CHAPTER TEN

RELATION TO QUANTUM MECHANICS

In this chapter we begin a study of the relationship between observer theory and quantum mechanics. The first section presents an overview of the characterization of quantum systems initiated by von Neumann, Weyl, Wigner, and Mackey. For this section we have relied heavily on the book by V.S. Varadarajan (1985). The second section discusses the appearance of vector bundles in this context. In the third section we explore possible connections between these vector bundles and linearizations of the specialized chain bundles of **9**–5.

Our approach is based on the idea that theories of measurement, which form the basis of quantum formalism, must have a large overlap with theories of perception. Quantum interpretations rest entirely on the interaction between observer and observed, and on the irreducible effects on both of them subsequent to such interaction. Conversely, it is reasonable to require of a theory of perception that it provide some illumination on the paradoxes that have dogged measurement theory to date. We must, however, make clear that in this chapter our intention is to provide neither a scholarly treatment of quantum measurement theory, nor a full and rigorous grounding of that theory in observer mechanics. Rather, we initiate a line of enquiry into their relationship, making a first attempt at setting up a language within which quantum measurement and perception-in-general may both be discussed.

There are other stochastic-foundational formalisms which seek to ground quantum theory, such as those of Nelson (1985) and Prugovecki (1984). We here make no comparisons with these theories.

1. Quantum systems and imprimitivity

The description of a "physical system" involves various constituents. First is a "set of propositions," or empirically verifiable statements. On this set is a "logic" obeyed by its elements. There is a notion of the possible "states" of the system and of the "dynamical evolution" of these states. There is a group of "symmetries" compatible with the logic and leaving the dynamics of the system invariant. There is usually a "configuration space" on which this group also acts. Finally, there is a specification of the possible "observables" of the system. In this section we discuss these concepts, and how they lead to the idea of a system of imprimitivity.

For our purposes, a logic $\mathcal{L}$ is a set Π of propositions together with a syntax in which the notions of "implies," "and," "or," and "not" are given, along with rules for their application. Quantum systems are distinguished from classical ones by their logics: a classical system obeys a "Boolean" logic, while a quantum system obeys a "standard logic."

More precisely, let Π be the set of propositions of a physical system. We call the system ***classical*** if there is a measurable space $(Y, \mathcal{Y})$ and a bijective function $\Phi \colon \Pi \to \mathcal{Y}$, such that the logic $\mathcal{L}$ on Π is that induced by Φ from the Boolean algebra $\mathcal{Y}$. That is, if we denote "implies" by $\Rightarrow$, "and" by $\wedge$, "or" by $\vee$ and "not" by $-$, we have $\mathcal{L} = (\Pi, \Rightarrow, \wedge, \vee, -)$, where for a, $b \in \Pi$,

$$\begin{aligned} a \Rightarrow b &\text{ iff } \Phi(a) \subset \Phi(b), \\ a \wedge b &\equiv \Phi^{-1}(\Phi(a) \cap \Phi(b)), \\ a \vee b &\equiv \Phi^{-1}(\Phi(a) \cup \Phi(b)), \\ -a &\equiv \Phi^{-1}(Y - \Phi(a)). \end{aligned} \tag{1.1}$$

Here "$\equiv$" means "defined by." For the partial order $\Rightarrow$ on Π there is a least element $0 = \Phi^{-1}(\emptyset)$ and a greatest element $1 = \Phi^{-1}(Y)$. A logic is called a ***σ-logic*** if it is closed under countable applications of $\wedge$ and $\vee$.

The peculiarity of quantum systems is that their logics are non-distributive: e.g., the proposition "a and (b or c)" need not have the same truth value as "(a and b) or (a and c)." Hence the distributive, or "de Morgan," laws valid in Boolean logic must be abandoned in favor of weaker laws. It turns out that an appropriate logic, called a ***standard*** or ***quantum*** logic, may be described as follows. There is a separable Hilbert space $\mathcal{H}$ over $\mathbf{C}$. Denote the set of closed subspaces of $\mathcal{H}$ by $\mathcal{S}(\mathcal{H})$. There is a bijective function $\Phi \colon \Pi \to \mathcal{S}(\mathcal{H})$ such that, for $a,\, b \in \Pi$

$$\begin{aligned} a \Rightarrow b \text{ iff } \Phi(a) < \Phi(b) \quad & (\text{ i.e., } \Phi(a) \text{ is a subspace of } \Phi(b)), \\ a \wedge b \equiv \Phi^{-1}(\Phi(a) \cap \Phi(b)), & \\ a \vee b \equiv \Phi^{-1}(\Phi(a) \bigvee \Phi(b)), & \\ -a \equiv \Phi^{-1}(\Phi(a)^{\perp}). & \end{aligned} \tag{1.2}$$

Here $\bigvee$ means "join": the join of a collection of subspaces is their joint closed linear span. $\perp$ denotes orthogonal complement. We set $0 = \Phi^{-1}(\{0\})$ and $1 = \Phi^{-1}(\mathcal{H})$.

It is easy to see that this is a σ-logic, and that if the Hilbert space $\mathcal{H}$ is of dimension ≥ 2 then the standard logic is non-distributive. For any $\mathcal{H}$, the standard logic is a σ-logic. Since there is a bijective correspondence $V \leftrightarrow P_V$ between closed subspaces in $\mathcal{S}(\mathcal{H})$ and orthogonal projections, we may also model the standard logic in terms of these projections. From now on we simply identify Π with $\mathcal{P}(\mathcal{H})$, the set of orthogonal projections on $\mathcal{H}$, or with $\mathcal{S}(\mathcal{H})$, whichever is convenient.

Assumption 1.3. We restrict our discussion of quantum systems to those obeying a standard logic.[1]

In section three we consider how these systems might be naturally associated to specializations of symmetric frameworks.

We may now define a ***state*** of a physical system. It is a mapping $\sigma\colon \mathcal{P}(\mathcal{H}) \to [0,1]$, the unit interval, such that

(i) $\sigma(0) = 0,\ \sigma(I) = 1.$

(ii) If $\{P_{U_i}\}_{i=1}^{\infty}$ is a pairwise orthogonal sequence of projections then

$$\sigma(P_{\bigvee_{i=1}^{\infty} U_i}) = \sum_{i=1}^{\infty} \sigma(P_{U_i}).$$

Intuitively, a state is a way to assign a likelihood to each proposition in the logic.

The set of all states, denoted by Σ, is a convex subset of the space of all mappings $\mathcal{P}(\mathcal{H}) \to [0,1]$. The ***pure states*** are the extremal elements of Σ as a convex set. Nonpure states are termed ***mixtures***. If the dimension of $\mathcal{H}$ is greater than 2, a theorem of Gleason says that states are in one-to-one

[1] Some systems studied in physics obey logics which are (non-Boolean) sublogics of standard logics. We do not treat such systems here.

correspondence with nonnegative selfadjoint operators on $\mathcal{H}$ of trace unity, as follows. Let σ be a state. Then there exists such an operator D_σ such that

$$\sigma(P_V) = \mathrm{Tr}(D_\sigma P_V), \quad V \in \mathcal{S}(\mathcal{H}). \tag{1.4}$$

D_σ is called the ***density operator*** of the state σ. If σ is a pure state, D_σ is orthogonal projection onto a one-dimensional subspace V of $\mathcal{H}$. That is, to σ there corresponds a unit vector ψ in V such that $D_\sigma \phi = P_{[\psi]}\phi = \langle \psi, \phi \rangle \psi$ for all $\phi \in \mathcal{H}$ ($\langle \cdot, \cdot \rangle$ being the inner product of $\mathcal{H}$).

