotle's *philosophical* conception of mathematics (specifically his taxonomy of quantities) that exerted the most influence on later mathematicians, including Barrow, who took infinite divisibility to be the explanation of the Pythagorean *number theoretic* discovery of incommensurability. But if we distinguish geometry from arithmetic (as Berkeley does) and use the Pythagorean conception of mathematics (i.e., number-monism and its taxonomy of quantities) then one can plausibly deny the truth of infinite divisibility in geometry while accommodating the incommensurability results in number theory.[8]

Overall, the reason I focus on ancient geometrical practice and discuss in detail the alternative conceptions of mathematics, is that I want to offer evidence, on Berkeley's behalf, for the assertion in PHK §124:

> Ancient and rooted prejudices do often pass into principles; and those propositions, which once obtain the force and credit of a principle, are not only themselves, but likewise whatever is deducible from them, thought privileged from all examination.

For if I am right that infinite divisibility is an Aristotelian philosophical thesis, then Berkeley's denial of infinite divisibility becomes more plausible when these Aristotelian assumptions or "ancient prejudices" are

[7] I will not discuss the issue of whether Aristotle failed to distinguish between mathematical atomism from physical atomism, the view that all there is are atoms and void. The claim that he may have done so has been made by Garber (1992, 123) and the sources cited there. Aristotle argues as if a mathematical thesis — the convergence of geometric series — must be true of the real world in order to solve Zeno's paradoxes. See the discussion by Heath in Heath (1956, Vol. 1, 233f.) as well.

[8] See section 4 below for a more careful discussion.

examined.

In summary, my argument for a plausible and nuanced reading of Berkeley's denial of infinite divisibility involves showing that the long-held view of identifying infinite divisibility with continuity is a view due to Aristotle's alternative conception of mathematics and its practice (section 3 and 4 below). How much Aristotle's conception differed will become clear when I compare it with the Pythagorean conception in section 4 below. The upshot of this comparison for my plausible reading of the claims in PHK §§123 and 124 is that by denying that finite lines are infinitely divisible, Berkeley was drawing on a conception of geometry that was more faithful to *The Elements* than was Aristotle and Barrow's conception. The faithfulness involves, among other things, abandoning Aristotle's philosophical thesis of infinite divisibility, which involves *potentially* existing points; and adopting the alternative view of construing geometrical magnitudes as composed of *actual* points or what Berkeley called *geometrical minima*.

Here's how I have organized my paper. In the sections 2 – 3.2 I clarify the strategy or approach I will be taking in my paper and introduce the terms which set the debate regarding infinite divisibility. In sections 3.3 – 4.4, I show how Aristotle's conception of mathematics (i.e., his taxonomy of quantities into numbers and magnitudes) led him to reconceptualize continuity — a process which, among other things, culminated in the characterization of continuous quantities in terms of infinite divisibility. I also show how, and why, someone who ascribes to the Pythagorean conception of mathematics (only numbers are quantities) together with the theory of proportions by Eudoxus, can work without the philosophical thesis of infinite divisibility in order to meet the objection to my reading based on the existence of incommensurable magnitudes.

## 2 Some comments on my strategy

The strategy I have chosen is to go back to *The Elements* in order to look for any evidence that refutes or substantiates Berkeley's claim in PHK §123. The authoritative English translation of *The Elements* I have used is the three volume Heath (1956) based on the Greek edition of the Danish philologist Johan Ludvig Heiberg. There is one worry about this strategy, however. The historian writing today, as De Risi (2016) has shown, will have to come to terms with the numerous translations, modifications, and editions of *The Elements*. Consider, for example, the second postulate in *The Elements* Book I. Of this postulate, De Risi (2016, 15) writes:

> A few postulates changed their formulation in different editions of the Elements. In particular, Postulate 2 on the extendibility of a straight line states, in its original form, that the straight line may be produced continuously (*κατα το συνηχες*). Already in the translation of Adelard of Bath (from the Arabic), however, the postulate is altered so as to make it into a statement about the extendibility of a straight line to an arbitrary length ("assignatam lineam rectam quantolibet spacio directe protrahere"), dropping the aspect of continuity and stressing rather the length of

the extension.

As is well known, some version of this postulate was referred to by Aristotle in his arguments against the actual infinite in *Physics* Book III 207b28 – 207b34:

> Our account does not rob the mathematicians of their science, by disproving the actual existence of the infinite in the direction of increase, in the sense of the untraversable. In point of fact they do not need the infinite and do not use it. They postulate only that a finite straight line may be produced as far as they wish. It is possible to have divided into the same ratio as the largest quantity another magnitude of any size you like. Hence, for the purposes of proof, it will make no difference to them whether the infinite is found among existent magnitudes.

Given that Aristotle's views on the actual and potential infinite are intertwined with his views of continuity and infinite divisibility, this is an example of how modifications of the original text make it difficult to say what ancient geometrical practice assumed. This difficulty is compounded by the worry about whether Aristotle, Barrow and Berkeley were even using the same textbook I am calling *The Elements*. Aristotle views on infinite divisibility certainly predate the text thought to have been compiled by Euclid into what is now *The Elements*. Whereas I will argue below that infinite divisibility of finite lines does not follow from *The Elements* Book I Proposition 10 (To bisect a given finite straight line), it is quite possible that the geometry textbook Aristotle used to develop his views was a different geometry textbook authored Theudius, which drew the conclusion of infinite divisibility. We may never know how Theudius's textbook compares to Euclid's because the Theudius textbook is lost. As we shall see later, there is also the worry about whether there was a common conception of mathematics and mathematical practice in ancient geometry. Aristotle's conception of mathematics, for example, is developed in opposition to the Pythagoreans. According to the Pythagoreans, there was only one species of quantity: number. The Pythagoreans sought to give their number theory a geometric foundation but failed because of their discovery of incommensurability. We may speculate that it is these and allied reasons that led geometers (and philosophers like Aristotle) to begin distinguishing between two species of quantity: magnitudes (dealt with in geometry) on the one hand and numbers (dealt with in arithmetic) on the other. Infinite divisibility, it was thought, applies to magnitudes (due to the incommensurability results) not numbers. Centuries later, Barrow and Berkeley pick up this issue of the relationship between geometry and arithmetic. On the one hand, one of Barrow's central claims in his *Mathematical Lectures* was that there is no distinction between arithmetic and geometry. Berkeley, on the other hand, believed that there is a distinction between arithmetic and geometry. I return to this problem in section 4 because I believe that the conception one has of mathematics matters in evaluating Berkeley's claim.

Given these worries, I have made the following decisions in my paper. First, I will *not* assume that the textbook used by Aristotle in developing his views of continuity and infinite divisibility is identical, in terms of mathematical content, with the textbook, whose authorship we now attribute to Euclid. But this need not deter us from pushing ahead with this inquiry. Relying on historical sources on the practice of ancient geometry such as Simplicius and Proclus, we can get a sense of what the common core was across all the different texts. Second, with regard to Berkeley and Barrow I will assume that they both had access to the same geometry textbook. Here are the editions we can assume they both had access to. First, Barrow's own 1655 Latin *Euclidis Elementorum Libri XV breviter demonstrati* or its 1660, 1705, 1722 and 1732 English Editions. These Barrow translations are based on the singularly important Latin translation by Commandinus (1509 – 1575) of Urbino.[9] Commandinus followed the original Greek more closely than his predecessors and most of the subsequent translations (for example into Latin by Clavius) are based on Commandinus's work. If the common textbook they both had access to was not any of Barrow's Latin or English translations, then I will assume that it was the 1703 Oxford edition by David Gregory which, until the issue of Heiberg and Menge, was still the only Greek edition of the complete works of Euclid. Of this Gregory edition, De Risi (2016, 75) says, "The system of principles [...] is left unmodified." Thus, if we assume that it was the Gregory edition that Berkeley would have used in the claim at PHK §123, then I am justified in saying that Berkeley had a reliable source of what ancient geometry entailed comparably similar to the one we have today in Heath (1956).

# 3 What does infinite divisibility mean for Berkeley?

## 3.1 Do infinitely many parts of a whole W imply that W is infinite?

Following his remark that the thesis of infinite divisibility is neither assumed nor proved in *The Elements*, Berkeley gives a surprising argument for this view. As we shall see below, the argument has roots in the Epicureans. It claims to show that what is infinitely divisible must contain infinitely many parts and consequently be infinitely large. I propose that we start here and work backwards to substantiate Berkeley's initial remark in PHK §123.

> If the terms extension, parts, and the like, are taken in any sense conceivable, [...] then to say a finite quantity or extension consists of parts infinite in number, is so manifest a contradiction. PHK §124 (W2: 99)

> [W]hen we say a line is infinitely divisible, we must mean a line which is infinitely great. PHK §128 (W2: 101)

There are really two arguments here although in the second quotation it is implicit in the talk of meaning. In the first argument Berkeley is arguing that from the supposition that a finite line (he uses 'extension') consists of infinitely many lines, it follows that the original line is infinite. But the original line is finite.

[9] See Heath (1956, Vol. 1, p. 104 – 110).

