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A Comment on ‘Democratic Theory: A Preliminary Mathematical Model.’

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In a recent paper in this journal, Raphael Karmann1 has independently rediscovered a model of group decision-making in a dichotomous choice situation whose implications were first realized in the eighteenth century by the French mathematician and social theorist Nicholas Charles de Condorcet.2 Condorcet's work was, as far as I know, first brought to the attention of the English speaking world by the noted economist Duncan Black,3 and the result described by Karmann (which I shall label the Condorcet Jury Theorem), has been discussed by

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Black, and in publications by two political scientists as well as by myself in some unpublished work.

Consider a group of N members such that each member has some probability p of reaching a correct judgment in some dichotomous choice situation (e.g., with respect to the innocence or guilt of a defendant in a criminal trial). Let us initially assume that this probability is the same for all group members. Also, let us assume that each member arrives at his decision independently of the choice of the other members and that there exists a secret ballot mechanism for ascertaining these choices. Finally, let us assume that the group decides by simple majority vote.

Let m be a majority of the group, defined as $m = \frac{N}{2}$. The probability (for N odd) that a majority of the group will reach a correct judgment is simply (as Kasmann points out):\(^{11}\)

$$
\begin{align*}
1) \quad & \sum_{h=m}^{N} \binom{N}{h} p^h (1-p)^{N-h} \\
& \text{if } 0 \leq p \leq \frac{1}{2}, \\
& \text{if } \frac{1}{2} < p \leq 1.
\end{align*}
$$

Theorem 1: (Condorcet’s Jury Theorem): For N odd, $1 > p > \frac{1}{2}$ the larger the group, the more likely it is that a majority of the group will reach a correct judgment, while for N even, $0 < p < \frac{1}{2}$, the larger the group, the less likely it is that a majority of the group will reach a correct verdict; and for N odd, $p = \frac{1}{2}$, the likelihood of a majority of the group reaching a correct judgment is independent of N and is equal to $\frac{1}{2}$, i.e.,

$$
\binom{N}{m} p^m (1-p)^{N-m} = \frac{1}{2} \quad \text{for } N \text{ odd}.
$$

Moreover, as $N \to \infty$ the probability that the group’s judgment will be correct $\to \frac{1}{2}$ if $p > \frac{1}{2}$ and 0 if $p < \frac{1}{2}$.

A rather straightforward proof of this theorem has been found by Gilles Aschenmeyr.\(^{6}\)

9 Kasmann, op. cit., p. 22.


11 Kasmann does not answer the question of how the least competent members of the group are to be determined. One possibility is as follows: Consider a group of size N with mean competence $p$ binomially distributed. The group takes a vote and some external mechanism determines what is the “Correct” answer. Those who get it wrong are then dropped from the group. This process is then repeated if $p_{K}$ is the mean competence of the group after the Kth round of this weeding out then I believe (although I haven’t formally proved) that

$$
F_{K} \approx \frac{K+1}{K+2}
$$

Similarly, if $p_{K}$ is the size of the group after the Kth round of this weeding out process, then I believe that

$$
N_{K} \approx \frac{K+1}{K+2} N_{K-1} \frac{K+1}{K+2} \frac{K+2}{K+3} \frac{K+3}{K+4} \cdots \frac{(K-2)+1}{K-1} \geq \frac{(K-1)(K+1)}{K+2} > 1
$$

where $\tau = 1$.

$K = 0$

\(^{6}\) Black, op. cit., pp. 145-145.

\(^{12}\) Kasmann, op. cit., p. 20.

\(^{6}\) Aschenmeyr and Grofman, op. cit.
Theorem 2: Groups of size \( N + y \) and mean (binomially distributed) competence \( p - x \) are identical in expected group (majority) verdict to a group of size \( N \) and mean competence \( p \) if

\[
y = N \left[ \frac{.25x (2p - 1 - x)}{p(1-p)(p-x-.5)^2} \right]
\]

Proof: By the normal approximation to the binomial, we wish to find \( x, y \) such that

\[
\Phi \left( \frac{p - .5}{\sqrt{p(1-p)/N}} \right) = \Phi \left( \frac{p-x-.5}{\sqrt{(p-x)(1-p+x)/N+y}} \right)
\]

The desired result follows from some simple algebraic manipulation.

We show in Figure 1 incompetence curves for groups of various sizes for \( p = .55, .6, .7, .8, \) and \( .9 \).

These incompetence curves shed interesting light on the relative attractiveness (judgmental competence) of democracy and dictatorship (or oligarchy) as a function of the mean competence of the dictators (oligarchs) versus the mean competence of the larger (and presumably less competent, on the average) democratic mean. If the mean competence of the democratic electorate is \( > .5 \), then majority rule (for \( N \) large enough) may indeed be regarded as "divisely" improved, and to be preferred to the judgments of any dictator or any band of oligarchs who are not themselves infallible.

One final point: Kastor indeed rightly has called attention to the potential improvement in the accuracy of group verdicts which occurs when less competent members of a group are expelled. However, as Figure 1 makes clear, sometimes adding new members to a group who are less competent on average than the group's previous average member, may actually increase the group's probability of a correct judgment. This rather counterintuitive phenomenon occurs as long as the increase in group size compensates for the decrease in mean group competence. (See Equation (3)).
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