The states of a physical system change, in general, with time. Let us write the state at time t as σ_t, assuming it was σ at time 0. It is reasonable to assume that this change is linear: if $\{c_i\}_{i=1}^n$ are positive numbers whose sum is unity, and if $\{\sigma^i\}_{i=1}^n$ are states, then

$$(\sum_{i=1}^{n} c_i \sigma^i)_t = \sum_{i=1}^{n} c_i \sigma_t^i. \tag{1.5i}$$

It is also reasonable to suppose that, for any $V \in \mathcal{S}(\mathcal{H})$,

$$t \to \sigma_t(P_V) \text{ is a Borel function} \tag{1.5ii}$$

from $\mathbf{R}$ to $[0, 1]$. Finally, it is clear that the evolution has the structure of a one-parameter group:

$$\sigma_{t_1+t_2} = (\sigma_{t_1})_{t_2}, \tag{1.5iii}$$

called the ***dynamical group*** of the system. The conditions in 1.5 are summarized by saying that $t \to \sigma_t$ is a ***representation*** of the additive group of real numbers in $\mathrm{Aut}(\Sigma)$, the group of convex automorphisms of Σ. By Stone's theorem, to this evolution there corresponds a selfadjoint operator H on $\mathcal{H}$, unique up to additive constants, such that the density operator D_{σ_t} is related to D_σ by

$$D_{\sigma_t} = e^{-itH} D_\sigma e^{itH}. \tag{1.6}$$

If σ is a pure state with density operator $P_{[\Psi]}$, this reduces to

$$D_{\sigma_t} = P_{[e^{-itH}\Psi]}, \quad D_\sigma = P_{[\Psi]}.$$

The operator H, which determines the evolution of states, is called the ***hamiltonian*** of the system.

The result of a "physical measurement" is a proposition stating that a certain quantity takes values in some subset of, say, the real numbers. An ***observable*** of a quantum system is, then, the association of a projection to each (Borel) subset of the real numbers in a manner consistent with such measurements. Precisely, an observable is a ***projection-valued measure,*** i.e., a mapping χ from the Borel σ-algebra $\mathcal{B}$ of $\mathbf{R}$ into $\mathcal{P}(\mathcal{H})$, such that

(i) $\chi(\emptyset) = 0$, $\chi(\mathbf{R}) = I$.

(ii) If $E, F \in \mathcal{B}$ and $E \cap F = \emptyset$ then $\chi(E) \perp \chi(F)$.

(iii) If $\{E_i\}_{i=1}^{\infty}$ is a sequence of pairwise disjoint sets in $\mathcal{B}$,

$$\chi(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \chi(E_i).$$

The meaning of (i) is clear. The second condition is the requirement that the propositions

$\mathcal{E}$: The observation takes a value in E and

$\mathcal{F}$: The observation takes a value in F,

are logically contradictory statements if $E \cap F = \emptyset$. The third condition states that the proposition "the observable quantity takes value in at least one of the E_i" corresponds, in the logic, to the join of the subspaces corresponding to the E_i.

More generally, given a measurable space Y, a ***Y-valued observable*** of the system is a projection-valued measure based on Y (i.e., satisfying (i), (ii), and (iii) above, with $\mathbf{R}$ replaced by Y).

If σ is a state and χ is an observable, $\sigma \circ \chi$ is a Borel probability measure on $\mathbf{R}$. Quantum theory prescribes for $\sigma \circ \chi$ the interpretation that it is the distribution of observed values for the observable χ in the state σ. A customary way of saying this employs the spectral calculus to associate to χ the selfadjoint operator A_χ given by

$$A_\chi = \int_{\mathbf{R}} \chi(d\lambda)\lambda. \tag{1.7}$$

It follows that the ***expected value of the observable χ in the state σ*** is then

$$\langle\chi\rangle_\sigma \equiv \mathrm{Tr}(D_\sigma A_\chi). \tag{1.8}$$

In particular, for a pure state with $D_\sigma = P_{[\Psi]}$,

$$\langle\chi\rangle_\sigma = \langle\Psi, A_\chi \Psi\rangle \tag{1.9}$$

where $\langle , \rangle$ is the inner product in $\mathcal{H}$.

This is the point at which the theory makes contact with experiment.

We have seen above how a group representing the "time-axis" defines the dynamics of a system. Physical systems have, however, a deeper geometrical character arising out of the requirement of the "objectivity" of experimental results. This requirement is framed, within the scientific paradigm, in terms of the ***relativity*** of conclusions arrived at by different experimenters viewing the same phenomenon, as follows.

Consider a physical system, together with a class of "experimenters" which take measurements in the system. Suppose that the set of meaningful statements (with its given syntax) is, for each of these experimenters, the same: namely, the given logic $\mathcal{L}$. Intuitively, this means that each conceivable "physical" phenomenon for any one experimenter is a conceivable phenomenon for any other. However, at each instant of time the various experimenters have different ***ways*** to use the propositions of Π to describe these phenomena. Let us call a particular experimenter's way of doing this his "frame of reference" at time t. Let Ω denote the set of all the frames of reference for these experimenters. (We allow different experimenters to have the same frame of reference.) In looking at a phenomenon, an experimenter with frame of reference ω_i would describe it with a proposition, say $p(\omega_i) \in \Pi$; an experimenter looking at the same phenomenon, but with frame of reference ω_j, would describe it with a proposition $p(\omega_j)$. If $\omega_i \neq \omega_j$ then, in general, $p(\omega_i) \neq p(\omega_j)$. In order to objectively relate propositions in ω_i to those in ω_j we would expect that there exist bijective mappings

$$T_{\omega_j,\omega_i}\colon \Pi \to \Pi, \quad \forall\, \omega_i, \omega_j \in \Omega, \tag{1.10}$$

such that $T_{\omega_j,\omega_i}(p(\omega_i)) = p(\omega_j)$. Thus T_{ω_j,ω_i} provides a dictionary that translates propositions about any phenomenon made with frame of reference ω_i into propositions about that phenomenon made with frame of reference ω_j. Now what makes Π useful is the logic $\mathcal{L}$; thus these T_{ω_j,ω_i} should preserve the syntax of the logic, i.e., the operations of 1.2. Such a bijective mapping is called an ***automorphism of the logic*** $\mathcal{L}$. Notice that the identity automorphism of $\mathcal{L}$ is included: $T_{\omega,\omega}$ is the identity mapping of Π, for any $\omega \in \Omega$. Also, T_{ω_i,ω_j} is the inverse automorphism of T_{ω_j,ω_i}. The requirement of objectivity may then be expressed as follows:

Assumption 1.11. (*Objectivity*). The set

$$J = \{T_{\omega,\omega'} \mid \omega, \omega' \in \Omega\}$$

is a subgroup of the group of automorphisms of $\mathcal{L}$. Given $g \in J$ and $\omega \in \Omega$, there exists a unique $\omega' \in \Omega$ such that $g = T_{\omega',\omega}$. If we denote this ω' by

$g\omega$, then $\omega \to g\omega$ is a transitive action of the group J on the set Ω. The automorphisms $T_{\omega',\omega}$ depend only on the frames ω and ω' and, in particular, have no explicit dependence on time.

We call J the ***group of (physical) symmetries*** of the system (for the given class of experimenters). The transitivity of the action means that no pair of frames of reference are isolated from each other, i.e., T_{ω_i,ω_j} exists for each pair (ω_i, ω_j). Furthermore, the transitivity on Ω implies that the dictionary translation between ω_i and ω_j can be effected through any intermediary ω_k: $T_{\omega_i,\omega_j} = T_{\omega_i,\omega_k} T_{\omega_k,\omega_j}, \quad \forall\, \omega_i,\, \omega_j,\, \omega_k \in \Omega$.

Objectivity is a property of a class of experimenters on the system; it expresses the mutual consistency of descriptions of the system by the various experimenters in the class. At this level of analysis, the group J is associated to the class of experimenters; one does not need to have a "configuration space" for the system (see below) in order to make sense of the group.

At this point we note some connections with observer theory. The situation we have been discussing corresponds to a symmetric framework (X, Y, E, S, G, J, π). The "experimenters" are participators in the framework; the class of experimenters under consideration are the participators in a particular environment supported by the framework. The "frame of reference" of an experimenter at time t is the perspective of the participator at time t; thus we can think of the set Ω of frames of reference as being isomorphic to the set of distinguished perspectives $\{\pi_e \mid e \in E\}$ (or isomorphic to E itself). The group J is the distinguished structure group of the framework; the action of J on E in the framework corresponds to the action of J on Ω in 1.11. Notice that the logic $\mathcal{L}$ of the physical system is not explicitly in evidence at this level of description. However, recall that the original meaning of a frame of reference is a "way of using the propositions of the logic to describe physical phenomena." Such a way, then, corresponds to a way of mapping E to S. We should expect, therefore, that the logic $\mathcal{L}$ itself has meaning in the observer theory and, conversely, that the fundamental map π and the premise space S have meaning in the quantum mechanics. And the quantum mechanical notion of ***state*** must have an observer-theoretic interpretation consistent with these meanings. These interpretations will emerge most clearly when we realize the framework above as a ***specialization***. The goal of the chapter is to lay some groundwork for this level of connection between the two theories. In this section, however, we continue to use the terminology "experimenter," "frame of reference" and "physical symmetry group" rather than "participator," "perspective" and "framework group."

