Notes on "Quantum Logic"*

Version 1.0

David B. Malament

Department of Logic and Philosophy of Science

University of California, Irvine

dmalamen@uci.edu

Contents

*Thanks to John Manchak for creating a TeX file from my notes.

1 Formal (sentential) quantum logic

First we introduce a formal language L for sentential logic. The primitive symbols are:

(a) an infinite supply of sentential letters: $A_1, A_2, A_3, ...$

(b) connectives: $\neg$ & $\vee$

(c) left and right parentheses: ( )

The set of *sentences in* L, $Sent(L)$, is defined by induction:

(1) Every sentential letter A_i is a sentence in L.

(2) If φ and ψ are sentences in L, then so are: $(\neg\varphi)$ $\quad (\varphi \vee \psi)$ $\quad (\varphi \& \psi)$.

(We shall use lower case Greek letters as metalinguistic variables ranging over sentences in L, and upper case Greek letters as variables ranging over sets of sentences in L. Where confusion cannot arise, we sometimes drop outer parentheses.)

Now we define the relations of logical implication (or entailment) that characterize classical logic and "quantum logic". To make the two look as much alike as possible, we present the first definition (for the classical case) in a slightly non-standard form.

Let a *CL-valuation* be a pair (S, t), where S is a non-empty set and t is a map from $Sent(L)$ to $\mathcal{P}(S)$, the power set of S, satisfying the following conditions:

(C1) $t(\neg\varphi) = t(\varphi)^c$ (= the complement of the set $t(\varphi)$ relative to S)

(C2) $t(\varphi \& \psi) = t(\varphi) \cap t(\psi)$

(C3) $t(\varphi \vee \psi) = t(\varphi) \cup t(\psi)$

for all sentences φ and ψ in L. We say that Γ *classically implies (or entails)* ψ, and write $\Gamma \models_{CL} \psi$, if, for all CL-valuations (S, t),

$$\cap\{t(\varphi) : \varphi \in \Gamma\} \subseteq t(\psi).$$

So, in particular, if Γ is finite with elements $\varphi_1, ..., \varphi_n$, $\Gamma \models_{CL} \psi$ if

$$t(\varphi_1) \cap ... \cap t(\varphi_n) \subseteq t(\psi).$$

In the usual fashion, we write $\varphi \models_{CL} \psi$ rather than $\{\varphi\} \models_{CL} \psi$, and write $\models_{CL} \psi$ if ψ is classically implied by the empty set of sentences in L. Notice that $\varphi \models_{CL} \psi$ iff, for all CL-valuations (S, t), $t(\varphi) \subseteq t(\psi)$; and $\models_{CL} \psi$ iff, for all CL-valuations (S, t), $t(\psi) = S$. (The latter assertion follows from the fact that the intersection of the empty set (of subsets of S) is S.)

In parallel, we take a *QL-valuation* to be a pair $(\mathbf{H}, v)$ where $\mathbf{H}$ is a Hilbert space[1] and v is a map from $Sent(L)$ to $L(\mathbf{H})$, the set of all (closed) subspaces of $\mathbf{H}$, satisfying the following three conditions

[1] I will review basic facts about Hilbert spaces in class.

(Q1) $v(\neg\varphi) = v(\varphi)^\perp$ (= the orthocomplement of the subspace $v(\varphi)$ in $\mathbf{H}$)

(Q2) $v(\varphi \& \psi) = v(\varphi) \cap v(\psi)$

(Q3) $v(\varphi \vee \psi)$ = the span of $v(\varphi)$ and $v(\psi)$, i.e., the smallest subspace of $\mathbf{H}$ containing both $v(\varphi)$ and $v(\psi)$

for all sentences φ and ψ in L. We say that Γ *quantum logically implies (or entails)* ψ, and write $\Gamma \models_{QL} \psi$, if, for all QL-valuations $(\mathbf{H}, v)$,

$$\cap\{v(\varphi) : \varphi \in \Gamma\} \subseteq v(\psi).$$

The expressions $\varphi \models_{QL} \psi$ and $\models_{QL} \psi$ are understood in parallel to their counterparts above. So, for example, $\models_{QL} \psi$ is equivalent to the assertion that, for all QL-valuations $(\mathbf{H}, v)$, $v(\psi) = \mathbf{H}$.