Hence it follows by reductio that our supposition was wrong. The second implicit argument in the claim in the second quotation is that if a line is infinitely divisible, then we must mean a line which is infinitely long. Since it is common ground that there are no infinitely long lines, it follows by reductio again, that our supposition was wrong.[10]

In evaluating Berkeley's argument, most scholars point out that Berkeley is missing the obvious property of infinite geometric series: *convergence*.[11] A geometric sequence is a sequence $s_n$ with $n \in \mathbb{N}$ of terms with a common ratio $|\frac{s_{n+1}}{s_n}| = |r| < 1$ between successive terms ($s_n$ and $s_{n+1}$). An infinite geometric series involves summing all the terms in an infinite geometric sequence. We say that the infinite geometric series *converges* to finite sum or number if the sequence of partial sums $S_n = \sum_1^n s_n$ converges. Today, with a point-set conception of the continuum, the convergence of an infinite sequence is proved by showing that after a *finite* large number $M$, all subsequent terms $s_m$ with $m > M$ are so close to each other that they are virtually indistinguishable, i.e., the distance between them is almost negligible or there is "no part" between them. So beyond $M$, the Euclidean distance $d(s_m, s_{m+1})$ between any two successive terms adds nothing significant to the already total finite units of distance of the terms before $M$. This is why a geometric series converges to a finite number. For example, the sum of the terms in the geometric sequence $\langle 1, \frac{1}{2}, \frac{1}{4}, \ldots, \frac{1}{2^n}, \ldots \rangle$ for $n \in \mathbb{N}$ is 2. This series converges since the common ratio $|r| = \frac{1}{2} < 1$.

But I don't think that Berkeley is ignorant of the existence of convergent geometric series or denying the theorems which support them. First, in query 53 of *The Analyst* he asks, "Whether, if the end of geometry be practice, and this practice be measuring, and we measure only assignable extensions, it will not follow that unlimited approximations completely answer the intention of geometry?" The convergence of geometric series is essentially an approximation or *limit* procedure, where a mathematician claims that the sum of the infinite series can be approximated by a *finite* partial sum to any given error $\epsilon > 0$. Therefore, the query in *The Analyst* suggests to me that Berkeley would assent to the existence of geometric series.

Second, *The Elements* does not discuss the notions of convergence and divergence of infinite series. The mathematics of infinite series was made precise in the 19th century with the work of Cauchy and Weierstrass aimed at rigorously reformulating analysis.[12] So pointing out the existence of convergent sequences, supports rather than refutes Berkeley's argument that infinite divisibility is not in *The Elements*.

Moreover, according to commentators, it is Aristotle, not *The Elements*, who was one of the first to use

[10]Initially I had thought that Berkeley wanted to give a complicated or novel instrumentalist interpretation of infinite divisibility based on his doctrine of signs. I have come to see that was not his intention at all. In PHK §§125 – 128 *he is offering one possible explanation*, on the basis of his representative theory of generality, for what might have led the mathematicians, erroneously, to suppose the thesis of infinite divisibility. It does not mean that this explanation is the right one nor does it mean that this is the only way to construe the thesis of infinite divisibility or lead to its acceptance within mathematics. Cf. Jesseph (1993, 72 – 74) who thinks Berkeley is offering an instrumentalist account.

[11]This is a point that Fogelin (1988, 52 – 53) and Franklin (1994) make.

[12]The terms 'convergent' and 'divergent' were used by James Gregory in 1668 but he did not develop the ideas. Newton only affirmed that power series converge for small values of the variable and for the geometric series. Leibniz showed that series whose terms alternate in sign and decrease in absolute value monotonically to zero converge. See Kline (1972, Vol. 2, 461) and for Cauchy and Weierstrass see Kline (1972, Vol. 3, 948, 952, 963ff.).

"convergence of a geometric sequence" synonymously with "potentially infinitely divisible" as a response to Zeno's paradoxes.[13] Zeno in supposing that magnitudes are infinitely divisible, intended this to imply that the magnitudes are actually divided into infinitely many parts. He did not intend it in the restricted Aristotelian sense of merely potentially infinitely divisible. One way of understanding his paradoxes was that a *supertask* would have to be completed. A supertask is a task involving actually infinitely many steps completed in finite time.[14] For example, in the Dichotomy Paradox, the motion can never begin because to start from the beginning of an interval to the half-way point of the interval, one would have to traverse an actually infinite number of monotonically decreasing intervals of space approaching the beginning of the interval. So if someone merely pointed out to him that a geometric series converges as Aristotle did — speaking in terms of potential infinite divisibility — Zeno would have been unconvinced that this solves the Dichotomy Paradox. In other words, Aristotle shows *why* motion is possible (namely that the sum of a convergent series is finite) not *how* it is possible (how can actually infinite many steps be completed in a finite time). Here we see one way in which Aristotle's *philosophical* thesis of potential infinite divisibility became associated with a *mathematical theorem* that asserts the existence of convergent geometric sequences. The other way has to do with Aristotle's reconceptualization of continuity which I discuss below.

The Port Royal Logicians (Antoine Arnauld and Pierre Nicole), Isaac Barrow and John Keill[15] continued this Aristotelian thought in the late seventeenth century and early eighteenth century, arguing that the convergence of geometric series counts as a reason in favor of infinite divisibility. Berkeley clearly read these mathematicians' work as evidenced by his notebook entries.[16] The intellectual (albeit virtual) exchange between Barrow and Keill on the one side supporting infinite divisibility in mathematics and Berkeley on the other denying it has been discussed in detail in Jesseph (1993, 63 – 67). However, one argument that Jesseph does not discuss is the argument based on the convergence of geometric series. Barrow (1734, 157), in *Lecture IX*, puts it this way:

> [I]t is plainly taught and demonstrated by Arithmeticians, that an infinite series of fractions, decreasing in a certain proportion, is equal to a certain number; e.g. that such a series of fractions decreasing in a *subsesquialter* proportion is equal to two, in a *subduple* proportion to unity, in *subtriple* to one half; from whence it is not inconsistent for something finite to contain in it an infinity of parts.

Since Berkeley read these authors, it is not true that he was ignorant of the possibility that convergent geometric series would be counterexamples to his view that any magnitude that contains infinitely many parts must be infinite.

[13]See Heath (1956, Vol. 1, 233 – 234) for discussion. For the reference in Aristotle see *Physics* 206b4 – 206b12.

[14]For the supertasks reading of Zeno's paradoxes see Black (1967) and Manchak and Roberts (2016).

[15]Keill became the Savilian Professor of Astronomy in Oxford in 1712. For Keill's view I have used Keill (1745) *An Introduction to Natural Philosophy: or Philosophical Lectures Read in the University of Oxford 1700 A.D.*

[16]See PC 263 W1:33, PC 308 W1:38, PC 462 W1:57.

### 3.2 What Berkeley means by finite divisibility

Why, then, would Berkeley have been convinced that his arguments were sound? In the previous subsection I rejected the view that Berkeley was simply ignorant of convergent geometric series (a series with infinitely many parts yet finite in size). So in order to see the soundness of Berkeley's argument against infinite divisibility, we need to look more carefully at the distinction between parts and points, and what infinite divisibility (or its denial) by Berkeley even means. Let me begin with what Berkeley's denial of infinite divisibility of finite lines means.

Many have read Berkeley's denial of infinite divisibility as entailing that for every line $L$ there is a finite number of divisions $n \in \mathbb{N}$ that can be done on $L$ such that for all $m > n \in \mathbb{N}$, $L$ is not divisible further. I am indexing by the natural numbers $\mathbb{N}$ because I am assuming that infinite divisibility is decidable or a constructive procedure. But Book I Proposition 10 implies there are no indivisible lines (see the more detailed discussion in §3.4 below). Therefore, on this reading, Berkeley's denial of infinite divisibility contradicts this well established theorem.

The problem with this reading is that it assumes that Berkeley's indivisibles for $m > n \in \mathbb{N}$ are other lines with parts. But we know that in the seventeenth century, it was a matter of controversy what the indivisibles of a line were. From Leibniz's work on continuity, we know that there were at least three candidates for the indivisibles of a line: other lines, infinitesmals, or minima, i.e., Euclidean points.[17] Berkeley's indivisibles, or what he calls geometrical minima (minimal parts which compose of a geometrical magnitude), are *points with distance and order relations*.[18] Therefore, the well known problems of indivisible lines, which would arise on this reading, including the conflict with Book I Proposition 10, can be avoided because Berkeley doesn't believe there are indivisible lines.

At the same time, his denial of infinite divisibility doesn't entail that there are a finite number of points in a line. The notion of cardinality which we get later with Cantor — such that a line has *actually infinitely* many points — is not a notion Berkeley and his contemporaries would have been familiar with. Berkeley consistently uses the notion of parts, not points, when discussing the *number* of parts in a finite line. Recall the passage from NTV §112 quoted earlier:

> For by the distance between any two points, nothing more is meant than the number of intermediate points: If the given points are visible, the distance between them is marked out by the number of the interjacent visible points: If they are tangible, the distance between them is a line consisting of tangible points.