Returning to our overview of quantum mechanics, we assume that 1.11 is satisfied. We denote the action of J on $\mathcal{S}(\mathcal{H})$ by $(g, V) \mapsto gV$ and its action on $\mathcal{P}(\mathcal{H})$ by $(g, P_V) \mapsto {}^gP_V \equiv P_{gV}$.

Thus J may be viewed as a subgroup of the ***projective group*** $\mathrm{Aut}\mathcal{P}(\mathcal{H})$ of automorphisms of $\mathcal{P}(\mathcal{H})$. We assume henceforth that J satisfies the following assumptions.

Assumption 1.12.

(i) J has a locally compact, second-countable (lcsc) topology; the corresponding standard Borel structure will be denoted $\mathcal{J}$, and J is a measurable group with this structure.

(ii) If $\mathcal{P}(\mathcal{H})$ is given the ***strong*** topology, i.e., if $\{P_n\}_{n=1}^{\infty} \subset \mathcal{P}(\mathcal{H})$ then $P_n \to P \in \mathcal{P}(\mathcal{H})$ iff $P_n u \to Pu$ in $\mathcal{H}$ for all $u \in \mathcal{H}$, then J acts measurably on $\mathcal{P}(\mathcal{H})$.

These assumptions are summarized by saying that the action of J on $\mathcal{P}(\mathcal{H})$ gives a ***representation*** of J in $\mathrm{Aut}\mathcal{P}(\mathcal{H})$.

Let $\mathcal{U}$ be the group of ***unitary*** automorphisms of $\mathcal{H}$ (i.e., $B \in \mathcal{U}$ iff $B\colon \mathcal{H} \to \mathcal{H}$ is a surjective isometry). We have the following result from representation theory.

Theorem 1.13. Under Assumption 1.12, the action of J on $\mathcal{P}(\mathcal{H})$ arises from a ***unitary*** representation in the following manner. Let J^* be the ***universal covering group*** of J. Let $\delta\colon J^* \to J$ be the covering homomorphism. Then there exists a unique unitary representation of J^* in $\mathcal{U}$, say $g^* \mapsto U_{g^*}$, such that for any $V \in \mathcal{S}(\mathcal{H})$ and $g \in J$,

$$gV = U_{g^*}V \quad \text{for any } g^* \text{ with } \delta(g^*) = g, \quad \text{or equivalently}$$

$${}^gP_V = U_{g^*}P_V U_{g^*}^{-1} \quad \text{for any } g^* \text{ with } \delta(g^*) = g.$$

Since we assume that our symmetry group J satisfies (i) and (ii) of 1.12, it also satisfies the conclusions of 1.13. Examples of such a J include the group of additive reals (leading to the dynamical group above) and the groups of Galilean and Einsteinian relativity.

We imagine that to each experimenter, at each time t, is associated a state of the system (which describes the way the experimenter assigns probabilities to propositions). We think of this as the experimenter's *description* of the system at time t. This is distinct from the experimenter's frame of reference. In particular, consider two experimenters whose frames of reference at time t are ω and ω', where $\omega' = g\omega$ for some $g \in J$. Suppose that at time t the state associated to the first experimenter is $\sigma \in \Sigma$. Let σ^g denote the state that expresses in terms of the frame of reference $g\omega$ the same underlying probabilities that are expressed by the state σ of the first experimenter in terms of its frame of reference ω. In this way the action of J on $\mathcal{L}$ gives rise to an action on Σ. By definition this action has the property that, for any $P \in \mathcal{P}(\mathcal{H})$,

$$\sigma(P) = \sigma^g({}^gP), \text{i.e.,}$$

$$\sigma^g(\cdot) = \sigma({}^{g^{-1}}\cdot). \tag{1.14}$$

It is clear that $\sigma \to \sigma^g$ is in fact a *convex* automorphism of Σ (i.e., preserving convex combinations of states). We assume that, for each $\sigma \in \Sigma$ and $P \in \mathcal{P}(\mathcal{H})$, $g \mapsto \sigma^g(P)$ is a Borel map from J to $\mathbf{R}$. We then say that we have a representation of J in the collection of all convex automorphisms $\mathrm{Aut}(\Sigma)$ of states, a representation which is *covariant* with the representation in $\mathrm{Aut}(\mathcal{P}(\mathcal{H}))$, as indicated in 1.14.

Henceforth we assume that, at all times t, the descriptions of the system by the various experimenters are in agreement; we say that their descriptions are *covariant* with J:

Assumption 1.15. *Descriptive Covariance with* J. Let A and A' be any two experimenters (in the given class) whose frames of reference at time t are ω and $\omega' = g\omega$ respectively. Then the states σ and σ' associated to A and A' at time t are related by $\sigma' = \sigma^g$.

To relate this type of covariance to the dynamics given in 1.6 we first note that the requirement of time independence of the $T_{\omega,\omega'}$ (in Assumption 1.11) may be expressed as follows: for any $\sigma \in \Sigma$, $g \in J$, and $t \in \mathbf{R}$,

$$(\sigma^g)_t = (\sigma_t)^g.$$

This implies that the hamiltonian H commutes with the unitary action of J of 1.13: if we write for $g \in J$, $U_g = U_{g^*}$ for any $g^* \in J^*$ with $\delta(g^*) = g$, then

$$[H, U_g] = HU_g - U_gH = 0. \tag{1.16}$$

That is, the dynamical law is the same for each experimenter.

We have now described the essential features of quantum systems we shall need in the sequel. A useful characterization of such a system arises if it possesses a "configuration space." We say that a transitive measurable J-space Y is a ***configuration space*** for the quantum system if there exists a Y-valued observable, i.e., a projection-valued measure $P(\bullet)$ based on Y with the following property.[2] If we denote the action of J on Y by $x \to g \cdot x$,

$$P(g \cdot F) = U_g P(F) U_g^{-1}, \quad g \in J, F \in \mathcal{Y}, \text{ or equivalently}$$

$$P(F) = P_V \Rightarrow P(gF) = P_{gV}. \tag{1.17}$$

The word "covariant" is also used here: we say that the Y-valued observable $P(\bullet)$ is ***covariant*** with respect to the unitary representation of J. We note that a configuration space is not part of the intrinsic structure of the quantum system and class of experimenters, in contrast to the type of covariance expressed in 1.15.

If this situation obtains for $Y = \mathbf{R}^3$, we say that the system is ***localizable***: the position in space is an observable. Relativistic particles are localizable if they have nonzero mass; photons, e.g., are not localizable.

Given the understanding of observables, states, and their dynamics as above, we may capture the kinematical aspects of a quantum system with a configuration space by means of the following definition, due to G. W. Mackey:

Definition 1.18. Let $(Y, \mathcal{Y})$ be a standard Borel G-space, G an lcsc group, acting measurably on Y. Let $\mathcal{H}$ be a separable Hilbert space. A ***system of imprimitivity for*** G ***acting in*** $\mathcal{H}$ ***and based on*** Y is a pair (U, P), where

(i) U is a unitary representation of G on $\mathcal{H}$;
(ii) P is a projection-valued measure on $\mathcal{Y}$ with values in $\mathcal{P}(\mathcal{H})$;
(iii) $P(g \cdot E) = U_g P(E) U_g^{-1}, \quad \forall\, g \in G$ and $E \in \mathcal{Y}$.

We abbreviate "system of imprimitivity" by SOI.

