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THIRTEEN THEOREMS IN SEARCH OF THE TRUTH‡‡

ABSTRACT. We review recent work on the accuracy of group judgmental processes as a function of (a) the competences (judgmental accuracies) of individual group members, (b) the group decision procedure, and (c) group size. This work on individual competence and group accuracy represents an important contribution to democratic theory and a useful complement to the usual emphasis in the social choice literature on individual preference and preference aggregation mechanisms. The work reported on is rooted in a tradition which goes back to scholars such as Condorcet, Poisson, and Bayes.

> "Ye shall know the truth, and the truth shall make you free". *New Testament*, John 8: 13.
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> "The many, when taken individually, may be quite ordinary fellows, but when they meet together, they may well be found collectively better than the few". Aristotle, *Politics*, Book III.
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> "I do not believe in the collective wisdom of individual ignorance". Thomas Carlyle.

1. INTRODUCTION

The thirteen theorems in this paper derive from a tradition which goes back to Condorcet (1785), in which a group is confronted with a choice among a set of alternatives, and members of the group are assumed to each possess more or less reliable perceptions of which of these alternatives 'ought' to be chosen. Here, the force of the 'ought' comes from the underlying notion that there is a 'true' ordering of alternatives (for example, from best to worst in terms of some ideal standard or criterion, such as the public interest or justice or efficiency, etc.) and that the group decision should be judged by how likely it is to make the 'best' choice from among the set of alternatives available to it.

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The central question with which the literature which springs from this tradition has been concerned is "How likely are groups to reach correct judgments as a function of (a) the judgmental competence of the individual group members, (b) the decision rule/deliberation process which is used to aggregate individual choices into a group decision, and (c) the size of the group?"

Condorcet's ideas struck a responsive chord among pioneering statisticians such as Laplace and Poisson (see Gillispie, 1972; Gelfand and Solomon, 1973; and Baker, 1976 for historical details); but for over a hundred years after Poisson's work in the middle of the 19th century on the accuracy of majority verdict criminal juries (Poisson, 1837), concern for modelling the accuracy of group judgmental processes appeared dead. Condorcet's work in this area was forgotten until its rediscovery by Black (1958; see also Baker, 1967, 1976; Grainger, 1956; Grofman, 1975b) while Poisson's work wasn't rediscovered till even later (Gelfand and Solomon, 1973).

Our aim in this paper has been to review recent work on the accuracy of group judgmental processes. There has been a resurgence of interest in this question in the past decade, and important new results have been discovered. However, the findings are reported in widely scattered sources written by scholars in different disciplines, and a number are as yet unpublished; hence, the need for an overview. Because the results given in this paper are the product of many scholars operating singly and in a variety of permutations and because some of the theorems reported are not the work of any of the authors of this paper, we have specified for each result its original source. We hope that credit for results has not inadvertently been misallocated. In an important sense all of the work in this paper springs from the pioneering labors of Condorcet, Poisson, and Bayes. We should also note that while all of the theorems in this paper deal with dichotomous choice, analogous results for the polychotomous case can be derived. These will be reported in subsequent research. However, the limitation to dichotomous choice is much less important than it might first appear since many of the most important decision procedures for the multi-alternative case (e.g., standard amendment procedure) can be decomposed into sequences of pairwise choices. For reasons of space, proofs for the theorems are not included.

We view the research reported in this paper as primarily a contribution to two bodies of literature: (1) the literature on democratic theory, which has been concerned with the relative advantages of democratic vs. elitist forms of government, and (2) the literature on social choice, which, at least since Arrow's pioneering work, has been dominated by an emphasis on individual preferences and on preference aggregation mechanisms to which our emphasis on group accuracy and individual competence provides a useful complement. Moreover, because of the generality of many of the theorems we report, they may also be seen as a contribution to the statistical decision theory literature and the variables may be relabeled so as to make the results relevant to other areas (e.g., artificial intelligence, the theory of automata) as well.