Here Berkeley appears to say that distance is the number of intermediate points but also a line consisting of points. In my opinion, the most plausible, or charitable way of reading Berkeley's denial of infinite

[17] See Arthur (2001, xxxiii – xxxv)

[18] See especially NTV §§54 – 61, 112; PHK §127 and *De Motu* §14. This view, I argue has roots in Pythagoras, Epicurus and Gassendi. Gassendi is mentioned by Berkeley in NTV §75 and Epicureanism is mentioned by Berkeley in PHK §93.

divisibility given what he says in this passage consists of two parts. First, one must take Berkeley to be assuming an actual ordered dense point-set conception of a line. The notion of order and density (between any two points there is another point) is suggested by Berkeley's talk of "intermediate points." If Berkeley's assumption is granted, then one can say that Berkeley is analyzing the finite divisibility of lines in *metrical* terms (i.e., units of distance) not in terms of *cardinality* (i.e., number of points). This is the nuance in his view and why it is not obviously false. The metrical approach is implied by "distance between them is a line consisting of tangible points." The line represents the total units of distance or length according to some scale.

This reading of Berkeley's analysis can be justified by looking at *The Elements* as well. The actual ordered dense point-set conception of a line is justified by Postulate 1 (To draw a straight line from any point to any point) and the metrical approach is justified by Postulate 3 (To draw a circle with any centre and distance). Taking all of these assumptions together means that when Berkeley says that finite lines are *finitely* divisible, he means that for any two points $x, y$ in a finite line such that $x < y$, the sum of the Euclidean distance $d(x, y)$ or parts between these two points is some finite *number* of *units of distance*. As already mentioned, the order relation $<$ can be defined using Postulate 1 as follows: $x < y$ if the straight line is drawn from $x$ to $y$. The distance function $d(x, y)$ can be defined using Postulate 3 as the length of diameter $\overline{xy}$ of the circle. Therefore, one plausible way to read Berkeley's claim and argument in PHK §124 is that the number of units of distance, or parts, in a finite line can only be finite. Suppose otherwise and take the end points $a$, $b$ of a finite line, then there is an infinite *number* of units of distance between these points. But this is absurd unless the line is infinite. For example, take the points to be the boundaries of the finite line contained within the interval $[5, 10]$. Today we know that there are the cardinality of the real numbers many points in this interval. But once one adopts the Euclidean distance, then there is a total of 5 units of distance or parts.[19]

Thus, by denying infinite divisibility, Berkeley wants us to draw at least two conclusions. First, the sum of the $d(x, y)$ units of distance between any points $x, y$ such that $x < y$ in a finite line is finite — it is infinite only if the line is infinite. Secondly, Berkeley's denial of infinite divisibility presupposes that lines can be composed of indivisible points or geometrical minima since the philosophical thesis of infinite divisibility is equivalent to the thesis that continuous quantities are fundamentally non-atomic. Here's how Aristotle expresses this thesis:

> Nothing that is continuous can be composed of indivisibles: e.g. a line cannot be composed of points, the line being continuous and the point indivisible [...] [I]t is plain that everything continuous is divisible into divisibles that are always divisible; for if it were divisible into indivisibles,

[19]There's a strong parallel to Berkeley's thinking in Bolzano. Bolzano realized that there is a one-to-one correspondence between the real numbers in the interval $[0, 5]$ and the interval $[0, 12]$ given by $y = \frac{12x}{5}$ a function from $[0, 5]$ to $[0, 12]$ but was still reluctant to accept that these two intervals have the same "size." We can explain Bolzano and Berkeley's puzzlement because they were thinking about size in distance or metrical terms (assuming an ordered dense point-set conception of a line) rather than in terms of cardinality.

> we should have an indivisible in contact with an indivisible, since the extremities of things that are continuous with one another are one and are in contact. The same reasoning applies equally to magnitude, to time, and to motion: either all of these are composed of indivisibles and are divisible into indivisibles, or none [of these are]. If time is continuous, magnitude is continuous also [...] If time is infinite in respect of divisibility, length is also infinite in respect of divisibility.
>
> Aristotle, *Physics* Book VI 231a18 – 231a20; 231b16 – 232a17; 233a13 – 233a21

Here we see that Aristotle's thesis of infinite divisibility applied to magnitudes (or lines) is equivalent to the thesis that lines cannot be composed of points. This means that by denying infinite divisibility, Berkeley is rejecting at least this *particular* meaning of the Aristotelian view, which we called earlier the *philosophical thesis* of infinite divisibility. I have not yet discussed what implications this has for the mathematical thesis of infinite divisibility: Book I Proposition 10 (To bisect a given finite straight line) — I will return to it below. What I have shown is that when Berkeley says lines are finitely divisible, he presupposes, *contra* Aristotle, that there are *actual* and dense mathematical atoms or points in a line, and he is analyzing the number of divisions of a line (into parts) in terms of number of units of distance between its points. What we need to find out is whether the picture I am painting is consistent with *The Elements* as Berkeley or Barrow would have read it and whether Berkeley is right that infinite divisibility does not follow from anything in *The Elements*. The answer depends on what *The Elements* meant by parts and points.

### 3.3 What are parts and points?

Kline (1972, Vol. 3, 1008) notes that one criticism that Moritz Pasch made of *The Elements* had to do with *The Elements*'s definitions of 'point' and 'part.' In fact, the talk of parts ought to remind us of the starting point of *The Elements*. Famously, *The Elements* begins with the definition of a geometrical point (points, henceforth) as "that which has no part." But what does having "no part" mean? This can only be understood if we understand what "part" and "parts" are.

One possibility for understanding what these terms mean can be found in the geometrical Book V of *The Elements*, whose theory of proportions is attributed to Eudoxus. In this book we read (my emphasis), "A magnitude is a *part* of a magnitude, the less of the greater, when it *measures* the greater." Taking this geometrical characterization together with what we find in the arithmetical books (Books VII, VIII, IX of *The Elements*), we may say that *a part of a magnitude or number* is what we call today a *factor* or *integral divisor* according to some unit of distance or measure. *Parts* (plural) are what we call today a *fraction*, although ancient geometers did not make use of expressions which we use today when we talk about fractions.[20] In both the singular and plural case, what we have is a *number* that represents a measure

[20] Cf. Heath (1956, Vol. 2, 115) for Heath's discussion of parts of a magnitude and Heath (1981, Vol. 1, 42) for a discussion on fractions. The introduction of fractions as denoting quantities or real numbers had to wait until 1500 or so. See Kline (1972, Vol. 1, 251).

or what I am calling units of distance. Therefore, this first possibility of understanding what *The Elements* means by ‘part’ or ‘parts’ is consistent with how Berkeley understands the notion of part or parts of a line — it is a metrical notion.

The other mention *The Elements* makes of ‘parts’ is in the discussion of the *Common Notions*. Common Notions were self-evident truths with such widespread acceptance that most people adopted them without proof. Some of these common notions were “Things which are equal to the same thing are also equal to one another” (Common Notion 1) and “The whole is greater than the part” (Common Notion 5).[21] For my purposes, the relevant way of reading what *The Elements* means by the part-to-whole relation involves homogeneity, the property of two or more things being similar in some respect. We may say that $A$ is a part of a whole $B$ iff $A$ is homogeneous with $B$ but not equal to (strictly less than) $B$.[22] In *The Elements* and most ancient geometers, the part-to-whole relation in geometry is a relation between homogeneous *quantities* since it is only the category of quantity, according to Aristotle, that admits of the relation equal-to, less-than or greater-than.[23] For example, the parts of a (whole) line will be other (homogeneous) lines. A part of a (whole) multitude, such as a collection of coins, will be another smaller collection of coins.

Applying this to Berkeley, the relevant sense of homogeneity, I claim, is the measure or *unit of distance*. He writes in *De Motu* §14, “To prove that some quantity is infinite, one must show that some finite homogeneous part is contained in it infinitely many times.” Therefore, in NTV §61, PHK §§127 − 128, the relevant sense of ‘part’ in these passages can plausibly be read in terms of *number* of units of distance in a finite line (or whole).[24] The finite line represents a number of units distance, which is the union or sum of its parts. Therefore, the nuance in Berkeley’s denial of infinite divisibility, which makes it plausible, is that it requires a notion of ‘parts’ which presupposes a homogeneous definite measure or unit of distance, which we saw in the first possibility discussed earlier. It does not imply that a finite line has a finite number of points.

### 3.4 Can a magnitude be composed of points?

The preceding discussion of parts is relevant for my evaluation of the assumptions in Aristotle’s philosophical thesis of infinite divisibility and for lending some credibility to Berkeley’s denial of it. In this subsection, I show how this works.

First, *The Elements*’s definition of a point does not, *by itself*, tell us anything about the divisibility (or lack thereof) of points. The definition clearly does not mention divisibility. Second, *The Elements*’s

[21]Despite this widespread belief in antiquity, today it is hard to accept the fifth common notion. We know, for example, that in the case of infinite sets, the whole is not necessarily greater than the part. While this result would have been paradoxical to Galileo, Leibniz and Berkeley, it would be anachronistic to try to refute their view on the basis of modern developments in mathematics. See Mancosu (1996) for Galileo and Leibniz’s grapplings with the actual infinite. For the actually infinite point-set conception of the geometric continuum, which is justified by a *cardinality* approach, beginning most explicitly with Cantor and Dedekind, see Kanamori (2020).

[22]See *The Elements* Book V and Book VII also.

[23]“[The] most distinctive of a quantity would be its being called both equal and unequal.” *Categories* 6a26 – 6a36.