Example 1.19. ***Koopman system of imprimitivity.*** Let α be a positive, σ-finite measure on Y. Assume that α is ***quasi-G-invariant,*** i.e., the null sets of α form an invariant subset of $\mathcal{Y}$ for the action of G (equivalently, the measure class

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[2] We use the notation $P(\bullet)$ for the projection-valued measure, and $P_\bullet$ for the projections themselves. For example, for $F \in \mathcal{Y}$, $P(F) = P_V$ for a suitable closed subspace V of $\mathcal{H}$.

of α is G-invariant). Then it follows that the measures $\alpha(dx)$ and $\alpha^g(dx) = \alpha(d(g^{-1}x))$ are mutually absolutely continuous. Suppose

$$r_g(x) \text{ is a version of } \frac{\alpha(dx)}{\alpha^{g^{-1}}(dx)}. \tag{1.20}$$

Let $\mathcal{K}$ be a complex separable Hilbert space with inner product $\langle\!\langle, \rangle\!\rangle$. Let $\mathcal{H} = L^2(Y, \alpha; \mathcal{K})$, i.e., the Hilbert space of measurable functions $f: Y \to \mathcal{K}$ with finite norm, given the inner product

$$\langle f_1, f_2 \rangle = \int_Y \alpha(dx) \langle\!\langle f_1(x), f_2(x) \rangle\!\rangle. \tag{1.21}$$

For each $g \in G$, define U_g by

$$U_g f(x) = \sqrt{r_g(g^{-1}x)}\, f(g^{-1}x), \quad f \in \mathcal{H}. \tag{1.22}$$

Then $g \to U_g$ is a unitary representation of G in $\mathcal{H}$. Let the projection-valued measure P based on Y and taking values in $\mathcal{P}(\mathcal{H})$ be defined by

$$(P_E f)(x) = 1_E(x) f(x), \quad E \in \mathcal{Y}, f \in \mathcal{H}. \tag{1.23}$$

Then (U, P) is a system of imprimitivity for G acting in $\mathcal{H}$ and based on Y, called the Koopman system of imprimitivity. Systems of imprimitivity more general than the Koopman system may be constructed using the concept of "cocycles," discussed in the next section.

2. Cocycles and bundles

In this section we examine the one-to-one correspondence between systems of imprimitivity and certain "cohomology classes of cocycles." This correspondence leads to a classification of all SOI's based on a given space X and acting in a given Hilbert space $\mathcal{H}$; this is part of the theory of Mackey. We go on to discuss the one-to-one correspondence between cohomology classes and equivalence classes of "transitive G-bundles." This allows us to describe SOI's based on X in terms of unitary Hilbert-space bundles on X and to consider the way in which SOI's arise in the "linearization" of arbitrary G-bundles.

Aside from its intrinsic interest, our reason for presenting this theory is that it provides some support for a bridge between observer theory and physics. We have noted that the mathematical structure of a system of imprimitivity embodies the kinematical aspects of a quantum system with a configuration space and a physical symmetry group J (c.f. section one). We want to realize some general principles according to which this structure arises from observer theory. One approach is as follows. We consider a chain bundle symmetric framework as in **9**–5, with distinguished framework group J. Now a chain bundle is a principal bundle, not a unitary Hilbert space bundle, but it gives rise to a collection of Hilbert bundles by linearization; we discuss bundle linearization in this section. Among all linearizations of the given chain bundle there are certain ***canonical linearizations*** which contain information about the asymptotics of the participator-dynamical chains which appear in the chain bundle; we describe this in section three. We can then consider the systems of imprimitivity which are embodied in these canonical linearizations. We may view the quantum systems associated to these systems of imprimitivity as being "linearizations" of the specialized perception expressed in the original chain-bundle observer framework. We emphasize that while the primary meaning of the group J in physics is as the group of symmetries of the configuration space, in observer theory it is as the group of symmetries of the set of observer perspectives in the specialized framework. The role of physical configuration space itself is not primary in observer theory. As a matter of terminology note that the ***physical configuration space*** is not the same as the ***observer theoretic configuration space*** (e.g., the spaces E or X of the specialized framework). In fact, the physical configuration space, or at least the orbits of J in it, corresponds to the distinguished premise space S of the specialized framework.

In what follows we assume that G is an lcsc group. For such a group there exists a (nonzero, σ-finite) ***left-invariant,*** or ***Haar,*** measure λ on G:

$$\lambda(gA) = \lambda(A), \quad g \in G, A \in \mathcal{G}.$$

Denote the measure class of λ (cf. **2**–1) by $\mathcal{C}_G$. Suppose G acts measurably on a measurable space $(X, \mathcal{X})$. Let $\mathcal{C}$ be a measure class on $(X, \mathcal{X})$.

Definition 2.1.

(a) If M is a measurable group, a (G, X, M)***-cocycle related to*** $\mathcal{C}$ is a measurable function $\varphi: G \times X \to M$, such that

  (i) $\varphi(e, x) = 1$ for almost all $x \in X$ (e is the identity of G and 1 is the identity of M);

(ii)

$$\varphi(g_1g_2, x) = \varphi(g_1, g_2x)\varphi(g_2, x) \tag{2.2}$$

for almost all $(g_1, g_2, x) \in G \times G \times X$. (Here the null set is determined by $\mathcal{C}_G \times \mathcal{C}_G \times \mathcal{C}$.)

(b) Two (G, X, M)-cocycles φ and φ' are ***cohomologous*** if there exists a measurable function $b\colon X \to M$ such that, for almost all $(g, x) \in G \times X$,

$$\varphi'(g, x) = b(gx)\varphi(g, x)b(x)^{-1}. \tag{2.3}$$

This is an equivalence relation on the set of (G, X, M)-cocycles. Its equivalence classes are called ***cohomology*** classes. The collection of all (G, X, M) cohomology classes is denoted $\mathbf{H}^1(G, X, M, \mathcal{C})$, or simply $\mathbf{H}^1$ when there is no danger of confusion. Cocycles cohomologous to the trivial cocycle $\varphi(g, x) \equiv 1$ are called ***coboundaries.***

(c) If the cocycle φ satisfies (i) and (ii) of (a) for ***all*** values of the arguments, we call φ a ***strict cocycle.*** If φ and φ' are strict cocycles satisfying (b) for all (g, x), we call them ***strictly cohomologous.***[3]

As an example of a $(G, X, \mathbf{R}^+)$-cocycle ($\mathbf{R}^+$ is the multiplicative group of positive reals) we have

$$\varphi(g, x) = r_g(x)$$

where $r_g(x)$ is as in 1.19.

Definition 2.4.

(a) A SOI (U, P) for G acting in $\mathcal{H}$ is ***equivalent*** to an SOI (U', P') for G acting in $\mathcal{H}'$ if

(i) They are both based on the same space X;

[3] If there are invariant measure classes on G and X, we have the cocycles defined in 2.1, as well as strict cocycles. Mackey showed how the cohomology classes (with respect to the measure classes) are in one-to-one correspondence with strict cohomology classes. For details, see Varadarajan chapter five. In what follows, we are not careful to distinguish between strict cocycles and cocycles related to measure classes.

(ii) There exists a unitary isomorphism $W\colon\mathcal{H} \to \mathcal{H}'$ such that for all $g \in G$ and $E \in \mathcal{X}$,

$$U'_g = WU_gW^{-1}$$

and

$$P'(E) = WP(E)W^{-1}.$$

(b) A projection-valued measure P based on X and with values in $\mathcal{H}$ is ***homogeneous*** if it is unitarily equivalent to the projection-valued measure $\bar{P}$ based on X acting in $L^2(X, \mathcal{K}; \alpha)$, ($\mathcal{K}$ is a separable Hilbert space, α is a σ-finite measure on X) given by

$$\bar{P}(E)f = 1_E f, \quad f \in L^2(X, \mathcal{K}; \alpha).$$

If (U, P) is a SOI and P is homogeneous, we say that (U, P) is ***homogeneous.***

(c) For a SOI (U, P) the set

$$\{E \in \mathcal{X} \mid P(E) \text{ is the 0 operator}\}$$

is G-invariant and so defines a G-invariant measure class. We call this the ***measure class of*** P.