The relation $\models_{QL}$ behaves just like $\models_{CL}$ in many respects. For example, the following hold for all sentences φ, ψ, and θ in L.

(1) $\varphi \models_{QL} (\varphi \vee \psi)$ and $\psi \models_{QL} (\varphi \vee \psi)$

(2) $(\varphi \models_{QL} \theta$ and $\psi \models_{QL} \theta) \implies (\varphi \vee \psi) \models_{QL} \theta$

(3) $\{\varphi, \psi\} \models_{QL} (\varphi \& \psi)$

(4) $(\varphi \& \psi) \models_{QL} \varphi$ and $(\varphi \& \psi) \models_{QL} \psi$

(5) $\models_{QL} \neg(\varphi \& \neg\varphi)$

(6) $\models_{QL} (\varphi \vee \neg\varphi)$

(7) $\varphi \models_{QL} \neg(\neg\varphi)$ and $\neg(\neg\varphi) \models_{QL} \varphi$

(8) $((\varphi \& \psi) \vee (\varphi \& \theta)) \models_{QL} (\varphi \& (\psi \vee \theta))$ and $(\varphi \vee (\psi \& \theta)) \models_{QL} ((\varphi \vee \psi) \& (\varphi \vee \theta))$

(9) $\neg(\varphi \& \psi) \models_{QL} (\neg\varphi \vee \neg\psi)$ and $(\neg\varphi \vee \neg\psi) \models_{QL} \neg(\varphi \& \psi)$

(10) $\neg(\varphi \vee \psi) \models_{QL} (\neg\varphi \& \neg\psi)$ and $(\neg\varphi \& \neg\psi) \models_{QL} \neg(\varphi \vee \psi)$

There are, however, significant differences. In particular, the "other half" of the distributive law fails, i.e., it is *not* the case that

$$(\varphi \& (\psi \vee \theta)) \models_{QL} ((\varphi \& \psi) \vee (\varphi \& \theta))$$

for all φ, ψ, and θ. For example,

$$(A_1 \& (A_2 \vee A_3)) \not\models_{QL} ((A_1 \& A_2) \vee (A_1 \& A_3)).$$

To see this, consider a two-dimensional Hilbert space, and take $v(A_1)$, $v(A_2)$, and $v(A_3)$ to be any three (distinct) one-dimensional subspaces. Then we have

$$v(A_1\&(A_2 \vee A_3)) = v(A_1) \cap (v(A_2) \text{ span } v(A_3)) = v(A_1) \cap \mathbf{H} = v(A_1)$$

but

$$v((A_1\&A_2) \vee (A_1\&A_3)) = v(A_1\&A_2) \text{ span } v(A_2\&A_3) = \mathbf{0} \text{ span } \mathbf{0} = \mathbf{0},$$

where $\mathbf{0}$ is the 0-dimensional space. So the first is not a subset of the second.

Here is one more example of a difference. It is *not* the case that

$$\Gamma \cup \{\varphi\} \models_{QL} (\psi\&\neg\psi) \implies \Gamma \models_{QL} \neg\varphi.$$

for all Γ, φ, and ψ (though it *is* the case that

$$\varphi \models_{QL} (\psi\&\neg\psi) \implies \models_{QL} \neg\varphi$$

for all φ and ψ). In particular, we have

$$\{A_1, (\neg A_1 \vee \neg A_2), A_2\} \models_{QL} ((A_1\&A_2)\&\neg(A_1\&A_2)),$$

but

$$\{A_1, (\neg A_1 \vee \neg A_2)\} \not\models_{QL} \neg A_2.$$

To verify the latter, negative claim, consider a two-dimensional Hilbert space, and let $v(A_1)$ and $v(A_2)$ be any two (distinct) one-dimensional subspaces that are not orthogonal. Then $v(A_1)^\perp$ and $v(A_2)^\perp$ are also distinct one-dimensional subspaces. So

