Geometry and Spacetime
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Model Solutions for Odd-Numbered Problems in Section 2.4

Problem 2.4.1 Prove that for all vectors u and v in V,

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

Proof Let u and v be vectors in V. Then

$$\begin{aligned} \|u+v\|^2 + \|u-v\|^2 &= \langle u+v,\, u+v \rangle + \langle u-v,\, u-v \rangle \\ &= 2 \langle u,\, u \rangle + 2 \langle v,\, v \rangle \\ &= 2\,(\|u\|^2 + \|v\|^2). \qquad \square \end{aligned}$$

Problem 2.4.3 (The measure of a straight angle is π.) Let p, q, r be (distinct) collinear points, and suppose that q is between p and r (i.e., $\overrightarrow{pq} = a\,\overrightarrow{pr}$ with $0 < a < 1$). Show that $\measuredangle(p,q,r) = \pi$.

Proof Since $\overrightarrow{pq} = a\,\overrightarrow{pr}$, we have $\overrightarrow{qp} = -a\,\overrightarrow{pr}$ and $\overrightarrow{qr} = (1-a)\,\overrightarrow{pr}$. Hence,

$$\langle \overrightarrow{qp},\ \overrightarrow{qr} \rangle = \langle -a\,\overrightarrow{pr},\, (1-a)\,\overrightarrow{pr} \rangle = -a\,(1-a)\,\|\overrightarrow{pr}\|^2$$

and, since $0 < a < 1$,

$$\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\| \ = \ \|-a\,\overrightarrow{pr}\|\,\|(1-a)\,\overrightarrow{pr}\| \ = \ a\,(1-a)\,\|\overrightarrow{pr}\|^2.$$

It follows that

$$cos(\measuredangle(p,q,r)) \ = \ \frac{\langle \overrightarrow{qp},\ \overrightarrow{qr} \rangle}{\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|} \ = \ -1.$$

The only number between 0 and π whose cosine is -1 is π. So, $\measuredangle(p,q,r) = \pi$. $\square$

Problem 2.4.5 (Right Angle in a Semicircle Theorem) Let p, q, r, o be (distinct) points such that (i) p, o, r are collinear, and (ii) $\|\overrightarrow{op}\| = \|\overrightarrow{oq}\| = \|\overrightarrow{or}\|$. (So q lies on a semicircle with diameter $LS(p,r)$ and center o.) Show that $\overrightarrow{qp} \perp \overrightarrow{qr}$, and so $\measuredangle(p,q,r) = \dfrac{\pi}{2}$.

Proof By (i), we have $\overrightarrow{op} = a\,\overrightarrow{or}$ for some a. Hence, $\|\overrightarrow{op}\| = |a|\,\|\overrightarrow{or}\|$ and, therefore, by (ii), $|a| = 1$. Now a cannot be 1. For if $\overrightarrow{op} = \overrightarrow{or}$, then

$$\overrightarrow{op} \ = \ \overrightarrow{or} + \overrightarrow{rp} \ = \ \overrightarrow{op} + \overrightarrow{rp}.$$

And so it would follow that $\overrightarrow{rp} = \mathbf{0}$, which is impossible since p and r are distinct. So $a = -1$ and $\overrightarrow{op} = -\overrightarrow{or}$. This implies that

$$\overrightarrow{qr} \;=\; \overrightarrow{qo} + \overrightarrow{or} \;=\; -\overrightarrow{oq} - \overrightarrow{op}.$$

We also clearly have

$$\overrightarrow{qp} \;=\; \overrightarrow{qo} + \overrightarrow{op} \;=\; -\overrightarrow{oq} + \overrightarrow{op}.$$

Hence, by (ii) again,

$$\begin{aligned} < \overrightarrow{qp}, \overrightarrow{qr} > \;&=\; < -\overrightarrow{oq} + \overrightarrow{op}, -\overrightarrow{oq} - \overrightarrow{op} > = < \overrightarrow{oq}, \overrightarrow{oq} > - < op, op > \\ &=\; \|\overrightarrow{oq}\|^2 - \|\overrightarrow{op}\|^2 \;=\; 0. \end{aligned}$$

Thus, $\overrightarrow{qp} \perp \overrightarrow{qr}$ and

$$cos(\measuredangle(p,q,r)) \;=\; \frac{< \overrightarrow{qp},\ \overrightarrow{qr} >}{\|\overrightarrow{qp}\|\,\|\overrightarrow{qr}\|} \;=\; 0.$$

The only number between 0 and π whose cosine is 0 is $\frac{\pi}{2}$. So, $\measuredangle(p,q,r) = \frac{\pi}{2}$.
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