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VOTING IN ONE'S HEAD AS A SOURCE OF NEARLY TRANSITIVE INDIVIDUAL PREFERENCES OVER MULTI-DIMENSIONAL ISSUES

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Abstract

We consider choices over alternatives in a multidimensional issue space. Individuals may make choices in a way that is analogous to majority rule processes in a group, i.e. individuals may act as if they had several different perspectives, each represented by a particular ideal point. Individuals then may express preferences that are derived from an "internal" majority rule process among individual's ideal points. This process of "internal" majority voting can give rise to intransitivities in much the same way as for majority rule in group decision making. However, the nature of these alternatives is constrained because of the geometry of the multidimensional issue space, so as to give rise to what we may characterize as a "fuzzily" transitive ordering. In that ordering, one alternative may be preferred by the individual to another if the former is closer to a point in the space that we call the center of the voter shell than is the other alternative.

1. Introduction

In the usual models of individual choice over alternatives in a multi-dimensional issue space, a voter is characterized by a quasi-concave utility function centered around the voter's ideal point ("bliss point") or, for nonsatiated preferences, by a family of convex indifference curves [1,8,9]. In either case, each individual voter's preferences over the alternatives in the space are necessarily transitive. In the simplest instance, where each voter is characterized by circular indifference curves, in any choice between *two* alternatives the alternative which is closest to the voter's ideal point is preferred.

In actuality, we might not expect voters to have fully transitive preferences, but we would expect that transitive preferences would predominate and that there would be clear patterns to the observed intransitivities.

We suggest that individuals may make choices in a way that is analogous to majority rule processes in a group. Individuals may act as if they had several different perspectives, each represented by a particular ideal point. Individuals then may express preferences that are derived from an "internal" majority rule process among the individual's ideal points. This process of "internal" majority voting can give rise to intransitivities in much the same way as for majority rule in group decision making [5]. We show, however, that we can characterize the nature of the intransitivities arising from an internal majority rule process in terms of a new construct we call "orthogonal preferences".

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Orthogonal preferences give rise to a "fuzzy" partial order among alternatives which generate a form of near-transitivity (cf. ref. [7]). In particular, in an individual's choice between any pair of alternatives *i* and *j*, if *j* is "sufficiently" further than is *i* from an analogue to the bliss point, the center of the voter's "shell", then the voter must prefer *i* to *j*. If *i*, *j*, and *k* are "sufficiently distinct" from one another, where distinctiveness is defined in terms of distance to the center of the voter's shell, then transitivity is guaranteed. The concept of orthogonal preferences is adapted from the majority rule context. The orthogonal preference shell is analogous to McKelvey's [6] concept of the "yolk".*

2. Orthogonal preferences

DEFINITION 1

An individual's preferences are *single-peaked* on a line if the individual can be treated as if he had an ideal point on the line, and prefers alternatives on that line less as they are further from that ideal point in either direction.

DEFINITION 2

An individual's preferences are *single-peaked* over a two-dimensional space if, over the alternatives on every line in the space, an individual has single-peaked preferences.

For simplicity of exposition, we shall confine our discussion to two-dimensional examples.

DEFINITION 3

An individual with single-peaked preferences over a space is said to have *orthogonal preferences* if, for every set of parallel lines, the individual's most preferred point on each of the lines falls on a line orthogonal to them.

DEFINITION 4

When an individual has orthogonal preferences, the orthogonal line passing through the individual's most preferred point on each line in a set of parallel lines we shall call an individual's *preference* line.

It should be apparent that circular indifference curves around a "bliss" point satisfy the orthogonal preference condition. However, orthogonal preferences is a much weaker condition than circular (or even convex) preferences. Indeed, a voter may have

*The radius of the yolk r is a measure of how close a spatial voting game is to a core situation [6,2–4]. If $r = 0$, then there is a core and majority rule is transitive. For $r > 0$, and two alternatives x and y, if $d_x + 2r < d_y$, then xPy; where d_q is the distance between a point q and the center of the yolk.

orthogonal preferences without a bliss point. Nonetheless, we can create a construct for each individual that serves as a "center" for the individual's preferences.

DEFINITION 5

For an individual with orthogonal preferences, the smallest circle that touches all preference lines we shall call the individual's *shell*. The center of that shell we call the individual's *center*.