In order to express our results, we shall specify a standard notation:

p_i = judgmental competence of the ith voter $(0 < p_i < 1)$ in a dichotomous choice situation, i.e., the probability that the voter will make the correct choice (i.e., the 'better' choice) of the two available to him;

N = number of voters in the group (for simplicity, N will generally be taken to be odd);

m = a majority $= (N + 1)/2$ for N odd;

$\bar{p}$ = average judgmental competence of voters in the group;

p = judgmental competence of a voter in a homogeneous group;

P_N = probability that at least a majority of voters will make the correct choice in a dichotomous choice situation, where N is the number of voters in the group;

w_i = weighted vote of the ith voter in a group using a weighted voting rule;

α = probability that any given voter will vote in accord with the choice of a specified 'opinion leader';

p_C = in a jury trial, the probability that a defendant will be convicted;

p_A = in a jury trial, the probability that a defendant will be acquitted;

p_H = in a jury trial, the probability that a jury will be unable to reach a verdict;

p_G = in a jury trial, the probability that the defendant is guilty of the offense charged;

$P_{k,N-k}$ = probability that a group choice is correct given k votes in its favor and $N-k$ votes against;

$L_{k,N-k}$ = ratio of the probability that a group choice is correct given k votes in its favor and $N-k$ votes against to the probability that the choice is incorrect, i.e.,

$$\frac{P_{k,N-k}}{1-P_{k,N-k}}.$$

2. MAJORITY RULE FOR THE CASE OF INDEPENDENT HOMOGENEOUS VOTERS (THEOREMS I–III)

Assumptions for Theorms I–III:

(1) Voters' choices are independent of one another.

(2) Voters are homogeneous, i.e., $p_i = p = \bar{p}$ for all i.

(3) The group decision rule is simple majority.

(4) There are exactly two alternatives, only one of which is correct (or equivalently, one of which is 'better' than the other).

(5) The prior odds as to which of the two alternatives is the correct (better) one are even.

THEOREM I (Condorcet Jury Theorem) (Condorcet, 1785; see also Moore and Shannon (1956a, 1956b); Nitzan and Paroush, 1980b). *If* $1 > p > \frac{1}{2}$, *then* P_N *is monotonically increasing in* N *and* $\lim_{N\to\infty} P_N \to 1$; *if* $0 < p < \frac{1}{2}$, *then* P_N *is monotonically decreasing in* N *and* $\lim_{N\to\infty} P_N \to 0$; *while if* $p = \frac{1}{2}$ *then* $P_N = \frac{1}{2}$ *for all* N. *Also*

$$(1) \qquad P_N = \sum_{h=m}^{N} \binom{N}{h} p^h (1-p)^{N-h}.$$

The rate of convergence to the asymptote is quite rapid. For example, if $p = 0.8$, then $P_{13} > 0.99$. One implication of this theorem is quite striking; if $p > \frac{1}{2}$, then 'vox populi, vox dei', i.e., the group judgmental accuracy under majority voting approaches infallibility as the group size grows larger.

COROLLARY 1 TO THEOREM I (Recursion Formula for Condorcet Jury Theorem) (Grofman, unpublished 1980):

$$(2)\qquad P_{N+2} = P_N + p^2 \binom{N}{\frac{N+1}{2}} p^{[(N-1)/2]}(1-p)^{[(N+1)/2]} - $$
$$-(1-p)^2 \binom{N}{\frac{N+1}{2}} p^{[(N+1)/2]}(1-p)^{[(N-1)/2]}.$$

COROLLARY 2 TO THEOREM I (Alternative Formula for Condorcet Jury Theorem) (Grofman, unpublished 1979; Feld, unpublished 1980):

$$(3)\qquad P_N = p + (2p-1) \sum_{h=3,\,5,\,\text{etc.}}^{N} \binom{h-2}{\frac{h-1}{2}} p^{[(h-1)/2]}(1-p)^{[(h-1)/2]}.$$

THEOREM II (Grofman Dummkopf–Witkopf Theorem) (Grofman, 1978). *For $p > 0.5$, a group of size $N + y$ whose members have competence $p - x$ is equivalent in judgmental competence to a group of size N whose members have judgmental competence p iff*

$$(4)\qquad y = N\left[\frac{0.25x(2p-1-x)}{p(1-p)(p-x-0.5)^2}\right].$$

This formula may be used to establish isocompetence curves which show the trade-offs between group size and individual accuracy needed to obtain a fixed level of group judgmental competence. (See Grofman, 1978.)

THEOREM III (Bigger is Better) (Owen, unpublished). *For $p > 0.5$, the larger the size of the majority in favor of an alternative, the more likely is that alternative to be the correct one. In particular*

$$(5)\qquad \log\left[\frac{P_{k,\,N-k}}{1-P_{k,\,N-k}}\right] = (2k-N)\log\left(\frac{p}{1-p}\right) = \log L_{k,\,N-k}.$$

This result need not obtain if competence is unequally distributed. (See Theorem VI.)