Suppose we have a homogeneous SOI (U, P). Then every SOI equivalent to it is also homogeneous. Let us suppose that $L^2(x, \mathcal{K}; \alpha)$ is as given in Definition 2.4(b) and denote by $\mathcal{U}$ the group of unitary transformation of $\mathcal{K}$. The following theorem is proved in Varadarajan, section 6.5.

Theorem 2.5. The SOI (U, P) for G, based on X and acting in $\mathcal{H}$ is homogeneous iff it is unitarily equivalent to an SOI $(\bar{U}, \bar{P})$ acting in some $L^2(X, \mathcal{K}; \alpha)$ where

$$\bar{P}(E)f(x) = 1_e(x)f(x) \quad \text{a.e. } x$$

and

$$\bar{U}_g f(x) = \sqrt{r_g(g^{-1}x)}\varphi(g, g^{-1}x)f(g^{-1}x) \quad \text{a.e. } x$$

for almost all g, every $f \in L^2(X, \mathcal{K}; \alpha)$ and where φ is a $(G, X, \mathcal{U})$-cocycle. This gives a one-to-one correspondence between, on the one hand, equivalence classes of homogeneous SOI's and, on the other hand, the set $\mathbf{H}^1$ of $(G, X, \mathcal{U})$-cohomology classes.

With this correspondence between homogeneous systems of imprimitivity and cohomology classes it is possible (using Hahn-Hellinger spectral multiplicity theory) to build up *any* SOI from inequivalent homogeneous ones. This is done by means of a direct sum construction; for details see Varadarajan, sections 6.4 and 6.5.

We turn now to a discussion of the relevance of cocycles to the structure of G-bundles. Recall the definition (given in **9**–5.3) of a *G-bundle* $(Z,\, p,\, W,\, G)$, where Z and W are sets, $p\colon Z \to W$ is a surjective function and G is a group acting on Z and W in such a way that p is a G-homomorphism, i.e., for $g \in G$ and $z \in Z$, $p(gz) = gp(z)$. (We write all actions as left actions, and assume that all sets, functions, groups, and actions are measurable.) Recall that $(Z,\, p,\, W,\, G)$ is a *transitive* G-bundle if Z is a transitive G-space.

Definition 2.6. A *G-bundle homomorphism* from the G-bundle $\mathcal{A} = (Z,\, p,\, W,\, G)$ into the G-bundle $\mathcal{A}' = (Z',\, p',\, W',\, G)$ is a measurable map $\Phi\colon Z \to Z'$ such that

(i) Φ preserves the G-actions: $\Phi(g(z)) = g\Phi(z)$, for $g \in G$ and $z \in Z$.

(ii) Φ respects fibres: $\Phi(p^{-1}\{w\})$ is contained in a single fibre of p'.

(We say that Φ is a *G-bundle isomorphism* if it is bijective and bimeasurable.)

Given such a Φ, there exists a well-defined function $\Psi\colon W \to W'$ such that $p' \circ \Phi = \Psi \circ p$. Also, $(\Phi(Z),\, p'|_{\Phi(Z)},\, \Psi(Z),\, G)$ is then a G-bundle. If $\mathcal{A}'$ is transitive then a G-bundle homomorphism from $\mathcal{A}$ to $\mathcal{A}'$ is surjective.

By means of cocycles, every transitive G-bundle may be viewed as a "twisting" of a trivial bundle (i.e., one whose total space Z is a product space $W \times F$, with $p =$ projection onto the first coordinate). To understand how this is so, we shall need some terminology. Let us suppose that (Z, p, W, G) is a transitive G-bundle. The fibre over $w \in W$ is called Z_w, and the stability subgroup of G for $w \in W$ is G_w. Then Z_w is a transitive G_w-space for each w and the fibres Z_w are mutually isomorphic. Fix $w_0 \in W$ and let $G_0 = G_{w_0}$. We expect that our bundle $(Z,\, p,\, W,\, G)$ is isomorphic to $(W \times Z_0,\, \mathrm{pr}_1,\, W,\, G)$ for a suitable action defined on the latter. The pursuit of this aim leads us to the association of $(G,\, W,\, G_0)$-cohomology classes to transitive G-bundles.

Definition 2.7. Let X, Y be measurable spaces and $f\colon X \to Y$ a measurable function. A measurable function $g\colon Y \to X$ is a *section of* f if $f \circ g = \mathrm{id}_Y$.

Because W is transitive, G/G_0 is in a one-to-one correspondence with W: $w \in W$ iff the set of elements of G transporting w_0 to w is a left coset gG_0. Define $\pi'\colon G \to W$ in terms of the canonical mapping $\pi\colon G \to G/G_0$ by

$$G \ni g \to \pi'(g) = \pi(g)w_0 \in W. \tag{2.8}$$

Then a section

$$\sigma\colon W \to G \tag{2.9}$$

of π' exists if G is lcsc and G_0 is a closed subgroup (Varadarajan, Theorem 5.1).

Lemma 2.10. Let W, G, w_0, and G_0 be as above. Let σ be a section as in 2.9. The function φ_σ, where

$$\varphi_\sigma(j,w) = \sigma(jw)^{-1} j\sigma(w) \tag{2.11}$$

is a (G,W,G_0)-cocycle. Moreover,

$$\sigma \to \varphi_\sigma$$

is a one-to-one correspondence between the set of sections and a (G, W, G_0)-cohomology class (the latter being determined solely by the action of G on W).

Proof. The set of group elements taking w_0 to jw is precisely the coset $\sigma(jw)G_0$. But $j\sigma(w)$ takes w_0 to jw. Hence $j\sigma(w) = \sigma(jw)g_0$ for some $g_0 \in G$. Thus $\varphi_\sigma\colon G \times W \to G_0$. The measurability of φ_σ is clear and it is immediate that $\varphi_\sigma(e,w) = e$ and $\varphi_\sigma(j_1 j_2, w) = \varphi(j_1, j_2 w)\varphi(j_2, w)$.

Now if σ' and σ are two sections, they define a measurable function $\alpha\colon W \to G_0$ by

$$\alpha(w) = \sigma'(w)^{-1}\sigma(w). \tag{2.12}$$

2.11 then gives

$$\varphi_{\sigma'}(j,w) = \alpha(jw)\varphi_\sigma(j,w)\alpha(w)^{-1},$$

so that φ_σ and $\varphi_{\sigma'}$ are cohomologous. Conversely, if $\varphi \approx \varphi_\sigma$, we have

$$\varphi(j,w) = \beta(jw)\varphi_\sigma(j,w)\beta(w)^{-1}$$

for some measurable $\beta\colon W \to G_0$. Then $\varphi = \varphi_{\sigma'}$, where $\sigma' = \beta\sigma$. ∎

Definition 2.13. Let the group G act transitively on W. Let $w_0 \in W$ and let G_0 be the stabilizer of w_0. Let Z_0 be a space on which G_0 acts transitively. Let φ be a (G, W, G_0)-cocycle. Then the G-bundle $\mathbf{B}^{\varphi}$ is defined to be $(W \times Z_0, \mathrm{pr}_1, W, G)$, with group actions given by

$$(g, w) \to gw \in W \text{ as before}$$

$$(g, (w, b_0)) \to (gw, \varphi(g, w) \cdot b_0) \in W \times Z_0. \tag{2.14}$$

The reader may check that 2.14 indeed defines an action, and that $\mathbf{B}^{\varphi}$ is G-bundle isomorphic to $\mathbf{B}^{\varphi'}$ iff φ and φ' are cohomologous.

Theorem 2.15. Let $\mathcal{A} = (Z, p, W, G)$ be a transitive G-bundle. Let $w_0 \in W$ and G_0 be the stabilizer of w_0. Then there exists a unique $(G,\ W,\ G_0)$-cohomology class $\xi_{\mathcal{A}}$ such that $\mathcal{A}$ is bundle isomorphic to any $\mathbf{B}^{\varphi}$ (as in Definition 2.13), $\varphi \in \xi_{\mathcal{A}}$.