[24]I am using “units of distance” as a general expression for Berkeley’s “inches” and “feet”.

definition does not deny that a point has what Plato called *onkos* (roughly, size or volume; more of this below). I will take this to mean that *The Elements*'s definition of a point does not deny that a point has *minimal size*.[25] So saying that there is nothing less than a point which is homogeneous to it does not entail that a point is nothing (i.e., that it has no size) as Hume famously thought in considering alternatives to his position. Hume considered his position (there are minima with color and solidity) to be the middle ground between infinite divisibility and mathematical points.[26] What it does entail is that there is nothing *smaller* than a point which is homogeneous to it.[27]

So where did this pervasive characterization of points as being indivisible originate? I claim that this identification of a point with the indivisible started with Aristotle's reconceptualization of continuity. This conception was developed in order to refute the physical atomists (there are indivisible physical atoms that compose matter). In doing so, Aristotle connects the physical atomist thesis (the view that all there is are atoms and void) with what I will call the *mathematical atomist* thesis (magnitudes are composed of actual points with distance and order relations between them). The sixth century CE Neoplatonist Simplicius, one of the few extant ancient commentators on Aristotle's *Physics*, has this to say in his commentary on Aristotle's *Physics* Book VI (this is the book that deals with continuity):

> Aristotle set up the logical division of the divisible into either indivisibles or forever divisibles, so that he might comprise the continuous in that which is divisible into forever divisibles. Simplicius (2014, 23f) Trans. lines 931, 5 – 10 in MSS.

More recently, Miller, Jr. (1982, 88) has written,

> Aristotle reformulated the old difficulties in his own terms and defined concepts in order to resolve them...He presents his own theory of the continuum as the only way out of an ancient dilemma which seeks to show the absurdity of continuous magnitudes.

I return to the dilemma in a moment. The important take away, for now, from Simplicius and Miller, Jr., is that Aristotle was reconceptualizing the debate with the physical atomists and that this involved identifying the continuous with the infinitely divisible.

The identification of the point with the indivisible is again stated in Aristotle's *Metaphysics* 1016b18 – 1016b30. Here he writes:

[25] There are complications with my attempt to reconstruct what Euclid might have meant by 'point.' These complications arise in view of recent developments in measure theory where a point is, indeed, assigned measure 0. I will simply flag this for now and direct interested readers to Skyrms (1983) for an excellent introduction to measure theory in the context of some of the issues I am discussing here.

[26] See *A Treatise on Human Nature* II.iv. The literature on Hume and infinite divisibility is vast. Good places to start are Jacquette (1996), Pressman (1997) and Holden (2002).

[27] It would be interesting to see whether my analysis is consistent with what the early modern philosophers, following Descartes, express in terms of lacking extension. See Arnauld, Antoine and Nicole, Pierre (1996, 231 – 232) where the words "zero extension" are used. In *Rules for the Direction of the Mind* Rule XIV it is noteworthy that Descartes disentangles his notion of 'extension' from 'quantity.' In Meditation V, quantity is only applied to continuous quantity and Descartes speaks of extended quantity. It is a thorny issue to try to understand what extension is for Descartes so I will not get into that here. See Garber (1992) especially Chapter 3 and 5.

> But everywhere the one is indivisible either in quantity or in kind. That which is indivisible in quantity and qua quantity is called a unit if it is not divisible in any dimension and is without position, a point if it is not divisible in any dimension and has position.

It is not clear who Aristotle's sources were for this characterization of points and units.[28] What we do get clearly from Aristotle is one way to conceive of mathematical points is that they are indivisible. But if they are indivisible, does it follow that the points have no *onkos* or are nothing? '*Onkos*' is a technical term used in different contexts — some of these contexts are theatrical. In the mathematical contexts, ancient scholars[29] vary in translating '*onkos*' as volume, measure or simply spatial extension and are divided on this question and what implications it has for our conception of points with respect to divisibility.[30]

Laying aside the difficulty of how to translate or understand *onkos*, this question raises a dilemma. On the one hand, if someone says that the mathematical points have no size or spatial extension, then they are "nothing" and cannot compose a magnitude. The assumption here is that a magnitude is composed of other (homogeneous) magnitudes with size. On the other hand, if one says that the mathematical points have size (i.e., they are proper parts of magnitudes), then they are not indivisible after all. Zeno, as presented by Aristotle, exploited this dilemma with relish in his paradoxes. On the one hand, he forced Aristotle to reject indivisible magnitudes in favor of infinite divisibility. On the other hand, the Epicureans and ancient atomists exploited the assumption that a magnitude is infinitely divisible into parts with size to argue that this would imply that the original magnitude is infinite in size. So they accepted indivisibles.[31] We've already met some version of this Epicurean argument in connection with Berkeley.

As mentioned, Aristotle got himself out of this dilemma by arguing that the mathematical points have no size. Faced with the conclusion that they cannot compose a magnitude or are "nothing", he argues that points exist potentially. That is, rather than accept that points are non-entities (since they lack *onkos*), Aristotle opted to say that a point is actualized whenever a magnitude is split, say, into two smaller magnitudes. Here's how Miller, Jr. (1982, 98) puts it:

> Aristotle refutes the nihilistic horn [the name Miller, Jr. gives for the first horn of the dilemma we've been discussing], used by atomists, by showing that even though division is possible and a point exists everywhere in the potential mode, it does not follow that magnitude reduces to points. For the existence of every actually existing point is conditional upon the existence of two segments with magnitude into which the subsection is divided.

Thus the same point is the limit or extremity of the two magnitudes resulting from the split. That is, the

[28]Cf. Proclus (1970, 78) Trans. lines 95.21 – 96.14. "A point is a unit that has position." The Pythagorean definition does not mention divisibility.

[29]See (Pfeiffer, 2018, 130 – 131) for example.

[30]See Vlastos and Owen discussed in Furley (1967, 67). Furley notes that Vlastos writes of Zeno's assumption, "that anything which does have size is at least logically divisible and has at least logically discriminable parts." But he also mentions Owen who writes that Zeno assumes without argument that the conjunction of size with theoretical indivisibility would be a contradiction.

[31]See Diogenes Laertius (2018, 507 – 522) for Epicurus' *Letter to Herodotus*.

point existed potentially before the split but now exists actually as a limit or extremity of the two separate lines after the split.

Let's evaluate the line of reasoning that led Aristotle to this view. In order to split a line $AB$, I presumably have to specify where I want to split it, say some location $C$ between $A$ and $B$. But if I can specify the location as a point $C$, then the point $C$ must already be there unless there is "gap" at that location. This means that Aristotle has to either prove that a magnitude (say a line) is continuous, as we do today, in either the dense or Cantor-Dedekind complete sense first; or assume that it is before he can argue that points exist potentially. In fact, Aristotle neither assumed nor proved any of these alternatives since for him a line was not composed of points. What Aristotle did is to assume that you can always bisect a line segment into two equal segments. Continuity for him consisted in the identity of the right limit of the left segment and the left limit of the right segment. On the basis of this analysis of the existence of points and continuity, Aristotle drew the conclusion that continuous magnitudes (such as lines) are infinitely divisible since bisections can be done an indefinite number of times.[32] Some geometers[33], following Aristotle, then understood the infinite divisibility of finite lines to be a consequence or assumption of what is now *The Elements* Book I Proposition 10. How warranted were geometers to draw this consequence or make this assumption? Let us look at this next.

### 3.5 Infinite Divisibility and *The Elements* Book I Proposition 10

Proposition 10 in *The Elements* Book I is the proposition 'To bisect a given finite straight line.' The proof is familiar to most people from elementary geometry using compass and straight-edge. The important point is that if one analyzes the proof, *The Elements* does not draw the conclusion that this process can be iterated infinitely many times. We know that Aristotle predated Euclid's textbook *The Elements* and that Aristotle and his students at The Lyceum had a different geometry textbook that according to historians (Heath, 1981, Vol. 1, 321) was authored by Theudius. There was also an arithmetical textbook *Elements of Arithmetic* apparently authored by Archytas (430 – 365 BCE) who also predates Aristotle.[34] We may never know how Theudius proved this theorem and what conclusion he drew because that textbook is lost. Thus, it is impossible to know definitively whether Euclid and Aristotle differed in their conception of ancient geometrical practice. Recent scholars Linnebo and Shapiro (2019, 164) speculate:

> Because of the structure of the geometric magnitudes (to echo Lear (1982)), we have procedures that can be iterated indefinitely, and we speak about what those procedures could produce, or what they will eventually produce if carried sufficiently (but only finitely) far. In holding that these geometric procedures can be iterated indefinitely, Aristotle again follows the mathematical

[32]See *Physics* 207b10.
[33]See the quotation from Proclus in the next section.
[34]See Heath (1956, Vol. 2, p. 295).

practice of the time, this time in opposition to his other major opponents, the atomists, who postulate a limit to, say, bisection.

Proclus (1970, 216 – 217, Trans. lines 278 – 279 in MSS.) — a commentator on Book I of Euclid and the best historical source on ancient geometry we have — disagrees with this speculation. Here's how Proclus sees this matter in his commentary on Book I Proposition 10 (my emphasis):

> [If a line] is not composed of indivisible parts, it will be divisible to infinity. This, they say, appears to be an agreed principle in geometry, that a magnitude consists of parts infinitely divisible. To this we shall give the reply of Geminus, that geometers do assume, in accordance with a common notion, that what is continuous is divisible. The continuous, we say, is what consists of parts that are in contact, and this can always be divided. *But they do not assume that what is continuous is also divisible to infinity*...it is an axiom that every continuum is divisible; hence a finite line, being continuous, is divisible. This is the notion that the author of the *Elements* uses in bisecting the finite straight line, not the assumption that it is divisible to infinity. That something is divisible and that it is divisible to infinity are not the same.

