Geometry and Spacetime
David B. Malament

Model Solutions for Odd-Numbered Problems in Section 2.1

Problem 2.1.1 Prove that for all vectors u in V, there is a *unique* vector v in V such that $u + v = \mathbf{0}$.

Proof Assume that for some vector u in V there are *distinct* vectors v_1 and v_2 in V such that $u + v_1 = \mathbf{0}$ and $u + v_2 = \mathbf{0}$. Then

$$\begin{aligned}
v_1 &= v_1 + \mathbf{0} && \text{by VS 3} \\
&= v_1 + (u + v_2) && \text{by our assumption that } u + v_2 = \mathbf{0} \\
&= (v_1 + u) + v_2 && \text{by VS 2} \\
&= (u + v_1) + v_2 && \text{by VS 1} \\
&= \mathbf{0} + v_2 && \text{by our assumption that } u + v_1 = \mathbf{0} \\
&= v_2 + \mathbf{0} && \text{by VS 1} \\
&= v_2 && \text{by VS 3}
\end{aligned}$$

Thus we have a contradiction (since we assumed that v_1 and v_2 are distinct), and may conclude that there can be only one vector v in V such that $u + v = \mathbf{0}$. □

Problem 2.1.3 Prove that for all vectors u in V, and all real numbers a,

(i) $0 \cdot u = \mathbf{0}$

(ii) $-u = (-1) \cdot u$

(iii) $a \cdot \mathbf{0} = \mathbf{0}$.

Proof Let u be a vector in V and let a be a real number. (i) By VS 6,

$$0 \cdot u = (0 + 0) \cdot u = 0 \cdot u + 0 \cdot u.$$

So, by problem 2.1.2, $0 \cdot u = \mathbf{0}$.

(ii) By VS 8 and VS 6,

$$u + (-1) \cdot u = 1 \cdot u + (-1)u = (1 - 1) \cdot u = 0 \cdot u.$$

But, by part (i), $0 \cdot u = \mathbf{0}$. So $u + (-1) \cdot u = \mathbf{0}$. Thus $(-1) \cdot u$ is the additive inverse of u, i.e., $(-1) \cdot u = (-u)$.

(iii) By VS 3 and VS 5,

$$a \cdot \mathbf{0} = a \cdot (\mathbf{0} + \mathbf{0}) = a \cdot \mathbf{0} + a \cdot \mathbf{0}.$$

So, by problem 1.1.2 again, $a \cdot \mathbf{0} = \mathbf{0}$. □

Problem 2.15 Let S be a subset of V. Show that $L(S) = S$ iff S is a subspace of V.

Proof Let X be the set of all subspaces of V that contain S as a subset. So $L(S) = \cap X$. Since S is a subset of each element of X, it is certainly a subset of the intersection of all those elements, i.e., $S \subseteq \cap X$. So $S \subseteq L(S)$. This much holds without any special assumptions about S. But if S is itself a subspace of V, i.e., $S \in X$, then it is also true that $\cap X \subseteq S$, and so $L(S) \subseteq S$. Thus, if S is a subspace of V, it follows that $L(S) = S$. Conversely, if $L(S) = S$, then S is certainly a subspace of V, since the linear span of *any* subset of V is a subspace of V. □

Problem 2.17 Show that two finite dimensional vector spaces are isomorphic iff they have the same dimension.

Proof Let $\mathbf{V} = (V, +, \mathbf{0}, \cdot)$ and $\mathbf{V}' = (V', +', \mathbf{0}', \cdot')$ be finite dimensional vector spaces. Assume first that there exists an isomorphism $\Phi : V \to V'$. Let $n = \dim(\mathbf{V})$. If $n = 0$, then $V = \{\mathbf{0}\}$ and $V' = \{\Phi(\mathbf{0})\} = \{\mathbf{0}'\}$. So, $\dim(\mathbf{V}') = 0 = \dim(\mathbf{V})$. Thus we may assume $n \geq 1$. Let $S = \{u_1, ..., u_n\}$ be a basis for $\mathbf{V}$. We claim that $S' = \{\Phi(u_1), ..., \Phi(u_n)\}$ is a basis for $\mathbf{V}'$ and therefore, in this case too, $\dim(V) = \dim(V')$.

First, we verify that S' is linearly independent. Assume to the contrary that there exist coefficients $a_1, ..., a_n$, not all 0, such that

$$a_1 \cdot' \Phi(u_1) \ +' \ ... \ +' \ a_n \cdot' \Phi(u_n) = \mathbf{0}'.$$

Since Φ is linear it follows that

$$\Phi(a_1 \cdot u_1 + ... + a_n \cdot u_n) = a_1 \cdot' \Phi(u_1) \ +' \ ... \ +' \ a_n \cdot' \Phi(u_n) = \mathbf{0}'.$$

Hence, since $\ker(\Phi) = \mathbf{0}$, $a_1 \cdot u_1 + ... + a_n \cdot u_n = \mathbf{0}$. But this is impossible since S is a basis (and, therefore, linearly independent). So S' is linearly independent, as claimed.