Proof. Let σ be any section of G/G_0 and let φ_σ be the $(G,\ W,\ G_0)$-cocycle defined in 2.11. We may lift φ_σ to a $(G,\ Z,\ G_0)$-cocycle by the projection p: define $\varphi^*(g, \cdot)$ to be $p^*\varphi(g, \cdot)$, i.e.,

$$\varphi^*(g, z) = \varphi(g, p(z)). \tag{2.16}$$

$\sigma^*(z)^{-1}$ transports z to the fibre B_0, φ^*_σ moves the resulting point within that fibre, and $\sigma^*(gz)$ transports to the fibre B_{gz}.

We now define a map $\Phi\colon Z \to W \times Z_0$ that effects an isomorphism of $\mathcal{A}$ with $\mathbf{B}^{\varphi}$. Let

$$\Phi(z) = (p(z), \sigma^*(z)^{-1} \cdot z), \tag{2.17}$$

It may be checked that the transitivity of the action of G on Z implies that Φ is an isomorphism.

Thus $\mathcal{A}$ is bundle-isomorphic to $\mathbf{B}^{\varphi'}$ for any φ' in the cohomology class $\xi_{\mathcal{A}}$ associated to $\mathcal{A}$ by Lemma 2.10; $\{\mathbf{B}^{\varphi}; \varphi \in \xi_{\mathcal{A}}\}$ is thus an isomorphism class among the “trivial” bundles of this form, as mentioned after Definition 2.13. ∎

Definition 2.18. Let $\mathbf{A} = (B,\ p,\ Y,\ G)$ be a transitive G-bundle. We denote the action of G on Y by $(g, y) \to gy$, and that of G on B by $(g, b) \to D(g)b$. Then $\mathbf{A}$ is called a *Hilbert bundle* (or a *unitary bundle*) if

(i) Each fibre $B_x = p^{-1}\{y\}$ is a separable Hilbert space, with inner product $\langle\cdot,\cdot\rangle_y$ and inner-product topology identical to that induced by B;

(ii) For each $g \in G$, $D(g)\colon B_y \to B_{gy}$ is a unitary isomorphism.

We can now discuss the linearizations of a given G-bundle. Suppose that $\mathcal{A}$ and $\mathbf{A}$ are G-bundles, with $\mathbf{A}$ Hilbert. Let $y_0 \in Y$ with stabilizer $G_0 < G$. Denote the fibre over y_0 in $\mathbf{A}$ by B_0, and the group of unitary transformations of B_0 by $\mathcal{U}$. As we have seen, to $\mathcal{A}$ is associated a (G, Y, G_0)-cohomology class $\xi_{\mathcal{A}}$, while to $\mathbf{A}$ is associated a $(G,Y,\mathcal{U})$-cohomology class $\zeta_{\mathbf{A}}$. If $\varphi \in \xi_{\mathcal{A}}$, then

$$g_0 \mapsto \varphi(g_0, y_0) \tag{2.19}$$

is a (measurable group-) homomorphism of G_0 into itself. Similarly, if $\Psi \in \zeta_{\mathbf{A}}$,

$$g_0 \mapsto \Psi(g_0, y_0) \tag{2.20}$$

is a unitary representation of G_0 in $\mathcal{U}$. Conversely, it was shown by Mackey[4] that every homomorphism class $G_0 \to M$ (M a group) corresponds to a unique (G_0, Y, M)-cohomology class. It is reasonable to call $\mathbf{A}$ a linearization of $\mathcal{A}$ only if the homomorphism 2.20 arises from 2.19 in a specified manner. Namely, there is a third homomorphism m from G_0 to $\mathcal{U}$, such that $\Psi(g_0, y_0) = m(\varphi(g_0, y_0))$. Recalling Definition 2.1, suppose that M, M' are measurable groups. Then every homomorphism $m\colon M \to M'$ induces a map from $\mathbf{H}^1(G,Y,M)$ to $\mathbf{H}^1(G,Y,M')$.

These considerations motivate the following definition.

Definition 2.21.

(i) Let ξ be a (G,Y,M)-cohomology class and ζ a $(G,Y,\mathcal{U})$-cohomology class for some group $\mathcal{U}$ of unitary operators on a Hilbert space. We say that ζ is a ***linearization*** of ξ if there is a Borel homomorphism $m\colon M \to \mathcal{U}$ such that ζ is the cohomology class of the $(G,Y,\mathcal{U})$-cocycle $m(\varphi(\cdot,\cdot))$, for each $\varphi \in \xi$.

(ii) If $\mathcal{A}$, $\mathbf{A}$ are transitive G-bundles over the same base space Y, we say that $\mathbf{A}$ is a ***linearization*** of $\mathcal{A}$ if the associated cohomology class $\zeta_{\mathbf{A}}$ associated to $\mathbf{A}$ (by Theorem 2.15) linearizes the cohomology class $\xi_{\mathcal{A}}$ (associated to $\mathcal{A}$).

[4] See Varadarajan, Theorem 5.27.

It is straightforward to verify that $\zeta_{\mathbf{A}}$ is well-defined by the above procedure; in fact

$$\zeta_{\mathbf{A}} = \{(G, Y, \mathcal{U})\text{-cocycles } \Psi \mid \exists \varphi \in \xi_{\mathcal{A}} \text{ and } \exists k \in \mathcal{U} \text{ s.t. } k\Psi k^{-1} = m(\varphi)\}. \tag{2.22}$$

We stated above the correspondence between homogeneous SOI's and $(G, Y, \mathcal{U})$-cohomology classes ($\mathcal{U}$ a group of unitary operators on some Hilbert space B_0). We also saw that such cohomology classes are in correspondence with equivalence classes of transitive G-bundles with total space $Y \times B_0$, which we now recognize as Hilbert bundles. Thus to Hilbert bundles are associated SOI's and vice versa. To complete the circle of ideas we ask: given a homogeneous SOI, what relationship obtains between the Hilbert space it acts in and the Hilbert bundle to which it is associated? The answer is given in Theorem 2.30 below.

We assume, as usual, that G is a lcsc group with (left) Haar measure. The projection of this measure to Y is σ-finite and G-invariant; we denote it λ. If a measure α on Y is quasi-G-invariant, it is in the same measure class as λ, as long as Y is a homogeneous G-space.

Definition 2.24. Let f be a measurable section of the Hilbert bundle $\mathbf{A}$ (notation as in 2.18). Let α be a quasi-G-invariant measure on Y. Define

$$\|f\|^2 = \int_Y \langle f(y), f(y)\rangle_y \alpha(dy). \tag{2.25}$$

The *Hilbert space $\mathcal{H}_{\mathbf{A}}$ associated to* $\mathbf{A}$ (and α) is the collection of all α-equivalence classes of sections f with $\|f\| < \infty$ and with inner product

$$\langle f_1, f_2\rangle = \int_Y \langle f_1(y), f_2(y)\rangle_y \alpha(dy). \tag{2.26}$$

The measurability of the integrands in 2.25 and 2.26 follows from the existence of a measurable section σ of $\pi\colon G \to G/G_0$, where G_0 is the stabilizer of $y_0 \in Y$. We have

$$\langle f_1(y), f_2(y)\rangle_y = \langle D(\sigma(y))^{-1} f_1(y), D(\sigma(y))^{-1} f_2(y)\rangle_{y_0},$$

which is clearly a measurable complex-valued function on Y. It is straightforward to verify that

$$V_\sigma\colon \mathcal{H}_{\mathbf{A}} \to L^2(Y, B_0; \alpha) \text{ by}$$

$$(V_\sigma f)(y) = D(\sigma(y))^{-1} f(y), \quad y \in X, f \in \mathcal{H} \tag{2.27}$$

is a unitary isomorphism.