$$v(\neg A_1 \vee \neg A_2) = v(A_1)^\perp \text{ span } v(A_2)^\perp = \mathbf{H},$$

and therefore

$$v(A_1) \cap v(\neg A_1 \vee \neg A_2) = v(A_1) \cap \mathbf{H} = v(A_1).$$

But $v(A_1)$ is *not* a subspace of $v(A_2)^\perp$, since $v(A_2)$ is not orthogonal to $v(A_1)$. Thus it is *not* the case that

$$v(A_1) \cap v(\neg A_1 \vee \neg A_2) \subseteq v(\neg A_2).$$

We have not included the conditional '$\rightarrow$' as a primitive symbol. Let's consider three ways we might introduce it as a defined symbol. Let's construe:

$$(\varphi \rightarrow_1 \psi) \text{ as an abbreviation for } (\neg\varphi \vee \psi)$$
$$(\varphi \rightarrow_2 \psi) \text{ as an abbreviation for } \neg(\varphi\&\neg\psi)$$
$$(\varphi \rightarrow_3 \psi) \text{ as an abbreviation for } (\neg\varphi \vee (\varphi\&\psi)).$$

The formulas on the right are equivalent in classical logic for all φ and ψ. But they are not in quantum logic. It turns out that there $(\varphi \rightarrow_1 \psi)$ and $(\varphi \rightarrow_2 \psi)$ are equivalent for all φ and ψ, but they are not equivalent to $(\varphi \rightarrow_3 \psi)$ in all cases. It also turns out that $\rightarrow_3$ is, in a sense, a better choice for the conditional in "quantum logic" than the others because "it behaves more like the material conditional in classical logic". To make this assertion precise, we consider three basic rules of inference that the material conditional satisfies in classical logic. They are:

Modus Ponens (MP): From $\Gamma_1 \models \varphi$ and $\Gamma_2 \models (\varphi \rightarrow \psi)$ infer $\Gamma_1 \cup \Gamma_2 \models \psi$.

Modus Tollens (MT): From $\Gamma_1 \models \neg\psi$ and $\Gamma_2 \models (\varphi \rightarrow \psi)$ infer $\Gamma_1 \cup \Gamma_2 \models \neg\varphi$.

Conditional Proof (CP): From $\Gamma \cup \{\varphi\} \models \psi$ infer $\Gamma \models (\varphi \rightarrow \psi)$.

The status of the three rules in quantum logic, when formulated in terms of $\rightarrow_1$, $\rightarrow_2$, and $\rightarrow_3$, is as follows.

	MP	MT	CP
$\rightarrow_1$	no	no	no
$\rightarrow_2$	no	no	no
$\rightarrow_3$	yes	yes	no

Table 1: Do the rules hold in quantum logic?

Thus, CP does not hold for any of candidates! But MP and MT hold, at least, in the case of $\rightarrow_3$.

Problem 1.1. *Provide counterexamples for all the negative entries in the table. (Hint: All can be handled using the sort of simple valuation over a two-dimensional space that has been used repeatedly to this point. For example, MP fails in quantum logic for the conditional $\rightarrow_1$ because we have $A_1 \models_{QL} A_1$ and $\neg A_1 \vee A_2 \models_{QL} \neg A_1 \vee A_2$, but **not** $\{A_1, (\neg A_1 \vee A_2)\} \models_{QL} A_2$. Thus we get a violation of MP if we take $\Gamma_1 = \{A_1\}$, $\Gamma_2 = \{(\neg A_1 \vee A_2)\}$, $\phi = A_1$, $\psi = A_2$.)*

2 The interpretation of quantum logic

Putnam's View (as presented in [3])

(a) All propositions considered in QM have truth values at all times.

(b) The connectives in QL have the same meaning as in classical logic.