This is a difficult text to make sense of and I do not claim that my reading is the only possible one here. Here is my proposal. Proclus is urging us to distinguish what we are in fact bisecting at a given stage $n \geq 1 \in \mathbb{N}$. Again, I am indexing by the natural numbers $\mathbb{N}$ because I am assuming that infinite divisibility is decidable or a constructive procedure. Suppose we start with a line $AB$ and bisect it at $C$ in stage 1. After the $n = 1$ bisection stage, we are strictly speaking not bisecting *the original* line $AB$ but have a choice of bisecting either $AC$ or $CB$. So it is at least misleading if not false to say that bisections of *the same original* $AB$ line can be done infinitely many times. Proclus's point seems to be that the theorem says that for *each* line the bisection can be done *once*. The continuity assumption, as a common notion, guarantees that any line segment is divisible, i.e., there are no indivisible lines. Berkeley will not object to any of this. But Proclus also points out that using *The Elements* Book I Proposition 10 as a proof for infinite divisibility of the original line is an extrapolation. Aristotle was one of those people who made the extrapolation from the bisection of *each* line (which the proposition shows) to potential infinite bisection of the same original line (which the proposition doesn't show). While it is true that one needs to assume continuity as a common notion to argue for the actual existence of a point as the limit of the two resulting line segments from bisection at a potentially existing midpoint; Aristotle, eager to refute the physical atomists, extrapolated from the claim that *every line* is bisectable *once* (which the proposition shows) to the claim that *the* (same) line is bisectable at *every stage* $n > 1 \in \mathbb{N}$ (which the proposition doesn't show). If my reading of Proclus's commentary is right, then this commentary is another source of evidence supporting Berkeley's claim that (potential) infinite divisibility is not a theorem or axiom in *The Elements*. Ancient geometrical practice could proceed simply by taking continuity to be a common notion, rather than defining continuity in terms of infinite divisibility, as Aristotle did.

## 4 Incommensurability in light of Aristotelian and Pythagorean views of Mathematics

However, there is the issue of incommensurable magnitudes. Many have taken this to be evidence for infinite divisibility. In fact, in this same commentary on Book I Proposition 10, Proclus (1970, 217) says that infinite divisibility follows from the existence of incommensurable magnitudes. Later philosophers such as the Port Royal Logicians took the existence of incommensurable magnitudes to be the definitive demonstration that there are no indivisible parts in magnitudes.[35] Incommensurability poses a threat for anyone who denies infinite divisibility (like Berkeley) only if such a person: (1) believes that there are indivisible lines; and (2) believes that the number of indivisible lines that a line can be divided into corresponds to its size. For if (1) and (2) are true, then suppose that the hypotenuse of a right triangle with side of unit length can only be divided into a finite number of lines $m$ and the side can only be divided into a finite number of lines $n$ where $m$ is larger than $n$ and $m$ and $n$ are in their least terms (having a greatest common divisor of one). Then the existence of the ratio $m : n$ would contradict the well known theorem that there are no numbers $m, n$ in their least terms such that the proportion $m : n :: \sqrt{2} : 1$ holds.[36]

But earlier I showed that Berkeley denies (1) because Berkeley doesn't believe there are indivisible *lines*. For Berkeley, the indivisibles in lines are *points*. So incommensurability doesn't pose a threat for Berkeley if he denies infinite divisibility. But does incommensurability really imply or presuppose that magnitudes are infinitely divisible? Recall that in the quotation that motivated pursuing the topic of this paper, Berkeley thought, *contra* Barrow, that deducing infinite divisibility from the existence of incommensurables is a *petitio principii*. Why did he think so? This issue needs to be investigated because it may shed light on what "ancient prejudice" Berkeley might have been alluding to in the passage at PHK §124.

I claim that the historical association of infinite divisibility with incommensurability follows from the Aristotelian conception of mathematics and its taxonomy of quantities into magnitudes and numbers (perhaps in light of the Pythagorean number theoretic discovery of incommensurable magnitudes). Since the Pythagorean conception had a different taxonomy, namely, that there was only one species of quantity — number — I will argue that incommensurability (which the Pythagoreans discovered from number theory or arithmetic) needs to be kept distinct from infinite divisibility (which arises in geometry). This will be another reason supporting my argument that Berkeley can avoid the objection that incommensurability implies infinite divisibility. That is, is it a *petitio principii*, to borrow Berkeley's words, to assume infinite divisibility as an explanation for incommensurability as Barrow had done (see section 4.3 below). Therefore, my discussion in this final section of my paper can be used not only to illuminate the issues involved in the

[35] Arnauld, Antoine and Nicole, Pierre (1996, 231)

[36] See §3.3 below for discussion of this proof.

Barrow-Berkeley debate, but also as a way of shedding light on what ancient prejudice Berkeley might have in mind.

### 4.1 Aristotelian Mathematics: Aristotle on Quantity

Aristotle's views on quantity in his collected works begin with the account of quantity in the *Categories* and is developed through the *Physics* and the *Metaphysics*. Throughout these accounts, Aristotle consistently distinguishes between discrete quantities *arithmos* (number) and continuous quantities *megethos* (magnitude). The genus term 'quantity' is the Greek word '*poson*.' But there's also the question of how to translate terms like *to pelikos* (how great), *onkos* or extension/volume and *metron* or measure as ways of discussing quantity. The taxonomy is complicated and opens up a lot of philosophical debate.[37] What is important for my purposes is that however this taxonomy ends up being sorted out, it is only one of the many other possible conceptions of mathematics that were available during Aristotle's time. At the heart of Aristotle's philosophical defense of infinite divisibility and the potential existence of points, I will argue, is that he held a different conception of mathematics. In doing so, he betrays an unfamiliarity with the import of the Pythagorean discoveries in mathematics; and the subsequent codification of these discoveries by Eudoxus in the theory of proportions in Book V and some of Theaetetus's discoveries that ended up being codified in Book X of *The Elements*. To be sure, Eudoxus and Aristotle were contemporaries and Theaetetus predated both of them. We may never know whether Aristotle was acquainted with Eudoxus's discoveries on the theory of proportions or whether Theaetetus's contribution, which we find in *The Elements* Book X, was included in the Theudius geometry textbook that was used in The Lyceum. In what follows (§4.2), Aristotle's remarks in the *Metaphysics* suggest an unfamiliarity with how to place Pythagorean number theoretic discoveries on rigorous geometrical foundations via Eudoxus's theory of proportions, which we find in Book V of *The Elements* and generalized in Book X.

### 4.2 Pythagorean Mathematics: all there is are numbers

One difficulty in assessing what Pythagoras actually believed is that there is no extant work written by Pythagoras. Any attempt to reconstruct what Pythagoreans actually believed cannot therefore be substantiated by anything from Pythagoras himself. To get a sense of the Pythagorean view of mathematics we have to rely on second hand accounts from philosophers like Plato and Aristotle some of whom, unfortunately, had an axe to grind; and commentators like Iambilichus, Proclus and Diogenes Laertius. In *Metaphysics* 985b23 – 986a13, Aristotle, for example, writes:

> Contemporaneously with these philosophers and before them, the Pythagoreans, as they are called, devoted themselves to mathematics; they were the first to advance this study, and having been brought up in it they thought its principles were the principles of all things.

[37] See Pfeiffer (2018) for the most up to date philosophical discussion of this taxonomy.

Here Aristotle shows agreement with the view that the history of arithmetic begins in Greece with Pythagoras, who is believed to have lived during the sixth century BCE. Historians speculate that Pythagoras was led to his *number-monism* (all there is are numbers and proportions between numbers) by his discovery in music theory of the *harmonical* proportion. That is, the fifth and the octave of a note could be produced on the same string by stopping at $\frac{2}{3}$ and $\frac{1}{2}$ of its length, respectively. Gow (1968, 68) writes about how led by such considerations,

> Pythagoras considered number to be the basis of creation: he looked to arithmetic for his definitions of all abstract terms and his explanation of all natural laws.

Thus, beginning with number-monism, Pythagoreans went on to develop number theory by classifying numbers as: odd, even, square, cube, triangular, perfect, defective, amicable etc. Proportions were either arithmetical, geometrical or harmonical.[38]

### 4.3 Comparing the two conceptions of mathematics

Given this Pythagorean number-monism, the first distinction we can make between the two conceptions of mathematics is that for the Pythagoreans there are no species of quantity. Aristotle is aware of this, writing in *Metaphysics* 1080b17 – 1080b21:

> Now the Pythagoreans, also, believe in one kind of number — the mathematical; only they say it is not separate but sensible substances are formed out of it.