If the $(G, X, \mathcal{U})$-cocycle φ_σ is defined by

$$\varphi_\sigma(g, y) = D[\sigma(gy)^{-1} g\sigma(y)], \tag{2.28}$$

then there is a corresponding G-bundle isomorphism $\Phi\colon \mathbf{A} \to \mathbf{B}^{\varphi_\sigma}$ (as in Theorem 2.15, where the total space of $\mathbf{B}^{\varphi}$ is $Y \times B_0$). We have the diagram:

$$\begin{array}{ccc}
\mathbf{A} & \longrightarrow & \mathcal{H}_{\mathbf{A}} \\
\Big\downarrow{\scriptstyle \Phi_\sigma} & & \Big\downarrow{\scriptstyle V_\sigma} \\
\mathbf{B}^{\varphi_\sigma} & \longrightarrow & L^2(Y, B_0; \alpha) = \mathcal{H}_{\mathbf{B}^{\varphi_\sigma}}
\end{array}$$

DIAGRAM 2.29. *The horizontal arrows are associations of Hilbert spaces of sections to bundles; the vertical arrows are isomorphisms of the relevant structures.*

Theorem 2.30. Let $\mathbf{A} = (B, p, Y, G)$ be a Hilbert bundle. Let $\mathcal{H} = \mathcal{H}_{\mathbf{A}}$ be the Hilbert space (of square-integrable sections) associated to $\mathbf{A}$ and α. Let the projection-valued measure P in $\mathcal{H}$ and the unitary representation U of G on $\mathcal{H}$ be defined by

$$P(E)f(y) = 1_E(y)f(y) \quad \text{and} \tag{2.31i}$$

$$U_g f(y) = \sqrt{r_g(g^{-1}y)}\, f(g^{-1} \cdot y), \tag{2.31ii}$$

for $y \in Y$, $E \in \mathcal{Y}$, $g \in G$, and $f \in \mathcal{H}$; r_g is as in 1.19. Then (U, P) is a SOI acting in $\mathcal{H}$. Furthermore, if $\zeta_{\mathbf{A}}$ is the unique $(G,\ Y,\ \mathcal{U})$-cohomology class associated to $\mathbf{A}$, then for each $\Psi \in \zeta_{\mathbf{A}}$, the SOI (U, P) is equivalent to the SOI (U^Ψ, P^Ψ) acting in $L^2(Y, B_0, \alpha)$, where

$$P^\Psi(E)h(y) = 1_E(y)h(y), \tag{2.32i}$$

and

$$U_g^\Psi h(y) = \sqrt{r_g(g^{-1}y)}\, \Psi(g, g^{-1}y) h(g^{-1}y), \tag{2.32ii}$$

for $y \in Y$, $E \in \mathcal{Y}$, $g \in G$, and $h \in L^2(Y, B_0, \alpha)$.

Conversely, let $(\bar{U}, \bar{P})$ be a SOI based on Y and acting in $\bar{\mathcal{H}}$. Suppose $\bar{P}$ is homogeneous (as in Definition 1.26 (c)). Then there exists a Hilbert bundle $\mathbf{A}$ such that $(\bar{U}, \bar{P})$ is equivalent to the (U, P) of $\mathbf{A}$ given in 2.32.

Proof. That $\{U_g\}_{g\in G}$ is a unitary representation follows from Definition 2.18 and Equation 1.20. Moreover, it is straightforward to compute that

$$U_g P(E) U_g^{-1} = P(g \cdot E).$$

Hence 2.25 defines a SOI.

Now suppose that $V_\sigma : \mathcal{H} \to L^2(Y, B_0; \lambda)$ is as given in 2.22. The corresponding $(G, Y, \mathcal{U})$-cocycle is

$$\varphi_\sigma(g, y) = D[\sigma(gy)^{-1} \cdot g \cdot \sigma(y)]$$

(recalling 2.11). We claim that

$$\begin{aligned} &\text{(a)} \quad P^{\varphi_\sigma}(E) = V_\sigma P(E) V_\sigma^{-1}; \\ &\text{(b)} \qquad\quad U_g^{\varphi_\sigma} = V_\sigma U_g V_\sigma^{-1}. \end{aligned}$$

(a) holds since $P(E)$ is a (scalar) multiplication operator. As for (b), we have by 2.26 and 2.27 that

$$\begin{aligned} U_g^{\varphi_\sigma} h(y) &= \sqrt{r_g(g^{-1}y)} D(\sigma(y))^{-1} \cdot D(g) \cdot D(\sigma(g^{-1}y)) h(g^{-1}y) \\ &= \sqrt{r_g(g^{-1}y)} D(\sigma(y))^{-1} \cdot D(g) \cdot (V_\sigma^{-1} h)(g^{-1}y) \\ &= (V_\sigma U_g V_\sigma^{-1}) h(y). \end{aligned}$$

For a general $\Psi \in \zeta_{\mathbf{A}}$ we have $\Psi = k^{-1}\varphi_\sigma k$, where $k \in M$ and φ_σ is as in 2.27. Then $P^\Psi = k^{-1} P^{\varphi_\sigma} k = P^{\varphi_\sigma}$ and $U_g^\Psi = k^{-1} U_g^{\varphi_\sigma} k$. For the converse, note that by definition of homogeneity for $\bar{P}$, we may assume $(\bar{U}, \bar{P})$ is of the form 2.12. $\mathbf{A}$ may then be taken to be $\mathbf{B}^\Psi$, with action

$$D(g)(y, b_0) = (gy, \Psi(g, y) b_0). \quad \blacksquare$$

Thus vector bundle structures are in turn naturally associated to physical systems (of the sort we have been considering). On the other hand, as we show in section three, vector bundles arise in the "canonical linearization" of the chain bundles of chapter nine (structures associated to the asymptotics of participator dynamical chains). This is, in our opinion, the nexus of the two theories of observer mechanics and quantum mechanics, the conceptual point at which our observer-theoretic allusions to systems of experimenters may be concretely realized. We give more indications of this in section three.

We conclude this section with a few remarks about bundle linearization in terms of the "induced representation" theory of G. W. Mackey. Mackey's classification of the irreducible unitary representations of a lcsc group G may be summarized as follows:

2.33. Let Y be a standard Borel G-space on which G acts transitively. Let $y_0 \in Y$, and let G_0 be the stabilizer of y_0. Then the equivalence classes of irreducible unitary representations of G_0 are in one-one correspondence with the equivalence classes of irreducible systems of imprimitivity for (U, P) for G based on Y. Moreover, all representations U arise in this manner, up to equivalence.

In this theory, which has come to be called the "Mackey machine," G_0 is called the "little group," G the "big group." Thus 2.33 may be paraphrased by saying that all the unitary representations of the big group are associated with systems of imprimitivity (for that group), which are induced by unitary representations of the little group. Note that both the system of imprimitivity and the corresponding representation of G are said to be "induced" by the given representation of the little group.

One of the main technical components of this theory is a result about the description of (G, Y, M)-cocycles for an arbitrary lcsc group M, in terms of representations of G_0 in M. First note that if $\gamma\colon G \times Y \to M$ is a cocycle, then the restriction of γ to $G_0 \times \{y_0\}$, when viewed as a map $\tilde{\gamma}\colon G_0 \to M$ is in fact a group homomorphism. We can now state the result:

2.34. (c.f. Varadarajan, Theorem 5.27): With the assumptions of 2.33, the correspondence $\gamma \to \tilde{\gamma}$ induces a 1-1 correspondence between (G, Y, M)-cohomology classes and conjugacy equivalence classes of homomorphisms $G_0 \to M$.

One proves 2.33 by applying 2.34 in the case where $M = \mathcal{U}$ is group of unitary transformations of some Hilbert space $\mathcal{K}$, and using the representation of systems of imprimitivity by cocycles (Theorem 2.5).

Since we know that cocycles also classify bundles (2.15), the above theory can also be described in terms of bundles. The interpretation of bundle linearization in this context is given in the following result, which is obtained by applying 2.34 to the linearization definition 2.21.

Theorem 2.35. Let $\mathcal{A}$ be a transitive G-bundle with base Y, where Y is as above. Then $\mathcal{A}$ corresponds to a unique (G, Y, G_0)-cohomology class $\xi_{\mathcal{A}}$ as in Theorem 2.15. Consider the set $L_{[\mathcal{U}]}(\mathcal{A})$ of all linearizations of $\mathcal{A}$ such that the unitary group of their fibres over y_0 is isomorphic to $\mathcal{U}$. Then the distinct (G-bundle) equivalence classes in $L_{[\mathcal{U}]}(\mathcal{A})$ are indexed by the distinct equivalence classes of representations $\alpha\colon G_0 \to \mathcal{U}$ which factor through $\tilde{\gamma}$ for some $\gamma \in \xi_{\mathcal{A}}$. (This means that $\alpha = \alpha' \circ \tilde{\gamma}$ for some $\alpha'\colon G_0 \to \mathcal{U}$.)