(c) $\Gamma \models_{QL} \psi$ is to be understood as the assertion that, for every state of the system in question, if all $\varphi \in \Gamma$ are true in that state, then so is ψ.

Possible Alternative View

(a′) The "propositions" considered in QM only have truth values under appropriate conditions of *realization* (possibly conditions of measurement, but not necessarily so). For this reason, the term 'proposition' is not really appropriate. One might use 'eventuality' instead.

(b′) The connectives in QL do not have the same meaning as in classical logic. For example, to assert $(\varphi \& \psi)$ is to assert that φ and ψ are *co-realizable*, and that (in some common conditions of realization) both are true.

(c′) $\Gamma \models_{QL} \psi$ is to be understood as the following assertion: every realization of (all the "propositions" in) Γ is a realization of ψ, and for every state of the system in question, and every realization of Γ, if all $\varphi \in \Gamma$ are true in that state (and that realization), then so is ψ.

For example, consider the conjunction "The photon is linearly polarized in direction d_1 and is linearly polarized in direction d_2", where d_1 and d_2 are oblique to one another. On Putnam's view, the assertion is false (no matter what the state of the photon). In contrast, one might maintain that the assertion simply makes no sense. It is not even false.

Remarks

(1) On the second view, quantum logic is not incompatible with classical logic. The two have different subject matters.

(2) One might try to argue against Putnam's second claim (b) by citing the fact that the relation $\models_{QL}$ does not respect the classical truth tables for negation, conjunction, and disjunction. But a stronger point can be made. $\models_{QL}$ does not respect any truth tables for these connectives. (This point has been stressed by Geoffrey Hellman [2]. We discuss it further in part 4.)

(3) One should not think that the "alternate view" is without difficulties of its own. Indeed, the EPR argument can be construed as an argument against it. For any direction d, one can bring about the conditions under which it is meaningful to attribute linear polarization in direction d to the right side photon by performing an experiment on the left side that in no way causally affects it. So how can one avoid the conclusion that it was meaningful to attribute linear polarization in direction d to the particle "all along"?

3 The "twin slit paradox"

In this section, we first reconstruct Putnam's attempted resolution of the "twin slit paradox", and then present a response by Peter Gibbins [1].

Let

'A_1' stand for "The particle passes through slit 1."

'A_2' stand for "The particle passes through slit 2."

'A_3' stand for "The particle arrives in (small) region R on the screen."

On Putnam's analysis, the "twin slit paradox" consists in the fact that we seem to be able to derive

$$pr_{QM}(A_3|(A_1 \vee A_2)) = \frac{1}{2}\, pr_{QM}(A_3|A_1) + \frac{1}{2}\, pr_{QM}(A_3|A_2), \tag{3.1}$$

even though, as a matter of experimental fact, $pr_{QM}(A_3|(A_1 \vee A_2))$ does not satisfy this additivity condition. Instead $pr_{QM}(A_3|(A_1 \vee A_2))$ exhibits interference effects. The derivation makes use of the following assumptions.

(a) $pr_{QM}(A_1 \& A_2) = 0$.

(b) $pr_{QM}(A_1) = pr_{QM}(A_2)$.

(c) Conditional probabilities are defined by the classical quotient formula, i.e., for all φ and ψ,

$$pr_{QM}(\varphi|\psi) = \frac{pr_{QM}(\varphi \& \psi)}{pr_{QM}(\psi)}.$$

(d) For all φ and ψ, if $pr_{QM}(\varphi \& \psi) = 0$, then $pr_{QM}(\varphi \vee \psi) = pr_{QM}(\varphi) + pr_{QM}(\psi)$.