3. JURY DECISION MAKING (THEOREM IV)

Assumptions for Theorem IV:

(1), (2), (4)

(3)′ The group decision process is a Davis (1973) social decision scheme. If the first ballot verdict distribution is given, a social decision scheme is a matrix which provides a mapping from these predeliberation preferences to final verdict outcomes.

(5)′ The proportion of defendants who are guilty is given by p_G.

THEOREM IV (Two-Parameter Model of Jury Decision-Making) (Poisson, 1837; Gelfand and Solomon, 1973, 1974, 1975; Grofman, 1974, 1980a): *In a series of homogeneous jury trials if the jury social decision scheme is specified and p_C and p_A are known, then we can solve to obtain p and p_G.*

This theorem tells us that if we know the rule used by the group to reach its decisions and we know outcomes, we can infer both how competent are the members of the group and what proportion of defendants are in fact guilty (and also what proportion of verdicts are in fact, correct). In other words, by positing a social decision scheme, we can move directly from observables (e.g., p_C, p_A, p_H) to unobservables (e.g., p, p_G). This is a rather counterintuitive result.

In this model, the probability that an individual juror votes for conviction can be expressed as

(6) $$pp_G + (1-p)(1-p_G).$$

Similarly, the probability that a jury of size N will achieve exactly r votes for conviction on the first ballot can be expressed in terms of the binomial theorem in an expression involving N, r, p, and p_G.

COROLLARY 1 TO THEOREM IV (For Majority Rule Juries, 12 is Better than 6) (Gelfand and Solomon, 1973). *In a series of homogeneous jury trials, if $p > \frac{1}{2}$ and if the de facto or de jure jury social decision scheme is simple majority, then 12-member juries are superior to 6-member juries in terms of reducing both Type I and Type II errors.*

COROLLARY 2 TO THEOREM IV (12-Member Juries are Expected to be Better than 6-Member Juries) (Gelfand and Solomon, 1977). *In a series of homogeneous jury trials, if* $p > \frac{1}{2}$ *and if the de facto jury social decision schemes for* 6*-member and* 12*-member unanimous verdict requirement juries are specified in terms of social decision schemes which have been observed to have good fit to jury and/or mock-jury data, then* 12*-member juries are superior to* 6*-member juries in terms of reducing both Type I and Type II errors.*

COROLLARY 3 TO THEOREM IV (Majority Rule Verdicts are Expected to be Better than De Jure Unanimous Verdicts) (Gelfand and Solomon, 1977)[1]. *In a series of homogeneous jury trials, if the de facto jury social decision schemes for* 6*-member and* 12*-member unanimous verdict requirement juries are specified in terms of social decision schemes which have been observed to have good fit to jury and/or mock jury data (see Gelfand and Solomon,* 1977; *Grofman,* 1979, 1980b), *then majority rule verdicts for* 12- (6-) *member juries are superior to de jure unanimous verdicts for* 12- (6-) *member juries in terms of reducing both Type I and Type II errors.*

COROLLARY 4 TO THEOREM IV (For Symmetric Social Decision Schemes, Majority Rule Verdicts are Better than De Jure Unanimous Verdicts) (Klevorick and Rothschild, 1978 unpublished). *In a series of homogeneous jury trials, if the social decision scheme for unanimous verdict requirement juries is symmetric with respect to convictions and acquittals (and certain other reasonable assumptions are met), then majority rule verdicts are superior to de jure unanimous verdicts in terms of reducing both Type I and Type II error.*

Intuition might suggest that the fewer the votes needed for conviction, the more likely is conviction and hence that smaller juries would convict more defendants than larger juries and majority verdict juries would convict more defendants than juries requiring unanimous verdicts. Intuition turns out to be misguided and the actual likely verdict implications of changes in jury size/jury decision rule are rather difficult to pin down and turn out in general to be quite small in the aggregate (see Grofman, 1980a).