For an individual with orthogonal preferences, there will always be a smallest circle that touches all of the individual preference lines. A useful way to represent the complete set of relevant preferences of an individual with orthogonal preferences is to use the center of that circle as an origin and indicate the individual's most preferred point on each line going through that center. Of course, if an individual has circular indifference curves, the shell will be a single point.

The radius of the individual's shell indicates how far from the voter's center an individual ideal point on a line through the center can be. We expect that the radius of an individual's shell will generally be relatively small; for instance, individuals will generally have something close to an ideal point at their center.

Consider an individual with several ideal points, i.e. preferred points on various lines.

THEOREM 1

If an individual chooses between pairs of alternatives by determining which alternative is closer to a majority of his ideal points, then the individual will exhibit orthogonal preferences.

Proof

For alternatives along any line, points closer to the median ideal point projection on that line are closer to a majority of ideal points than points further away, under the informal "majority choice among ideal points proximity" voters' rule. Thus, the individual's choice satisfies the definition of orthogonal preferences. □

Theorem 1 also holds if voters use an internal majority rule process based upon any "set" of ideal points.

3. Constrained individual intransitivities

THEOREM 2

With orthogonal preferences, if alternatives x and y are at distances d_x and d_y, respectively, from the center of an individual's shell, then, if the radius of the shell is r and $d_y > d_x + 2r$, we must have xPy.

Proof

The proof of theorem 2 is directly analogous to the proof of the equivalent result about majority preference and distance from the center of the yolk (see [6] or [2]). □

In order for there to be a cycle among any number of alternatives, if we neglect the essentially technical complications of ties, it is well known that there must be at least one cycle containing exactly three alternatives. Thus, to find necessary conditions for a cycle, we need merely find necessary conditions for a cycle of three. We show that, for any given individual, for any three alternatives to be in a cycle, the center of the circumscribing circle around the three alternatives must be very close to the center of the individual's shell. We can find the center of the circumscribed circle by finding the intersection of the three perpendicular bisectors of the sides of a triangle determined by the three alternatives. One *necessary* (but not sufficient) condition for a cycle is the following.

THEOREM 3

For there to be a cycle among any set of three alternatives, the individual's shell must intersect at least two of the three perpendicular bisectors of the triangle determined by the points (see fig. 1).

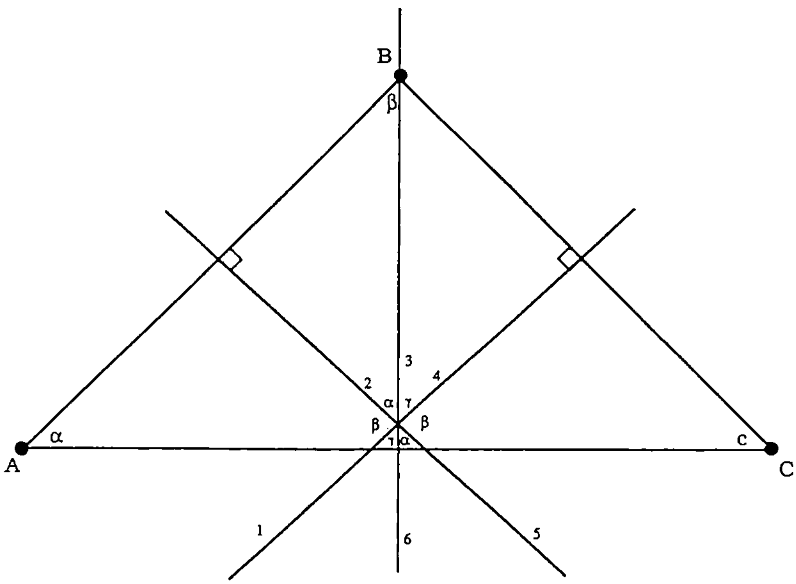

Fig. 1. Construction for theorem 3 and construction for three intersecting lines used in the proof of the corollary to theorem 3.

Proof

Figure 1 shows three alternatives A, B, and C, and the perpendicular bisectors of the line segments between each pair of them. These intersect at a point, defining six "spokes" (line segments radiating from that point), which we have labeled 1 through 6. If the voter's shell intersects only one bisector, this is equivalent to saying that it touches only one spoke. If the shell touches only one spoke, then the voter must have transitive preferences. We show the result for spokes 1 and 2; the remaining cases follow analogously.