(e) For all φ and ψ,

$$\varphi \models_{CL} \psi \implies pr_{QM}(\varphi) \leq pr_{QM}(\psi).$$

It goes as follows. First, by (c),

$$pr_{QM}(A_3|(A_1 \vee A_2)) = \frac{pr_{QM}(A_3 \& (A_1 \vee A_2))}{pr_{QM}(A_1 \vee A_2)}. \tag{3.2}$$

Since $(A_3 \& (A_1 \vee A_2))$ and $((A_3 \& A_1) \vee (A_3 \& A_2))$ are classically equivalent, it follows by two invocations of (e) that

$$pr_{QM}(A_3 \& (A_1 \vee A_2)) = pr_{QM}((A_3 \& A_1) \vee (A_3 \& A_2)). \tag{3.3}$$

Since $((A_3 \& A_1) \& (A_3 \& A_2)) \models_{CL} (A_1 \& A_2)$, it follows by (a) and (e) that

$$pr_{QM}((A_3 \& A_1) \& (A_3 \& A_2)) = 0,$$

and hence, by (d),

$$pr_{QM}((A_3 \& A_1) \vee (A_3 \& A_2)) = pr_{QM}(A_3 \& A_1) + pr_{QM}(A_3 \& A_2). \tag{3.4}$$

So, combining (3.2), (3.3), and (3.4), we have

$$pr_{QM}(A_3|(A_1 \vee A_2)) = \frac{pr_{QM}(A_3 \& A_1)}{pr_{QM}(A_1 \vee A_2)} + \frac{pr_{QM}(A_3 \& A_2)}{pr_{QM}(A_1 \vee A_2)}. \tag{3.5}$$

But now, by (a), (d), and (b), we also have

$$pr_{QM}(A_1 \vee A_2) = pr_{QM}(A_1) + pr_{QM}(A_2) = 2\, pr_{QM}(A_1) = 2\, pr_{QM}(A_2).$$

So we express (3.5) in the form

$$pr_{QM}(A_3|A_1 \vee A_2) = \frac{pr_{QM}(A_3\&A_1)}{2\,pr_{QM}(A_1)} + \frac{pr_{QM}(A_3\&A_2)}{2\,pr_{QM}(A_2)}.$$

The additivity condition (3.1) now follows by (c).

There is no question that the additivity condition follows from the five listed assumptions. The derivation is correct in that sense. Putnam's resolution of the "paradox" is to deny assumption (e). Without it, one cannot get to line (3.3). The sentences $(A_3\&(A_1 \vee A_2))$ and $((A_3\&A_1) \vee (A_3\&A_2))$ are not equivalent in quantum logic, he argues, and there is no reason to assume, indeed there is good empirical reason to deny, that their respective probabilities in QM are equal.

Gibbins has a very simple, but it would appear devastating, objection to this proposed resolution. Presumably, Putnam cannot object to the weakened assumption

(e$'$) For all φ and ψ,

$$\varphi \models_{QL} \psi \implies pr_{QM}(\varphi) \leq pr_{QM}(\psi).$$

And it *is* the case that a one-way version of the distributive law holds in QL:

$$((A_3\&A_1) \vee (A_3\&A_2)) \models_{QL} A_3\&(A_1 \vee A_2).$$

So it would seem that he cannot object to the following weakened version of (3.3):

$$pr_{QM}(A_3\&(A_1 \vee A_2)) \;\geq\; pr_{QM}((A_3\&A_1) \vee (A_3\&A_2)).$$

Moreover, since $(A_3\&A_1)\&(A_3\&A_2) \models_{QL} (A_1\&A_2)$, the full strength of (e) is not needed for (3.4); (e$'$) suffices. So (by the remainder of the proof, which nowhere uses (e)), we see that assumptions (a) - (d), (e$'$) collectively imply

$$pr_{QM}(A_3|A_1 \vee A_2) \geq \frac{1}{2}\,pr_{QM}(A_3|A_1) + \frac{1}{2}\,pr_{QM}(A_3|A_1).$$

But *even this one way version of the additivity condition is in conflict with experiment.* As a result of destructive interference, there will always be regions R on the screen for which the probability $pr_{QM}(A_3|A_1 \vee A_2)$ is strictly *less* than the sum on the right hand side!

4 The connectives in quantum logic are not truth functional

We remarked in part two that the connectives in quantum logic are not truth functional. Here we make the claim precise and prove it.