Intuition (and elementary statistics courses) might also insist that it is impossible to simultaneously reduce Type I and Type II errors – rather all we can do is to trade off one type of error reduction against the other. In this context, Corollaries 1 and 2 are quite startling in their assertion that 12 is better than 6 in terms *both* of reducing the likelihood that the guilty will be freed and in reducing the likelihood that the innocent will be convicted. What drives these results is the notion of juror competence – larger juries are simply less likely to make mistakes (of any kind). (See Theorem I.) Corollaries 3 and 4 are even more striking in their support for majority verdicts vs. the unanimous verdicts commonly held to be the safeguard of wrongly accused defendants. Space does not permit us a full discussion of the realism of the assumptions which produce these results, but one key assumption is that of symmetry, which requires that a minority (of a given size) which is in the right be no more likely to persuade the majority who hold the opposite view to change their minds than is a minority (of that same size) which is in the wrong.[2] Related theorems on the superiority of majority rule as a social decision rule are found in Taylor (1969), Rae (1969), Badger (1972), Schofield (1971, 1972), and Grofman (1974, 1980a).

4. MAJORITY RULE FOR THE CASE OF HETEROGENEOUS INDEPENDENT VOTERS (THEOREMS V–XI)

Assumptions for Theorems V–XI:

(1), (3), (4), (5)
(2)′ Voters are heterogeneous, i.e., $p_i \neq \bar{p}$ for all i.

THEOREM V (Feld and Grofman, unpublished; see Grofman, Owen, and Feld, 1981). *If the distribution of p_i is symmetric, then we obtain results analogous to the Condorcet Jury Theorem (Theorem I) with $\bar{p}$ substituting for p.*

COROLLARY 1 TO THEOREM V (Grofman, 1978). *If judgmental competence is normally distributed with mean $\bar{p}$ and variance given by $[\bar{p}(1-\bar{p})/N]$, then we obtain results essentially identical to the Condorcet Jury Theorem (Theorem I), with $\bar{p}$ substituting for p.*

These results simply generalize the Condorcet jury Theorem (Theorem I) for the case of heterogeneous voters and symmetric competence distribution.

THEOREM VI (Feld, unpublished; see Grofman, Owen, and Feld, 1981). *For heterogeneous groups, if $p_i > 0.5$ for all i, then the greater the size of the majority in favor of an alternative, the more likely is that alternative to be the correct choice.*

To see that the result need not hold if $p_i < 0.5$ for some i, consider the distribution (0.8, 0.8, 0). If exactly 2 voters are in agreement, the conditional probability that they are correct is 0.67. If all 3 voters are in agreement, they are correct with probability zero. Note that for this distribution, $\bar{p} > 0.5$.

One implication of Theorem VI is that in general (i.e., $p_i > 0.5$) we would expect that large majorities are more likely to be right than small ones. Hence, especially in smaller assemblies we might want a supramajoritarian decision rule. There is some empirical evidence, drawn from U.S. state legislatures, that there is an inverse correlation between legislative size and special majority requirements for legislative decision-making (Crain and Tollison, 1977).

THEOREM VII (Correcting a True–False Exam Without an Answer Key) (Feld, unpublished). *Let r_i be the proportion of time that an individual with competence p_i agrees with the majority verdict of a group of which he is a part. First,*

$$(7) \qquad (r_i - 0.5) \propto (p_i - 0.05)$$

and, if $0.55 < p_i < 0.76$, then

$$(8) \qquad p_i - 0.5 \propto \ln\left(\frac{p_i}{1 - p_i}\right).$$

Hence, we can approximate an individual's true score (p_i) on a true–false exam by scoring the percentage of his agreement with the majority choices.

This is a very important result because it implies that even if p_i values are unknown, we can estimate them by comparing an individual's choices with those made by the group majority!

THEOREM VIII (Stupidity Can Sometimes Be Offset by Numbers) (Grofman, 1975b; see also Margolis, 1976). *Under certain circumstances, lowering $\bar{p}$ but increasing the size of the group by adding new members whose competence is less than that of the existing average member can raise the group's judgmental competence.*

Again, this is a rather counterintuitive result. However, the new members must still have average competence greater than 0.5.