Equivalently, we can then say that A is a ***linearization*** of $\mathcal{A}$ if the SOI on $\mathcal{H}_A$ associated by Theorem 2.30 is induced from a unitary representation of G_0 which factors through a $\tilde{\gamma}$.

3. Canonical Linearization

We have seen that quantum systems with configuration space Y and symmetry group J correspond to systems of imprimitivity, which in turn correspond to unitary Hilbert J-bundles with base Y. We propose that these "physical" bundles arise as linearizations of specialized chain bundles (c.f. **9**–5). This means that the phenomenology of the physical system is a linearized version of information about the asymptotics of a family of participator-dynamical chains on some "lower level" observer framework, which may itself have no evident physical interpretation. In fact, according to this viewpoint, the physics resides in the specialized perception of the asymptotics of these lower level dynamical systems, not in the systems themselves. We may take the proposal as representing a mathematical strategy for the embedding of certain aspects of physics in a more general hierarchical analytic context. Since examples have not been worked out in detail the ideas are speculative. Nevertheless we believe that the viewpoint is of sufficient interest to present at top level.

In particular, we may describe the essential mathematical idea as follows. Let be given a J-bundle (Z, p, W, J). Represent Z as a family of dynamical systems, say homogeneous Markov chains (on a fixed state space $\mathbf{E}$). Represent

W as a family of "asymptotic characteristics" (such as stationary measures) of those systems, in such a way that $p(z)$ is an asymptotic characteristic of z. Choose a complex number m, and construct a unitary Hilbert bundle B_m over W, which we might call the "m-linearization" of (Z, p, W, J), as follows. Each fibre $B_{m,w}$ of B_m is the subspace of functions on the state space **E**, generated by the eigenfunctions for eigenvalue m of the transition probability operators (i.e., the Markovian kernels) associated with the various Markov chains in the fibre Z_w of the original bundle. Thus we can think of the Hilbert bundle B_m as providing a canonical linearized picture of the "m-part" of the asymptotics of our J-family of dynamical systems. One thinks of the collection of all the linearizations B_m (as m varies) as giving a picture of the entire asymptotic structure of the dynamics. (Intuitively, the eigenvalue m corresponds to a characteristic frequency of the asymptotic behavior).

Notice that the family $\{B_m\}_m$ of linearizations is canonically associated to the bundle (Z, p, W, J) ***together with*** the particular representation of Z as a family of dynamical systems. In this section we consider this procedure for the case of the specialized chain bundles. In fact the chain bundle is, abstractly, the principal bundle $(Z, p, W, J) = (J, p, J/J_0, J)$, where J_0 is a subgroup of J, and $p: J \to J/J_0$ is the canonical map. To call it a "chain bundle" signifies precisely that we are representing Z as a particular family of participator-dynamical Markov chains, so that in principal we may consider the associated family of canonical linearizations.

Imagine that we are in the situation of the specialized chain bundle of **9**–5. Such a bundle, representing a specialized preobserver, arises from certain asymptotic regularities of an instantiation. Namely, a group J' acts on a class of stationary measures, as well as on a class of participator dynamical kernels, as in **9**–5.6. We now sketch the procedure which gives the canonical linearization, or "quantal description" of the chain bundle.

Let us first recall some definitions.

Notation 3.1. Let P_0 be a markovian kernel with state space **E** and let ν_0 be a stationary measure for P_0. Let J' be a group acting on **E**, with induced actions on kernels and measures as described in **9**–5.1. Let $\mathcal{A}$ be the J'-bundle (E_1', π_1', S', J') where

$$\begin{aligned} E_1' &= \{({}^{\gamma}P_0,\ {}^{\gamma}\nu_0) \mid \gamma \in J'\}, \\ S' &= \{{}^{\gamma}\nu_0 \mid \gamma \in J'\} \quad \text{and} \\ \pi_1'(P, \nu) &= \nu. \end{aligned}$$

$\mathcal{A}$ is a chain bundle if it arises from a participator dynamical system with sufficient regularities, as described in **9**–5. In that instance $\mathbf{E} = E^k$ for some natural number k, and E is the configuration space at the instantiated level.

Henceforth we assume that everything has all the topological and measurable properties assumed in section two of this chapter.

Now suppose μ is a quasi-J-invariant measure on $\mathbf{E}$. P_0 is then a selfadjoint operator on $L^2(\mathbf{E}, \mu)$. If P_0 has an eigenvalue m with eigenfunctions g,

$$P_0 g = mg, \tag{3.2}$$

then for any $\gamma \in \Gamma$, ${}^\gamma g$ is an m-eigenfunction of ${}^\gamma P_0$:

$${}^\gamma P_0 \; {}^\gamma g = m \; {}^\gamma g. \tag{3.3}$$

moreover, ${}^\gamma g$ lies in $L^2(\mathbf{E}, \mu)$ since μ is quasi-invariant.

We chose for μ the following measure. Suppose J' has a (left)-invariant Haar measure λ. Let

$$\mu = \int_{J'} \lambda(d\gamma) \; {}^\gamma \nu_0 \tag{3.4}$$

(note that $K(\gamma, de) = \; {}^\gamma \nu_0(de)$ is a kernel on $J' \times \mathcal{E}$, so that this integral is well-defined).

Definition 3.5. For each unimodular eigenvalue m of P_0, with μ as in 3.4 and ν_0 a stationary measure for P_0, let $\mathbf{B}_m = \mathbf{B}_m(\mu, \nu_0, P_0)$ be the J'-bundle with

$$\begin{aligned} \text{base space } &= S' \\ \text{total space } &= \{[\; {}^\gamma f; P_0 f = mf, f \in L^2(\mathbf{E}, \mu)] \times \{\; {}^\gamma \nu_0\}; \gamma \in J'\} \\ \text{projection } \quad &\pi\colon (\; {}^\gamma f, \; {}^\gamma \nu_0) \to \; {}^\gamma \nu_0. \end{aligned}$$

Here $[\cdot]$ means closed linear span in $L^2(\mathbf{E}, \mu)$. $\mathbf{B}_m$ is called the *canonical m-linearization* of the chain bundle $\mathcal{A}$ of 3.1.

The fibres of $\mathbf{B}_m$ may be described simply. For $\nu \in S'$, let

$$J^{(\nu)} = \{\gamma \in J' : \; {}^\gamma \nu_0 = \nu\}. \tag{3.6}$$

Then the fibre $B_{m,\nu}$ over $\nu \in S'$ is

$$\begin{aligned} B_{m,\nu} &= [g \in L^2(\mathbf{E}, \mu) : \ {}^{\gamma}P_0 g = mg \text{ for some } \gamma \in J^{(\nu)}] \\ &= \{ \ {}^{\gamma}f : \ f \in L^2(\mathbf{E}, \mu), P_0 f = mf \text{ and } \gamma \in J^{(\nu)} \}. \end{aligned} \tag{3.7}$$

To justify the designation "canonical linearization" for $\mathbf{B}_m$, note that if f_0 is any m-eigenfunction of P_0, a mapping $\Phi\colon \mathcal{A} \to \mathbf{B}_m$ can be defined by

$$\Phi(\ {}^{\gamma}P_0, \ {}^{\gamma}\nu_0) = (\ {}^{\gamma}f_0, \ {}^{\gamma}\nu_0),$$

and that this is a J'-bundle homomorphism.

The unimodular eigenvalues of P_0 play a fundamental role in the asymptotics of the Markov chain with T.P. P_0, in the instance where P_0 is a so-called *quasi-compact* operator. Specifically, to each such eigenvalue m is associated an asymptotic behavior of the dynamics (m is a root of unity). For details see Revuz, chapter six. The part of the spectrum of P_0 lying inside the open unit disk does not survive asymptotically: repeated iterations of P_0 send that part to zero. Hence our interest in the unimodular spectrum. We remark here that, for our present purposes, it is not important whether the spectrum of P_0 is pure point or not. A canonical "C-linearization" can be described analogously, where C is any measurable subset of the unit circle which intersects the spectrum of P_0.