Public Choice

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A Note on Some Generalizations of the Paradox of Cyclical Majorities.

Bernard Grofman*

Let us define a Condorcet (j, i) choice as a candidate which could receive at least j votes (\(j^0\), n) in a paired contest against at least k+1 (k0, k-2) of the other k-1 alternatives. It is easy to see that for \(m^0\) (where \(m^0\) is a majority) and for \(m^0\), the Condorcet (j, i) choice is simply the familiar Condorcet choice. We may also readily verify that if a candidate is a Condorcet (j, i) choice for some given \(j_0\), \(k_0\), then it is also such a choice for all \(I > j_0\), \(k > k_0\).

It is well known that there need not exist a Condorcet choice, and thus that there need not exist a Condorcet (j, 0) choice for \(j > m\). Is there always a Condorcet (j, 0) choice, \(j < m\)? More generally, for what values of j and k can we guarantee the existence of a C(j, i) choice? Let us consider this question for strong preference orderings. We shall assume n committee members (n odd) and k alternatives.

**Theorem 1:** If \(nk(k-1)/2 > k(j-1)\), then there exists a 
\[C(k, i)\] choice.

*Proof:* There are \(nk(k-1)/2\) votes to be distributed when there are \(n\) voters and \(k\) alternatives. If every candidate receives fewer than \(j\) votes against at least one other candidate, then at minimum there are \(k\) matrix entries less than \(j\); and in particular these \(k\) matrix entries contain at most \(kj-1\) votes. That leaves at least \(nk(k-1)/2 - kj-1\) votes to be distributed among the remaining \(k(k-2)\) possibly mono-coal (all diagonal entries are zero). But there cannot be more than \(n\) votes in any given cell. Hence if the quotient shown above were to exceed \(n\) this would conflict with the absence of a C(j, 0) choice. Q.E.D.

If the above quotient does not exceed \(n\), then there may be no C(j, 0) choice. We may readily show that the antecedent conditions of the above theorem are met for \(k^1, 2\) for all \(n\), and not met for \(k^3\) for all \(n\). Hence, we obtain, very loosely speaking, a voting analogue to Arrow's "Possibility Theorem for Two Alternatives."

**Theorem 2:** If \(nk(k-1)/2 > k(j-1) + 1\), then there exists a 
\[C(k, i)\] choice.

*Proof:* The proof of this theorem is analogous to the proof given for Theorem 1. The antecedent conditions of this theorem are met if \(j^0 < m\) and \(k^0 - 2\). Hence, the

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remarably unsurprising result that there will always exist a candidate who can receive a simple majority or less against at least one other candidate. On the other hand, for $j \geq m$, even $j-m-2$ is not sufficient to guarantee the existence of a C(j, k-2) choice, i.e., a choice which can receive a special majority of $j$ against even one other alternative. Moreover, even for $j < m$, only for $j = 1$, i.e., there necessarily exist a C(j, 1) choice.

Let us now consider what happens when we restrict ourselves to single-peaked preferences.

Lemma 1: Any subset of a single-peaked set of preference schedules is itself single-peaked.\(^2\)

Lemma 2: Any set of single-peaked preference schedules contains a Condorcet choice.\(^3\)

Theorem 3: For any single-peaked set of preference schedules there exist at least $m+1$ candidates who can receive at least $j$ votes ($j \geq m$) against at least $k-m+1$ other alternatives (including other members of the $m+1$ member set), i.e., there exist at least $m+1$ distinct C($j$, $j-m$) choices.

Proof: By Lemma 2 above there will exist a Condorcet choice and we know such a choice will be unique. Hence, Theorem 3 obtained for that choice. Now consider the set of preference schedules obtained when the Condorcet choice is deleted. By Lemma 1 this set will still be single-peaked and hence by Lemma 2 there will be a new Condorcet choice for the set. But the new Condorcet choice will, by definition, be able to receive at least $m-1$ votes against at least $k$ others. Thus, for $j \geq 1$, there will exist $m+1$ candidates who can receive at least $j$ votes against at least $k-m+1$ other alternatives. Thus, for $j \geq 1$, there will exist $m+1$ candidates who can receive at least $j$ votes against at least $k-m+1$ other alternatives. Thus, the old and new Condorcet choices—and so on. Q.E.D.

\(^2\) The proof of this lemma is trivial.

\(^3\) For a proof see Duncan Black, *The Theory of Committees and Elections*, Cambridge University Press, 1930. Recall that we have calculated $n$, and for $n$ even, the Condorcet Choice is that candidate, if any, who can receive at least $n$ votes in paired contests against each of the other $n-1$ other alternatives, and to the extent two candidates satisfy this condition, is the candidate favored by the chairman.

\(^4\) Defined only for $m+1 \leq k$.
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