THEOREM IX (Optimal Distribution of Competences) (Owen, Grofman, and Feld, 1981; cf. Sattler, 1966). *If the sum total of competence is fixed (which sum we may arbitrarily denote as $\bar{p}N$), then P_N is maximized*

(a) *if $\bar{p}N > (N+1)/2$, by setting a majority of the p_i's to one.*
(b) *if $(N+1)/2 \geqslant \bar{p}N \geqslant (N/2) - 0.2$, by setting $p_i = 0$ for $(N-1)/2$ members of the group and $\bar{p}_j = p[2N/(N+1)]$ for the remaining $(N+1)/2$ members of the group.*
(c) *if $\bar{p}N \leqslant (N/2) - 0.4$, by setting $p_i = \bar{p}$ for all i.*

Similarly, P_N is minimized

(a) *if $N(1-\bar{p}) \geqslant (N+1)/2$, i.e., if $1 > p[2N/(N-1)]$, by setting a majority of the p's to zero.*
(b) *if $(N+1)/2 > N(1-\bar{p}) \geqslant (N/2) - 0.2$ by setting $p_i = 1$ for $(N-1)/2$ members of the group and $(1-p_j) = (1-\bar{p})[2N/(N-1)]$ for the remaining $(N+1)/2$ members of the group.*
(c) *if $N(1-\bar{p}) \leqslant (N/2) - 0.4$ by setting $p_i = \bar{p}$ for all i.*

For $\bar{p}$ fixed, the distributions which maximize/minimize group accuracy are rather strange ones – where all the p_i's take on values of 0, 1, $\bar{p}[2N/(N+1)]$ or $(1-\bar{p})2N/(N+1)$. The values of 0.2 and 0.4 are only approximate. Intermediate cases take on maxima for distributions which concentrate all competence among exactly K members, $(N+1)/2 \leqslant K \leqslant N$. For details (relevant only to small values of N) see the Appendix to Owen, Grofman, and Feld (1981).

COROLLARY 1 TO THEOREM IX (Feld and Grofman, unpublished; see Owen, Grofman, and Feld (1981). *A necessary condition for $P_N > \frac{1}{2}$ is that*

(9) $\quad (\bar{p}[2N/(N+1)]^{[(N+1)/2]} > \frac{1}{2}$.

A sufficient condition for $P_N > \frac{1}{2}$ is that

(10) $\quad [(1-\bar{p})(2N/(N+1))]^{[(N+1)/2]} < \frac{1}{2}$.

This corollary implies the quite counterintuitive result that a group can have $\bar{p} < \frac{1}{2}$ and yet have $p_N > \frac{1}{2}$. For example: (a) (0.72, 0.72, 0); $p = 0.48$, yet $P_N = 0.5184$. (b) (0.8, 0.8, 0.8, 0, 0); $\bar{p} = 0.48$, $P_N = 0.512$. (c) (0.8, 9.0, 0.7, 0, 0); $\bar{p} = 0.48$, $P_N = 0.504$. Similarly, a group can have $\bar{p} > \frac{1}{2}$ and yet have $P_N < \frac{1}{2}$. For example: (a) (1, 0.28, 0.28); $\bar{p} = 0.52$, yet $P_N = 0.4816$. (b) (1, 1.0, 0.2, 0.2, 0.2); $\bar{p} = 0.52$ yet $P_N = 0.488$.

COROLLARY 2 TO THEOREM IX (Grofman, unpublished; see Owen, Grofman, and Feld, 1981). *A necessary condition for $P_N > \frac{1}{2}$ is that*

(11) $\quad \bar{p} > \sqrt{\tfrac{2}{3}} = 0.471$.

We might note that we can have a value as low as $\sqrt{\tfrac{2}{3}}$ only when $N = 3$.

COROLLARY 3 TO THEOREM IX (Feld and Grofman, unpublished; see Owen, Grofman, and Feld, 1981). *A sufficient condition for $P_N > \frac{1}{2}$ is that*

$$\bar{p} > \frac{3-\sqrt{2}}{3} = 0.529.$$

In general, as N gets large, these conditions become more and more restrictive, so that for large N, for all practical purposes $P_N > \frac{1}{2}$ if $p > \frac{1}{2}$ and $P_N < \frac{1}{2}$, if $p < \frac{1}{2}$. A much stronger result, however, is available.

THEOREM X (Generalized Condorcet Jury Theorem) (Owen, Grofman, and Feld, unpublished 1981). *If $\bar{p} < 0.5$ then as $N \to \infty$, $\lim_{N\to\infty} P_N \to 0$; if $\bar{p} > 0.5$ then as $N \to \infty$, $\lim P_N \to 1$; while if $\bar{p} = 0.5$, $1 - e^{1/2} < \lim_{N\to\infty} P_N < e^{1/2}$, i.e., $0.39 < P_N < 0.61$.*

This result provides an extension to the Condorcet jury theorem applicable to any competence distribution no matter how skewed!