On the other hand, for Aristotle, magnitude and numbers are both species of the genus quantity. The *differentia*, therefore, had to be sought. This difference was, for Aristotle, in terms of continuity and discreteness. Aristotle goes to great extent to defend his view of quantity first in the *Categories* and more fully in the *Physics*. In the *Physics*, he introduces subtle distinctions between whole and part; and between things being successive (next to each other), contiguous (touching), and finally continuous (*synechi* syn = together; echo = to have/hold) which in the Latin was translated *contenere* (con = together; tenere = hold). So the continuous is that which is "held-together." The depth and rigor of Aristotle's penetrating analysis going from weaker to stronger conditions for what is required for continuity is found in an extended discussion in *Physics* beginning in Book III all the way to Book VIII. Along the way, the association of infinity with continuity is made — an association that is with us to this very day. Further, Zeno's paradoxes of motion are considered and supposedly rebutted using the machinery developed until that point.[39]

One key difference between Aristotle and the Pythagoreans in this regard, is that for the Pythagoreans only numbers (i.e., positive integers greater than 1) can be answers to the question of quantity (*poson*).

[38]For details and historical references see Heath (1981, Vol. 1, 72 – 84) and Proclus (1970, 52 – 57).

[39]This is not the place to undertake a detailed analysis of Aristotle's analysis of continuity. For a good discussion see Miller, Jr. (1982) and Sorabji (1982). For a more recent discussion see Pfeiffer (2018).

These are questions that take the form “How many (much) X?” (*poson*) or the form “How great is X?” or “What size is X?” (*to pelikos*). Here is the important point, which Aristotle shows an unfamiliarity with. In the case of *magnitude*, numbers answer the question “How great is X?” or “What size is X?” in terms of *proportion* between *two* numbers.[40] Furley (1967, 52) writes:

> The Pythagorean method relied on finding proportions, and not on counting atomic constituents. It is the proportion 2:1 which constitutes the octave, no matter what the units may be.

Just as in the case of harmonics, the Pythagorean answer to the magnitude question “How great is X?” or “What size is X?” in geometry is a ratio or proportion (a proportion is an equality between ratios) involving *numbers* determined by *measuring* the two magnitudes with respect to size. Heath (1981, Vol. 1, 153) speculates that the Pythagorean theory of proportions was only applicable to commensurable magnitudes and that it was Eudoxus's work (which we find in Book V of *The Elements*) that generalized this theory to include incommensurables. Thus, unlike Aristotle, who sought to distinguish *arithmos* from *megethos*; for the Pythagoreans, there was only *arithmos* which was used to understand the *megethos*.

Let me put this in another way. The Pythagoreans started with number theory. Numbers were understood, for example, as even or odd; perfect; prime and so on. Corroborating Heath's claims, Van Der Waerden[41]speculates that Eudoxus's contribution found its way to *The Elements* Book V and Theaetetus's contribution found its way to Book X. They sought to place Pythagorean number theory (or arithmetic) on rigorous foundations (geometry). Eudoxus and Theaetetus's genius made it possible to embed the Pythagorean number theory into geometry using the general theory of proportions applicable to commensurable and incommensurable magnitudes. Euclid assembled these results in Book V and Book X respectively. The result is that on the Pythagorean conception of mathematics there was no need to have different answers to questions involving quantity (“How many (much)?”, “How great is X?” or “What size is X?”) in terms of discrete quantities and continuous quantities, as Aristotle thought. Rather, the answers are all in terms of numbers: positive integers or whole numbers in the case of “How many?”; or a ratio between two whole numbers in the case of “How great is X?” or “What size is X?”(magnitude). This is how Proclus (1970, 49 Trans. lines 61f) puts it:

> The theory of commensurable magnitudes is developed primarily by arithmetic and then by geometry in imitation of it. This is why both sciences define commensurable magnitudes as those which have to one another the ratio of a number to a number, and this implies that commensurability exists primarily in numbers.

We now can see why the discovery of incommensurable magnitudes (i.e., magnitudes that cannot be expressed (*irrational or alogos*) as a ratio between two integers one of which is their greatest common divisor or unit)

[40]Proclus (1970, 53) credits Pythagoras for discovering the doctrine of proportions.
[41]See Van Der Waerden (1961, 107 – 126; 141 – 146; 165 – 168; and 175 – 179).

was such an astonishing discovery. The astonishment was not, as is often suggested, that there were "gaps" in the rational numbers that had to be filled or completed by irrational numbers in order to get the real number continuum. The astonishment is that the Pythagorean number-monism was being threatened.[42]

We all know that the first discovery of incommensurability was of what we denote today by '$\sqrt{2}$.' The Pythagoreans would have used their number theory to say that there are no two whole numbers $m, n$ such that $m : n :: \sqrt{2} : 1$. In other words, $\sqrt{2}$ is incommensurable using 1 as the unit of measure. The proof is number theoretic since it is in terms of the distinction between odd and even *numbers*. Aristotle is clearly aware of this proof since he mentions it in *Prior Analytics* 41a26 – 41a27. But even though $\sqrt{2}$ was incommensurable, the Pythagoreans still had a way of expressing it in terms of a proportion between *known* magnitudes as follows: $\sqrt{2} : 1 ::$ **diagonal of right-isosceles triangle with side of length 1: one of the sides of the right-isosceles triangle**.

So, the Pythagoreans did not conclude that the rational numbers are incomplete ("gappy" or discontinuous) as we often hear. The Pythagoreans were not even thinking about these problems in terms of continuity or discontinuity at all. This can explain why *The Elements* is silent about its continuity assumptions except for Postulate 2 (To produce a finite straight line continuously in a straight line). The reason is that *The Elements* could never have doubted that magnitudes (such as lines) are continuous. We've already seen evidence from the commentary of Proclus that continuity was a common notion. But what the existence of incommensurables *did do* was to motivate a program in search of a rigorous theory of proportions between magnitudes in order to study, classify and ultimately understand what those newly discovered incommensurables were. This was the theory that was developed by the magisterial Eudoxus and Theaetetus and immortalized in *The Elements*'s Book V and Book X.

This brings us to the second difference between Aristotle and the Pythagoreans. Because Aristotle makes the distinction between continuous and discrete, he holds that there are indivisible units in discrete quantities (number) but not in continuous quantities (magnitude). Consequently he mistakenly attributes to the Pythagoreans the view that there are indivisible magnitudes. That is, that the Pythagorean units (or indivisibles) have spatial magnitude. He writes:

> For [Pythagoreans] construct the whole universe out of numbers only – not numbers consisting of abstract units; they suppose the units to have spatial magnitude. But how the first unit was constructed so as to have magnitude, they seem unable to say. *Metaphysics* 1080b17 – 1080b21

What evidence does Aristotle have to assert "how the first unit was constructed so as to have magnitude, they seem unable to say"? His claim is justified only because he held a different conception of mathematics from the Pythagoreans. Not only this, he also adds:

> The doctrine of the Pythagoreans in one way affords fewer difficulties than those before named,

[42]Cf. Heath (1981, Vol. 1, 155).

> but in another way has others peculiar to itself...[T]hat bodies should be composed of numbers, and that this should be mathematical number, is impossible. For it is not true to speak of indivisible magnitudes; and however much there might be magnitudes of this sort, units at least have no magnitude; and how can a magnitude be composed of indivisibles? But arithmetical number, at least, consists of abstract units, while these thinkers identify number with real things; at any rate they apply their propositions to bodies as if they consisted of those numbers. *Metaphysics* 1083b8 – 1083b19

Here, Aristotle is expressing his misgivings about the Pythagorean number-monism, which suggested that bodies are composed wholly of arithmetical numbers or units. He is arguing that this is impossible. First, it is not true to speak of indivisible magnitudes, he says. Since a body is a magnitude (meaning continuous), it cannot be composed of indivisible magnitudes (such as the arithmetical units which are discrete). This is an assertion he takes to have proven elsewhere. Secondly, on Aristotle's view geometrical units or points have no magnitude and so cannot be parts of (or compose) a magnitude. I have already discussed all of this in the previous section. Surprisingly, Kirk G.S. and J.E. Raven (1957, 246ff) point out that it is *the Pythagoreans* who are confused.

> The unfortunate consequence of their diagrammatic representation of numbers was that the Pythagoreans, thinking of numbers as spatially extended and confusing the point of geometry with the unit of magnitude, tended to imagine both alike as possessing magnitude...It is true that Aristotle, in discussing the views of earlier thinkers, often confronts them with such logical consequences of their doctrines as they themselves never either enunciated or foresaw...[Aristotle] leaves no doubt that the Pythagoreans did indeed assume, that units are spatially extended; and when we come to consider the paradoxes of Zeno we shall find that it is against this assumption, along with the confusion of points and units, that they have their greatest force.

I disagree with Kirk and Raven's attribution of confusion to the Pythagoreans. It is *Aristotle* who is confused or misunderstood the upshot of Pythagorean number theory. Remember he said, "But how the first unit was constructed so as to have magnitude, [Pythagoreans] seem unable to say." Aristotle has no grounds for making this claim. Here's why I think so. We know from historians that all the mathematics we find in *The Elements* — except for Book V (the theory of proportions) — was known before the time of Plato.[43] This mathematical knowledge includes the Pythagorean theory of proportions applicable to commensurables only, the discussion of arithmetical units and the mathematical knowledge in *The Elements* Book X on incommensurables. So we can reasonably expect the greatest student of Plato, Aristotle, to have known it. We may excuse Aristotle for being unfamiliar with the work of his contemporary Eudoxus, another student of Plato, who showed that magnitudes or bodies can be understood number theoretically

[43] See Heath (1981, Vol. 1, 216 – 217).

according to the theory of proportions we find in *The Elements* Book V. But I think it is nothing short of confusion for Aristotle to base his objection to the Pythagoreans on the claim that the unit has magnitude. It is a confusion because according to the Pythagoreans the unit has no magnitude (in Aristotle's sense) — the unit is a *number* or quantity, which is the common *measure* of commensurable magnitudes (in Aristotle's sense).