If a shell intersects spoke 1, then it is on the A side of both the AC and the AB bisectors. Therefore, the individual with the shell prefers A to B and A to C, and thus has transitive preferences regardless of whether or not he prefers B to C. If an individual's shell intersects spoke 2, then it is on the A side of the AC bisector and the B side of the BC bisector, and thus the individual has transitive preferences regardless of whether A is preferred to B or B to A. The same ideas go through for the remaining spokes.

If a shell intersects no spokes (i.e. no bisectors), the preferences of the voter among A, B, and C are determined by the specific location of the shell. Since the shell does not intersect any of the spokes, it must be located in one of the six "wedges" between them, labeled as 12, 23, 34, 45, 56, and 61. It follows that the voter would have transitive preference ordering ABC, BAC, BCA, CBA, CAB, or ACB, respectively, depending upon which of the six wedges the shell is located in. For example, if the shell is contained in the wedge labeled 12, then the shell is on the A side of the AB bisector, the B side of the BC bisector, and the A side of the AC bisector, establishing the voter as having the transitive ordering ABC. □

COROLLARY TO THEOREM 3

For a set of three alternatives, let d be the distance from the center of the individual's shell to the center of the circle passing through the three alternatives. For there to be a cycle we must have $d < r/\sin(\alpha/2)$, where α is the smallest angle in the triangle defined by the three alternatives. (For example, if the triangle is an equilateral triangle, then $\alpha = 60$ degrees, $\sin(\alpha/2) = 1/2$, and $d < 2r$.)

Proof of corollary to theorem 3

Consider two intersecting lines. The furthest the shell can be from the point of intersection and still intersect both lines is when the shell is the circle tangent to the two lines, as shown in fig. 2. In this case, the bisector of the angle made by the intersecting lines passes through the center of the shell. It follows that $\sin(\alpha/2) = r/d$. Thus, the maximum distance is given by

$$d = r/\sin(\alpha/2).$$

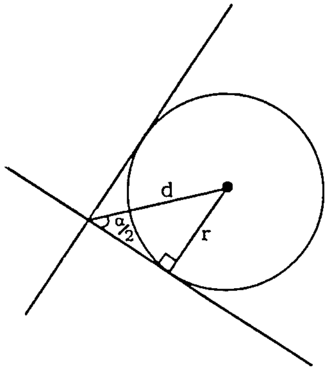

Fig. 2. Construction used to prove the corollary to theorem 3.

When there are three intersecting lines, they form a triangle, with angles as shown in fig. 1.

To find a circle (shell) which will touch any two lines, the distance will be $d = r/\sin(\alpha/2)$, where α is the smallest angle. □

THEOREM 4

A group composed of individuals each having orthogonal preferences, that makes decisions by majority rule, will have group preferences that are orthogonal.

Proof

Consider the points x and y on a line in the space. Each individual has an ideal point on that line. Suppose x is closer to the median ideal point than y; then a majority of voter ideal points are closer to x than to y. Since individuals have orthogonal preferences, that majority prefers x to y. For a set of parallel lines, the voters have a set of orthogonal preference lines. The median of these preference lines is a preference line for the group. □

COROLLARY TO THEOREM 4

If all individuals have Euclidean preferences, then the group majority has orthogonal preferences. For the group, the group shell corresponds to the yolk [6].

4. Discussion

The idea in this note is a very simple one. Even though intransitivities can be expected to occur, some types of intransitivities should be very rare or effectively impossible in spatial voting situations if voters choose among alternatives in a manner corresponding to majority voting. We have proposed to replace the idea of a unique bliss point with a circle based on induced ideal points which we refer to as a shell. The shell is related to the "yolk" in majority rule spatial voting games. It also provides a form of "fuzzy" ideal point [7]. The radius of the shell specifies the limits as to what intransitivities are feasible. In a like manner, if we use tennis rankings, for example, we would not expect to see $T_1 > T_{50}$, $T_{50} > T_{100}$ and then the intransitivity $T_{100} > T_1$.

If individuals can be characterized as having (strong) orthogonal preferences, this has important implications for majority rule decision making as well as for the nature of individual level intransitivities. In particular, if every individual has Euclidean preferences, then the group majority has orthogonal preferences. If every individual has orthogonal preferences, then the group majority has orthogonal preferences.

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