Group competence for the case $\bar{p} = 0.5$ is *quite* interesting. For example, for (0.75, 0.75, 0), $\bar{p} = 5$ while $P_N = 0.5625$; for $(\frac{5}{6}, \frac{5}{6}, \frac{5}{6}, 0, 0)$, $\bar{p} = 0.5$, yet

$P_N = 0.5787$; while for $(1, 0.25, 0.25)$, $\bar{p} = 0.5$ yet $P_N = 0.4375$. Of course, if $\bar{p} = 0.5$ and the p_i are symmetrically distributed, then $P_N = \frac{1}{2}$ (see THEOREM V).

THEOREM XI (Expected competence of the ith Best Member of the Group Relative to the Group Range) (Steiner and Rajaratnam, 1961). *If groups of size N are randomly assembled from a normally distributed population, the* ith *most competent members of the group will have a level of competence which corresponds to the* $[(100(N + 1 - i))/(N + 1)]$th *percentile score for the population.*

The expected competence of the ith most competent group member increases as a negatively accelerated function of group size. For example, if $i = 1$, then the most competent member of a 4-person group has expected competence at the 75th percentile, of a 5-person group at the 80th percentile, etc.

RESULT RELATED TO THEOREM XI (Majority Competence of the Group Compared to that of its Best Member) (Grofman, 1978). *If judgmental competence is normally distributed with mean $\bar{p}$ and variance given by $[\bar{p}(1 - \bar{p})]/N$, then it is more probable that the majority choice in small groups ($N \leqslant 35$) will be correct than the judgment of the most competent member of the group*

(a) *for* $\bar{p} < 0.55$ *for values of N as high as* 35.
(b) *for* $0.55 < \bar{p} < 0.59$ *for values of N up to* 21.
(c) *for* $0.59 < \bar{p} < 0.77$ *only for values of N as low as* 11.
(d) *for* $\bar{p} > 0.87$ *for values of N as high as* 35.

In other words, for low values of $\bar{p}$ and high values of $\bar{p}$, majority rule is preferred to rule by 'the best' for most small groups ($N < 35$), but for intermediate values of $\bar{p}$, only in relatively small groups is democracy preferred to rule by dictatorship of the most competent member. (Note: these results need some qualifications. See Grofman, 1978.)

5. GROUP CHOICES WHEN THERE IS AN OPINION LEADER (THEOREM XII)

Assumptions for Theorem XII:

(2), (3), (4), (5)

(1)′ Let α be the probability that a voter agrees with the choice of an opinion leader. Let $1-\alpha$ be the probability that a voter chooses independently of the preference of this opinion leader. It is assumed that there exists only one opinion leader and that α is the same for all voters. It is also assumed that the opinion leader also has competence p.

THEOREM XII (Think for Yourself, John) (Owen, unpublished 1980). *Consider a group of size N, whose members are of competence p if they cast an independent vote; but each of whom, with probability α, will vote in accordance with the views of one designated member of the group, the group leader or guru. Let $B = 1-\alpha$. When N is large and $\alpha \ll p$ (read α considerably less than p), the probability that group judgment will be correct drops from P_N to approximately*

$$(13) \qquad \frac{\alpha + BP_N}{1+\alpha}.$$

COROLLARY TO THEOREM XII (Owen, unpublished 1980). *If we observe a bloc of voters of size n, n large, casting identical votes, then the weight to be attached to these n votes, which would be $n \log(p/q)$ if each voter's decision was independently reached, should be reduced to approximately*

$$(14) \qquad \log\left(\frac{p}{q}\right) + (n-1)\log\left(\frac{\alpha + B_p}{\alpha + B_q}\right).$$

Note that the higher α the lower the judgmental competence of the group majority. In particular, it is easily seen that for $p > \frac{1}{2}$, $[(\alpha + B_p)/(\alpha + B_q)]$ is a decreasing function of α, approaching 1 as $\alpha \to 1$. Moreover, the effect of α can be dramatic if α is relatively large compared to p. For example, if $\alpha = 0.2$ and $p = 0.6$, then $E(P_N) = 0.6$ for all N; i.e., the group majority is only exactly as competent as the opinion leader (since the opinion leader's voting bloc can be expected to determine the election outcome) and the Condorcet Jury Theorem effect of raising the group competence toward 1 if $p > 0.5$ is lost entirely.[4]

6. THE BAYESIAN OPTIMAL GROUP DECISION RULE (THEOREM XIII)

Assumptions for Theorem XIII:

(1), (2)′, (4), (5)

(3)′ The group decision rule is a majority of the weighted votes, w_i, of its members.