Next, we verify that $L(S') = V'$. Let u' be any vector in V'. Since Φ maps V onto V', there is a vector u in V such that $\Phi(u) = u'$. Since S is a basis for V, there exist coefficients $a_1, ..., a_n$ such that $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$. Hence, by the linearity of Φ again,

$$u' = \Phi(u) = \Phi(a_1 \cdot u_1 + ... + a_n \cdot u_n) = a_1 \cdot' \Phi(u_1) \ +' \ ... \ +' \ a_n \cdot' \Phi(u_n).$$

Thus u' is in $L(S')$. Since u' was an arbitrary vector in V', $L(S') = V'$. Thus, S' is a basis for $\mathbf{V}'$, as claimed.

Conversely, assume that $\mathbf{V}$ and $\mathbf{V}'$ both have dimension n. If n = 0, then $V = \{\mathbf{0}\}$ and $V' = \{\mathbf{0}'\}$. So the trivial map Φ that takes $\mathbf{0}$ to $\mathbf{0}'$ qualifies as an isomorphism between the vector spaces. (It is certainly a bijection. And it is linear since, by VS 3,

$$\Phi(\mathbf{0} + \mathbf{0}) = \Phi(\mathbf{0}) = \mathbf{0}' = \mathbf{0}' + \mathbf{0}' = \Phi(\mathbf{0}) + \Phi(\mathbf{0})$$

and, for all real numbers a,

$$\Phi(a \cdot \mathbf{0}) = \Phi(\mathbf{0}) = \mathbf{0}' = a \cdot \mathbf{0}' = a \cdot \Phi(\mathbf{0}).$$

(Here we use the fact that $a \cdot \mathbf{0} = \mathbf{0}$, which we have from problem 1.1.3.) So we may assume that $n \geq 1$. Let $S = \{u_1, ..., u_n\}$ be a basis for V, and let $S' = \{u'_1, , ..., u'_n\}$ be a basis for V'. We define a map $\Phi : V \to V'$ as follows. Given any vector u in V, it can be expressed uniquely in the form $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$. We take $\Phi(u)$ to be $a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n$. We claim that, as defined, Φ is an isomorphism.

First, it is injective, i.e., $\ker(\Phi) = \{\mathbf{0}\}$. For suppose $\Phi(u) = \mathbf{0}'$ for some vector $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$. Then $\mathbf{0}' = \Phi(u) = a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n$. And therefore, since S' is linearly independent, all the coefficents a_i must be 0, i.e., $u = \mathbf{0}$. Thus, $\ker(\Phi) = \mathbf{0}$, as claimed.

Next, Φ maps V onto V'. For let u' be any vector in V'. It can be expressed as $u' = a_1 \cdot' u'_1 + ... + a_n \cdot' u'_n$. Hence, if $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$, $\Phi(u) = u'$. So $\Phi[V] = V'$, as claimed.

Finally, Φ is linear. For given any vectors $u = a_1 \cdot u_1 + ... + a_n \cdot u_n$ and $v = b_1 \cdot u_1 + ... + b_n \cdot u_n$ in V, and any real number a, it follows (by VS 1, VS 2, and VS 6) that

$$\begin{aligned}
\Phi(u + v) &= \Phi((a_1 + b_1) \cdot u_1 + ... + (a_n + b_n) \cdot u_n) \\
&= (a_1 + b_1) \cdot' u'_1 +' ... +' (a_n + b_n) \cdot' u'_n \\
&= (a_1 \cdot' u'_1 +' ... +' a_n \cdot' u'_n) + (b_1 \cdot' u'_1 +' ... +' b_n \cdot' u'_n) \\
&= \Phi(u) +' \Phi(v),
\end{aligned}$$

and (by VS 5 and VS 7) that

$$\begin{aligned}
\Phi(a \cdot u) &= \Phi(a \cdot (a_1 \cdot u_1 + ... + a_n \cdot u_n)) \\
&= \Phi((aa_1) \cdot u_1 + ... + (aa_n) \cdot u_n) \\
&= (aa_1) \cdot' u'_1 + ... + (aa_n) \cdot' u'_n \\
&= a \cdot' (a_1 \cdot' u'_1 + ... + a_n \cdot' u'_n) \\
&= a \cdot' \Phi(u).
\end{aligned}$$

So we are done. □