I have still not discussed how incommensurability and infinite divisibility became entangled. Recall that one of the main arguments for infinite divisibility was the existence of incommensurable magnitudes. So now we must face two questions: (1) What are magnitudes? and (2) What are incommensurable magnitudes?

### 4.4 Magnitudes and Incommensurables

Earlier we saw that Aristotle distinguished magnitudes from numbers by saying that magnitudes are continuous, which means they are infinitely divisible. We remarked that this identification of the continuous with the infinitely divisible is a philosophical thesis that does not follow from the bisection theorem. Although *The Elements* identifies *arithmos* (number) with the collection of units in Book VII, it does not follow from *The Elements* alone that *megethos* (magnitude) is not composed of units, where "not composed of units" is the definition of continuous. It is, after all, open for someone to construe the "units" as *actual* points, not parts, of a dense point-set continuum (something which Berkeley does). Commentators and historians of mathematics have noticed that it is hard to grasp the meaning of *megethos* because *The Elements* does not give us a definition that tells us what magnitudes *are*.[44] What *The Elements* does give us is a theory of proportions, going back to the Pythagoreans and Eudoxus, that tells us at least *how* to deal with *megethos* rigorously. This is the account that we get in Books V, VII, and X. But in order to for me to show this and in order to understand Books V, VII, and Book X, we need to inquire into the incommensurables more closely.

Recall that Aristotle says that the Pythagoreans were unable to say how the unit was constructed so as to have magnitude. In order to evaluate Aristotle's claim, we need to look at how incommensurability was discovered, under what assumptions, and what conclusions the discoverers drew. There are three competing accounts: (1) the number-theoretic proof regarding the incommensurability of the diagonal of a square of unit length; (2) the proofs in Plato's dialogues and the method of finding the mean proportional between two plane similar numbers; (3) the method in *The Elements* Book X.[45] Let us look at these accounts in turn. I will not seek to disentangle which of these methods was the one that was actually used. Here it is a matter of speculation. For my purposes, the question I shall be seeking to answer is this: is the infinite divisibility of magnitudes assumed or does it follow from the given proof in the method?

[44] I will not attempt to speculate what Euclid meant by 'magnitude.' Here's where examples work better than definitions: lines, areas, volumes are magnitudes. See the discussion in Mueller (1981, 121f, 136 – 138) for an attempt to sort out what magnitudes are.

[45] These competing accounts are discussed in detail in Knorr (1975, 22 – 49) with references to Von Fritz (1945). See also Knorr (1981), Unguru (1977), and compare with Heath (1981, 202 – 209).

1. *Number theoretic proof interpreted geometrically*

   The proof is familiar and proceeds by *reductio ad absurdum*. Let $ABC$ be a right isosceles triangle with side of unit length. Suppose that the diagonal $AC$ is commensurable to the side $AB$. Let $m : n$ be their ratio expressed in lowest terms (i.e., the greatest common divisor of $m$ and $n$ is 1). Now $AC^2 : AB^2 = m^2 : n^2$. Since $AC^2 = 2AB^2$ by the Pythagorean theorem (*The Elements* Book I. 47), it follows that $m^2 = 2n^2$. Hence $m^2$ is even and so is $m$. Since $m : n$ is in its lowest terms, it follows $n$ is odd. Let $m = 2a$ for some $a$; then $4a^2 = 2n^2$ and $n^2 = 2a^2$, hence $n$ is even. But this is impossible since $n$ was shown to be odd. Therefore, the diagonal $AC$ is incommensurable with the side $AB$.[46]

   Let us waive the difficulty that this proof (Proposition 117 in Euclid Book X) was actually an interpolation as Heath (1956, Vol. 3, 2) suggests. The important point to take away from this proof is that it is number theoretic and nowhere in the proof has the assumption that finite lines are infinitely divisible entered into the reasoning. Aristotle was familiar with this proof as I've mentioned.[47] So it is unclear on what basis he concluded that magnitudes are infinitely divisible from this theorem. If this was indeed the way that incommensurables were shown to exist, then Berkeley is right to say that it is a *petitio principii* to conclude from this that finite lines are infinitely divisible.

2. *The proofs in Plato's dialogues and the method of finding the mean proportional*

   This number theoretic proof did not generalize in an obvious way to incommensurable square roots greater than $\sqrt{2}$. The proofs that $\sqrt{3}$, $\sqrt{5}$, ..., $\sqrt{17}$ are incommensurable with 1 as the unit of measure are reported in Plato's *Theaetetus*, where it is said they were developed by the Pythagorean Theodorus. There is some controversy regarding exactly how Theodorus proved these incommensurability results since Plato does not tell us the method. For this reason, Heath (1981, Vol. 1, 202 – 209) offers three hypotheses. (1) The method of successively approximating $\sqrt{3}$ by a geometric sequence with common ratio $\frac{1}{2}$; (2) the traditional number theoretic approach used to show that $\sqrt{2}$ is incommensurable with 1 as the unit of measure; and (3) a proposal by Zeuthen based on the method for detecting incommensurability given by Proposition 2 in Euclid Book X.[48] In any case, these are hypotheses and as far as I can tell, there is no mention of infinite divisibility in the proofs according to the methods suggested by these three hypotheses. In method (3) in particular, it is the existence of a non-terminating *number theoretic* process that tells us that we are dealing with incommensurable magnitudes. I have found no evidence in Berkeley that he is objecting to this non-terminating number theoretic process in the case of incommensurable magnitudes.

   According to historians (Heath, 1981, Vol. 1, 89), the mathematics in Plato's *Timaeus* has Pythagorean

[46] See Heath (1981, 147 – 148) for discussion on how the Pythagoreans proved what is now Proposition 47 in *The Elements* Book I.

[47] See Heath (1956, Vol. 3, 2)

[48] Heath (1981, Vol. 1, 207) and Heath (1956, Vol. 3, 18) thinks that method (3) is similar to the Euclidean algorithm for finding the greatest common divisor. I return to a detailed discussion of this method in the next item.

themes and contains references to the existence of a geometric mean between two square numbers and two geometric means between two cube numbers.[49] Barrow thought that the theorem proving the existence of a mean proportional between two square numbers was the basis of incommensurability and that the method presupposed the infinite divisibility of quantities. Here's how he puts it in *Mathematical Lectures* XV (my emphasis):

> The principal reason of incommensurability seems to be founded in this, that since a mean proportional number may always be found between two plane similar numbers because the product made by the multiplication of plane similar numbers is always a square number, whose root is that mean proportional ... since I say, things are thus in similar numbers, and it is demonstrated in the *Elements*, that it happens quite otherwise in all dissimilar numbers; there is no mean proportional number between two dissimilar plane numbers. [H]ence, if two quantities are supposed to be to one another in the [ratio] of two dissimilar numbers, and a mean proportional be found between those quantities, which may perpetually be done, *because of the indefinite divisibility of every quantity*, there will be no number in universal nature which can represent or answer to this quantity, and consequently, those being supposed and expressed by numbers, this will be incommensurable.

Barrow's point here sounds a lot more complicated than it is. It is actually Book VIII. Proposition 11.[50] Let's put his point in more modern terms. A plane number $m$ is a number that is a product of two numbers $a$ and $b$, i.e., $m = ab$ (Book VIII. Proposition 5). According to Heath, plane similar numbers are what we call square numbers today. But it is possible to generalize plane similar numbers to include other numbers $m = ab$ and $n = cd$ such that the proportion $a : c :: b : d$ holds. Plane dissimilar numbers are numbers $m = ab$ and $n = cd$ such that the proportion $a : c :: b : d$ does not hold.[51] The mean proportional number between two numbers $m$ and $n$ is what we call today the geometric mean of $m$, $n$. That is, the number $x$, such that $m : x :: x : n$. So $x = \sqrt{mn}$ which is distinguished from their arithmetic mean $\frac{m+n}{2}$. Barrow's point, following Book VIII. Proposition 11, is that there is a rational mean proportional *number* between two plane numbers $m$ and $n$, just in case $m$ and $n$ are plane *similar* numbers. This is easy to see in the special case where $m$ and $n$ are square numbers since in that case $x = \sqrt{a^2b^2} = ab$. If $m$ and $n$ are plane dissimilar numbers, then in general $x = \sqrt{(ab) \cdot (cd)}$ is not a rational number. Barrow argues that this is the principal reason for incommensurability and that this follows because of the infinite (he uses the word 'indefinite') divisibility of every quantity. But nowhere in the proofs has the infinite divisibility of finite lines been

[49]See Heath (1956, 363)'s note to *The Elements* Book VIII Proposition 11.
[50]Cf. Book X. Proposition 9.
[51]See Heath (1956, 293 – 294) commentary on *The Elements* Book VII, Def. 21. Compare with Book VI Proposition 13 (To two given straight lines to find a mean proportional) and the geometrico-algebraic method given in *The Elements* Book II Proposition 14 (To construct a square equal to a given rectilineal figure) involving the extraction of a square root.

assumed or concluded. It is a *petitio principii* to conclude, on the basis of this argument, that finite lines are infinitely divisible because the proof has nothing to do with lines — it is number theoretic.