Our formulation of the semantics for classical sentential logic in part one was a bit non-standard. More often it is formulated in terms of "truth tables". One first takes a (classical) *truth value assignment* to be a map t from $Sent(L)$ to the set $\{T, F\}$ satisfying the following conditions

(C1′) $t(\neg\varphi) = T$ iff $t(\varphi) = F$

(C2′) $t(\varphi\&\psi) = T$ iff $t(\varphi) = T$ and $t(\psi) = T$

(C3′) $t(\varphi \vee \psi) = T$ iff $t(\varphi) = T$ or $t(\psi) = T$ (or both)

for all sentences φ and ψ in L. (The conditions, of course, reflect the standard truth tables for negation, conjunction, and (inclusive) disjunction.) Then one says that a set of sentences Γ in L *(classically) implies (or entails)* a sentence ψ in L if, for all classical truth value assignments t, if $t(\varphi) = T$ for all sentences φ in Γ, then $t(\psi) = T$. (It is easy to check that this formulation is equivalent to the one given earlier.)

Our task is to show that no parallel "truth table" characterization of $\models_{QL}$ is possible, i.e., it is not possible to move from $\models_{CL}$ to $\models_{QL}$ simply by changing conditions (C1′)-(C3′) so as to reflect the entries in some non-classical truth table.

Let $\mathcal{T}$ be a set of maps $t\colon Sent(L) \to \{T, F\}$. Let us say that $\mathcal{T}$ *respects the truth functionality of negation* if, for all t, t' in $\mathcal{T}$, and all φ, φ' in $Sent(L)$,

$$t(\varphi) = t'(\varphi') \implies t(\neg\varphi) = t'(\neg\varphi').$$

Similarly, let us say that $\mathcal{T}$ *respects the truth functionality of conjunction* if, for all t, t' in $\mathcal{T}$, and all φ, φ', ψ, ψ' in $Sent(L)$,

$$(t(\varphi) = t'(\varphi') \text{ and } t(\psi) = t'(\psi')) \implies t(\varphi\&\psi) = t'(\varphi'\&\psi').$$

Finally, let us say that $\mathcal{T}$ *respects the truth functionality of disjunction* if, for all t, t' in $\mathcal{T}$, and all φ, φ', ψ, ψ' in $Sent(L)$,

$$(t(\varphi) = t'(\varphi') \text{ and } t(\psi) = t'(\psi')) \implies t(\varphi \vee \psi) = t'(\varphi' \vee \psi').$$

Then we can formulate our claim this way.

Proposition 4.1. *Let $\mathcal{T}$ be a set of maps $t\colon Sent(L) \to \{T, F\}$. Then it is **not** the case that $\mathcal{T}$ satisfies both of the following conditions.*

(TF1) $\mathcal{T}$ respects the truth functionality of at least two of the three connectives in L.

(TF2) For all φ and ψ, $\varphi \models_{QL} \psi$ iff for all $t \in \mathcal{T}$, if $t(\varphi) = T$ then $t(\psi) = T$.

Proof. Assume that $\mathcal{T}$ satisfies condition (TF2). Since $A_1 \not\models_{QL} \neg A_1$, there exists a map t in $\mathcal{T}$ such that $t(A_1) = T$ and $t(\neg A_1) = F$. It follows that if $\mathcal{T}$ respects the truth functionality of negation, then for all t in $\mathcal{T}$ and all φ, $t(\varphi) = T \Longrightarrow t(\neg\varphi) = F$. Similarly, since $\neg A_1 \not\models_{QL} A_1$, there exists a map t in $\mathcal{T}$ such that $t(\neg A_1) = T$ and $t(A_1) = F$. Hence, if $\mathcal{T}$ respects the truth functionality of negation, it must be the case that for all t in $\mathcal{T}$ and all φ , $t(\varphi) = F \Longrightarrow t(\neg\varphi) = T$. Thus

(N) If $\mathcal{T}$ respects the truth functionality of negation, then for *all* t in $\mathcal{T}$ and all φ,

$$t(\neg\varphi) = T \text{ iff } t(\varphi) = F.$$

In effect, we have shown that *if* condition (TF2) holds, and *if* negation in "quantum logic" can be characterized in terms of *any* truth table, then it must be the standard classical truth table that characterizes it.