THEOREM XIII (Corollary to Bayes Theorem: The Bayesian Optimal Group Decision Rule) (Shapley, 1979 unpublished; see Shapley and Grofman, 1981; and Nitzan and Paroush, 1980a, see also Pierce, 1961; Minsky and Papert, 1971; and Duda and Hart, 1973, which contain the theorem, but in a different context). *In a heterogeneous group the decision rule which maximizes P_N is given by assigning weights, w_i*

$$w_i \propto \log\left(\frac{p_i}{1-p_i}\right). \tag{15}$$

Note that, once we pick a logarithmic base, the weight assignment we give to an individual is a function purely of his competence and is *independent* of the competence of the other members of the group. This result is a quite counterintuitive one. In the light of a proof of this theorem which shows it to be, in effect, a restatement of Bayes Theorem,[3] this result turns out to be equivalent to the well known fact that the posterior Bayesian probability is independent of the order in which evidence is inputted.

Some examples of this theorem will be useful in showing its counterintuitive power. The first example is due to Grofman and provided the incentive to Shapley's derivation of the theorem. Consider a group with competences (0.9, 0.9, 0.6, 0.6, 0.6). If we let the most competent members of the group decide, $P_N = 0.9$; if we let the group decide by majority rule, $P_N = 0.87$; but if we let the group decide using the weighted voting rule $(\frac{1}{3}, \frac{1}{3}, \frac{1}{9}, \frac{1}{9}, \frac{1}{9})$, then $P_N = 0.93$. This weighted voting rule is equivalent to giving the most competent members of the group 1 vote each and letting the three least competent share a vote among themselves which is to be cast by a majority vote among the three of them.

If we look at a three-member group with competences (0.55, 0.60, 0.70), then we may show that the optimal rule is to assign weights (0. 0, 1); on the other hand, if the competence of the first member is adjusted upward so that we get a competence vector of (0.65, 0.60, 0.70), then the optimal voting rule is simple majority, i.e., improving the competence of one voter dramatically affects the power of all the voters in the group under the weighted voting rule which optimizes group competence.

COROLLARY TO THEOREM XIII (Feld, unpublished). *We can approximate the optimal weights prescribed by Theorem XIII by scoring the percentage of agreement with the majority choices and assigning* $w_i \propto (r_i - 0.5)$. *(See Theorem VII.)*

Theorem XIII is a quite remarkable result, which is a fitting capstone to the theorems we have enumerated here. It tells us that, for some given type of judgmental processes where the p_i can be assumed to be stable and independent, then each individual can be assigned a weight proportional to the log odds of his or her competence, and should be assigned that same weight in any group in which he/she may take part. Automatically, so to speak, the aggregate weights will adjust each individual's powers to affect the group (weighted) majority decision so as to maximize the likelihood that group decision will be the correct one! In some groups an individual's weight assignment may give him dictatorial power; in other words, he or she may be powerless to affect outcome (in the language of game theory, a *dummy*).

This theorem sheds important new light on the issue of democracy vs. rule by the select few. In particular, it appears to be the case (unpublished work in progress) especially as N is large, that optimal weights do not improve substantially on simple majority rule.

7. CONCLUSIONS

We hope to have demonstrated how work by a variety of scholars in several different disciplines offers new and often striking results on the nature of group decision making in situations involving pairwise choice. In particular, these new results help us in explicating the link between the accuracy of the summary group judgment and the judgmental competences of the group's individual members.

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[1] Gelfand and Solomon (1977) did not actually note this property of their results. It was independently observed and reported in Klevorick and Rothschild (1978 unpublished), Penrod and Hastie (1979), and Grofman (1979, 1980b). See also Grofman (1974, 1980b).
[2] For a full discussion, see Grofman (1980b).
[3] The Shapley proof of the theorem is somewhat different from that of other authors and does not directly derive from Bayes' Theorem.
[4] This example is due to Feld.

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