3. *The Method of The Elements Book X Proposition 2*

This being said, there is a non-terminating method for detecting incommensurable magnitudes that is related to this method of finding the mean proportional.[52] Heath (1956, Vol. 3, 18) remarks that these propositions make essential use of the Euclidean division algorithm for finding the greatest common divisor between two numbers (I describe this method below). Von Fritz (1945) and Van Der Waerden (1961, 176f) call this method *anthyphairesis* and speculate that incommensurables were discovered by this method even though Heath (1981, Vol. 1, 207) finds it improbable. Let us call this method *epanalipsi-afairesis* (*repeated-subtraction*) in order to distinguish it from Aristotle's potential infinite divisibility.[53] Let's look at this method starting with Book X Proposition 2.

Book X Proposition 2

If, when the less of two unequal magnitudes is continually subtracted in turn from the greater, that which is left never measures the one before it, the magnitudes will be incommensurable.[54]

Compare this with the number theoretic proposition in Book VII. There is a strong analogy although the one is about incommensurables and the other is about relative primes.

Book VII Proposition 1

Two unequal numbers being set out, and the less being continually subtracted in turn from the greater, if the number which is left never measures the one before it until a unit is left, the original numbers will be prime to one another.

The *epanalipsi-afairesis* method for detecting incommensurables is this: To determine the proportion between two lengths $M$ and $m$ *representing numbers*, of which $M$ is the greater, first subtract $m$ from $M$ as many times as possible, leaving a remainder $m'$. Then subtract $m'$ from $m$ in the same way leaving a remainder $m''$. Then subtract $m''$ from $m'$ and so on until no remainder (if at all) is left. The first length which can be subtracted thus without leaving any remainder is *the unit* in terms of

[52]See Knorr (1975, 29f) for discussion although even he thinks that it is very unlikely that this was how incommensurables were discovered.

[53]There's a brief discussion in Furley (1967, 49) where he calls this process *antistrofi-afaireisis* (reciprocal-subtraction). I have chosen to call this process *epanalipsi-afairesis* (repeated-subtraction) in order to remain faithful or closer to the plain reading of the Greek text. In his discussion of the infinite in *Physics* III. 5 – 6 Aristotle uses the term *division*(*diaresis*) most frequently as the antithesis of *addition*(*synthesis*). He occasionally speaks of *subtraction*(*afairesis*) and *diminution*(*kathairesis*). See also Heath (1956, Vol 1, 232). Could this be the method that presupposes or concludes that magnitudes are infinitely divisible? Here I am less confident. For this reason, I withhold judgment and suggest opening this up for debate.

[54]This proposition depends on **Book X Proposition 1**: Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. This is a version of the Eudoxus-Archimedes Axiom in Book V. Definition 4.

which the ratio $M : m$ can be expressed. The unit will vary according to what these lengths $M$ and $m$ are. These units are not geometrical points but numerical measures.

Assuming that the method of *The Elements* Book X Proposition 2 was how incommensurables were first detected (and it is reasonable to do so since Book X is largely due to Theaetetus and predates Aristotle), then what the Pythagoreans called a unit (*monas*) is what we call today: (1) *the number one* (if the numbers are relatively prime from Book VII.1), *or* (2) the *greatest common divisor* of two composite numbers if it existed (from Book VII.2). This unit (of measure) can be used to measure (*metron*) the magnitude, i.e., how great (*to pelikos*) a homogeneous quantity is relative to another homogeneous quantity. If the answer to the question "How great is X?" could be expressed as a *ratio* (it is *logos*) or proportion, then the *numbers* were rational and the magnitudes representing them were commensurable. The answers which the Pythagoreans would give would always be in terms of proportions, $4 : 2 :: 2 : 1$ which means that 4 is 2 times as great as 2 using 2 as the unit. If there is no greatest common divisor, *including* 1, between two numbers, then the two magnitudes representing them are incommensurable. There is no common measure or no way of comparing them with respect to size (by Book V. Definitions 3 and 4). This would be the case if the process of *epanalipsi-afairesis* did not terminate after a finite number of steps. But it is one thing to say that this non-terminating number theoretic process is true for incommensurable magnitudes and it is another thing to conclude or assume on the basis of this, that the continuity of magnitudes consists in their being infinitely divisible.

To see why connecting the two is misleading, consider a line equal in length with the circumference of a circle and a line equal in length with the diameter of the same circle. It is common ground between the Pythagoreans and Aristotle that both these lines are continuous. Suppose that there are numbers which can be represented by these lengths, say $c$ (the circumfrence) and $d$ (the diameter). There is no greatest common divisor between these numbers (since this is the definition of the constant $\pi$). The process of *epanalipsi-afairesis* does not terminate in the case of these two numbers and many others like them. But this has nothing to do with continuity or infinite divisibility of the lengths representing these numbers as Aristotle thought. Thus, incommensurability does not show that the essence of the continuity of magnitudes is infinite divisibility. Rather, it shows that there are pairs of numbers for which this process of *epanalipsi-afairesis* does not terminate after a finite number of steps.

Of some of these incommensurables, there are those that cannot be represented as a ratio between known magnitudes (they are *alogos*, inexpressible or irrational, because of this; rational otherwise). Notice that even though the magnitude (*megethos*) representing $\sqrt{2}$ is incommensurable using 1 as the unit of measure, the number (*arithmos*) $\sqrt{2}$ is not irrational or *alogos* in the Pythagorean sense. $\sqrt{2}$ can be expressed as the ratio between known magnitudes, namely, the ratio between the diagonal of a right-isosceles triangle with side of length 1 and one of its sides. So incommensurability does not imply irrationality in the Pythagorean sense. This is how what we mean by irrational numbers today differs from how the Pythagoreans conceived of them. However, magnitudes representing numbers such as $\sqrt{19}$ are not only incommensurable with 1 as the unit of measure but also irrational. Thus irrationality implies incommensurability. I am not sure how to think of $\pi$ in Pythagorean terms. It seems to me that even though $\pi$ is incommensurable with 1 as the unit of measure, it is not irrational in the Pythagorean sense since it can be expressed as the ratio between the circumference of a circle and its diameter.

It is the conflation of the Aristotelian thesis of infinite divisibility with the non-terminating *epanalipsi-afairesis* characteristic of incommensurability that has stayed with mathematicians and philosophers for millennia. I have suggested that this is the ancient prejudice that Berkeley was alluding to in the passage in PHK §124. If the method that was first used for detecting incommensurable magnitudes representing numbers besides $\sqrt{2}$ was indeed Book X Proposition 2, then one way to read Book X is as a geometric (hence rigorous) translation or formulation of *number theoretic* facts. Incommensurability arises when number theoretic facts are being embedded in geometry, for example by trying to find the ratio or proportion between two magnitudes that represent certain numbers. This raises the question of the proper foundations for mathematics: is it geometry or arithmetic? If Van der Waerden is right, then according to the Pythagoreans, the way to place their number theoretic investigations on rigorous foundations was to cash them out geometrically. But this suggests not taking what Pythagoreans took to be a number theoretic fact (a non-terminating process) as evidence for a geometrical fact (the infinite divisibility of finite lines). Barrow, in his mathematical lectures (Lecture III and Lecture XV), famously argued for the identity of geometry with arithmetic. Aristotle objected to the use in geometry of Pythagorean number theoretic units that can be represented geometrically as spatially extended magnitudes. Aristotle raised the valid question, "If a unit is indivisible, how can it be spatially extended?" But this is the right question for Aristotle to ask only if one accepts his philosophical conception of mathematics; since for him spatially extended parts of magnitudes are divisible ad infinitum. But the Pythagoreans meant something completely different when they spoke of representing units as magnitudes in geometry. We've seen that for the Pythagoreans, these units are *units of measure* of the ratio between magnitudes, i.e., what we refer to today as the greatest common divisor (if it existed) of two numbers. These units have nothing to do with the infinite divisibility (or lack thereof) of magnitudes since continuity was a common notion.

## 5 Conclusion

In conclusion, let me recapitulate the main points of this paper. I have given evidence that the theory of proportions was motivated by number theoretic discoveries of incommensurability. The theory of proportions developed by Eudoxus and Theaetetus was given in order to place these discoveries on a rigorous foundation in *The Elements*'s Books V and X. In addition to historical evidence from Proclus that the bisection theorem was not always taken to imply infinite divisibility, I have given reasons for resisting the assimilation of a number theoretic process (the non-terminating *epanalipsi-afairesis* characteristic of incommensurability) with the philosophical thesis of infinite divisibility, which defines continuity in terms of infinite divisibility. If by infinite divisibility Aristotle means that *actual* indivisible points cannot compose a magnitude, then I hope to have shown that this view flows out of a different conception of mathematics; and that this view is not necessary to develop the theory of proportions along Pythagorean lines and hence to handle magnitudes (or continuous quantities). My whole discussion was meant to establish (or to at least open up the possibility for thinking) that ancient geometrical practice did not require infinite divisibility. Therefore, if we take of all of these reasons into account, we arrive at a more nuanced and plausible reading of Berkeley's claim in PHK §123 that the infinite divisibility of finite lines is neither an axiom nor theorem in the elements of geometry.

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