Analogous assertions hold for conjunction and disjunction. Since $A_1 \not\models_{QL} A_2$ there exists a map t in $\mathcal{T}$ such that $t(A_1) = T$ and $t(A_2) = F$. It follows that

$$\begin{aligned}
t(A_1\&A_1) = T &\quad \text{since} \quad A_1 \models_{QL} A_1\&A_1 \\
t(A_1\&A_2) = F &\quad \text{since} \quad (A_1\&A_2) \models_{QL} A_2 \\
t(A_2\&A_1) = F &\quad \text{since} \quad A_2\&A_1) \models_{QL} A_2 \\
t(A_2\&A_2) = F &\quad \text{since} \quad (A_2\&A_2) \models_{QL} A_2 \\
t(A_1 \vee A_1) = T &\quad \text{since} \quad A_1 \models_{QL} (A_1 \vee A_1) \\
t(A_1 \vee A_2) = T &\quad \text{since} \quad A_1 \models_{QL} (A_1 \vee A_2) \\
t(A_2 \vee A_1) = T &\quad \text{since} \quad A_1 \models_{QL} (A_2 \vee A_1) \\
t(A_2 \vee A_2) = F &\quad \text{since} \quad (A_2 \vee A_2) \models_{QL} A_2.
\end{aligned}$$

Hence

(C) If $\mathcal{T}$ respects the truth functionality of conjunction, then for *all* t in $\mathcal{T}$ and all φ, ψ

$$t(\varphi\&\psi) = T \;\text{ iff }\; t(\varphi) = T \text{ and } t(\psi) = T.$$

(D) If $\mathcal{T}$ respects the truth functionality of disjunction, then for *all* t in $\mathcal{T}$ and all φ, ψ

$$t(\varphi \vee \psi) = T \;\text{ iff }\; t(\varphi) = T \;\text{ and }\; t(\psi) = T \text{ (or both).}$$

(These are just the classical truth table characterizations for conjunction and negation.) It only remains to consider the three cases.

(case 1) Assume $\mathcal{T}$ respects the truth functionality of conjunction and disjunction. Then it follows from (C), (D), and condition (TF2), that

$$A_1\&(A_2 \vee A_3) \models_{QL} (A_1\&A_2) \vee (A_1\&A_3).$$

This is a contradiction since this inference is *not* valid in quantum logic.

(case 2) Assume $\mathcal{T}$ respects the truth functionality of negation and conjunction. Then it follows from (N), (C), and (TF2) that

$$A_1\&\neg(A_1\&A_2) \models_{QL} \neg A_2.$$

Again we have a contradiction since this inference is not valid in quantum logic.

(case 3) Assume $\mathcal{T}$ respects the truth functionality of negation and disjunction. Then it follows from (N), (D), and (TF2) that

$$\neg(\neg A_1 \vee \neg(\neg A_1 \vee \neg A_2)) \models_{QL} \neg A_2.$$

This inference is not valid in quantum logic either. So, again, we have a contradiction. $\square$

References

[1] P. Gibbins. Putnam on the twin-slit experiment. *Erkenntnis*, 16:235–241, 1981.

[2] G. Hellman. Quantum logic and meaning. In P. Asquith and R. Giere, editors, *PSA: Proceedings of the 1980 Biennial Meeting of the Philosophy of Science Association*, volume II, pages 493–511. Philosophy of Science Association, 1981. (Available online: http://www.jstor.org/stable/192607).

[3] H. Putnam. Is logic empirical. In R. Cohen and M. Wartofsky, editors, *Boston Studies in the Philosophy of Science*, volume 5, pages 216–241. Kluwer, 1969.