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Measures of Bias and Proportionality in Seats-Votes Relationships

Bernard Grofman

1. INTRODUCTION

There has been a good deal of recent interest in the functional relationship in a two-party, single-member district system between a party's aggregate vote share across all legislative districts (V) and the proportion of seats that it wins (S). While some early work simply regressed S on V (see, e.g., Dahl, 1956) and looked at the slope and intercept of the regression line, recent work has focused on nonlinear models of seats-votes relationships. Thøli (1969) and Taagepera (1973) have proposed a general functional relationship of the form

\[
\frac{S}{1 - S} = \left( \frac{V}{1 - V} \right)^{B_1}.
\]

(1)

Tuftt (1973) fitted a logarithmic transformation of this relationship to data from elections in Britain, New Zealand, and the United States, of the form

\[
\log \left( \frac{S}{1 - S} \right) = B_1 \log \left( \frac{V}{1 - V} \right) + B_0,
\]

(2)

where \(B_0\) is a stochastic error term.

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Linehan and Schrod (1978) have proposed an alternative specification of the relationship in Equation (1):

$$\frac{S}{1 - S} = \left( \frac{V}{1 - V} \right)^{B_1} + \epsilon,$$

(3)

where $\epsilon$ is again a stochastic error term hypothesized to have zero mean, but with a normal distribution rather than the lognormal distribution Tufte (1973) proposed.

At one time it was thought that three was the most likely value for $B_1$. This conjecture is known as the "cube law" of politics (Kendall and Stuart, 1950). With the exception of the estimates offered by Linehan and Schrod (1978) and Schrod (1981), recent work has found a wide variation in $B_1$, with only parliamentary elections in Great Britain closely approximating the magic number of three. Estimated values of $B_1$ (some from linear, some from nonlinear models) ranged from .7 (U.S. Congressional elections in the period 1966-1970 [Tufte, 1973]) to 4.4 (the U.S. Electoral College, 1928-1968 [Taagepera, 1973]). Most of the fitted values are, however, between 2 and 3.1,2

Our aim in this paper is limited to one aspect of the seats-votes relationship, specifying useful theoretical measures of the "bias" in seats-votes relationships for two-party, single-member district contests. We consider six definitions of "bias" offered in the literature, and propose a seventh and eighth definition of our own, inspired by the Gini index of inequality (see, e.g., Taagepera and Ray, 1977) and related to an index of maximum/minimum electoral bias (distortion) proposed by Grofman (1975). We also clarify the distinction between measures of "bias" and measures of "proportionality" in seats-votes relationships.

II. MEASURES OF PROPORTIONALITY

We propose two criteria that any measure of degree of proportionality of the seats-votes relationship ought to satisfy.

First Criterion of Measurement of Proportionality: If any set of election outcomes may be characterized by the function $S = V$, then any satisfactory measure of deviation from proportionality must assign a value of zero to that set. (Analogously, any satisfactory measure of degree of proportionality must assign a value of one to that set.)

Second Criterion of Measurement of Proportionality: If two sets of observations of seats-votes relationships are generated by the same functional relationship between seats and votes (including identical parameters of that function), then any satisfactory measure of deviation from proportionality (or degree of proportionality) must yield the same value for both sets of observations.

We henceforth denote these criteria as P1 and P2. In a single-member district system of elections, we would **never** expect to find complete proportionality between a party's vote share and its seat share. In general, we would expect that the graph of the seats-votes relationship will be an $S$-shaped curve such as is generated by the power function in Equation (1). (See Figure 1.) The parameter, $B_1$, which represents the slope of the seats-votes curve in the neighborhood of $V = .5$, has come to be known as the swing ratio (Tufte, 1973). It is an index of the proportionality of seats-votes relationships. Similarly, $|B_1 - 1|$ is an indicator of deviation from proportionality in seats-votes relationships. For the functional relationship shown in Equation (2), if $B_0 = 0$, only for $B_1 = 1$ will the percentage of seats won equal the percentage of votes received. Note that $|B_1 - 1|$ satisfies both criterion P1 and criterion P2 as a measure of deviation from proportionality and that $B_1$ satisfies P1 and P2 as a measure of degree of proportionality. Any measure that violates with $V$, (party 1's vote share at a particular election) will violate P2.

III. MEASURES OF BIAS

We follow the Niemi and Deegan (1978) definition of bias in a set of election outcomes. If the seat share $S$ earned by party 1 for a given vote share $V$ is the same as the seat share earned by party 1 for that identical vote share, then the election outcomes shall be said to be unbiased for that value of $V$. If an election system is unbiased for all values of $V$, we refer to it as completely unbiased. We propose the following criterion that any measure of bias ought to satisfy:

First Criterion of Measurement of Bias: If a set of election outcomes is unbiased for all elements of the set, then any satisfactory measure of bias must assign a value of zero to that set.

We propose a second criterion that any measure of bias in seats-votes relationships also ought to satisfy.

Second Criterion of Measurement of Bias: If two sets of observations of seats-votes relationships are generated by the same functional relationship between seats and votes (including identical parameters of that function), then any satisfactory measure of bias must yield the same value for both sets of observations.
We denote these criteria as B1 and B2, respectively. Any purported measure of bias that varies with \( V_+ \) will violate B2.

B2 and P2 are examples of what Rae (1981) has called "lot-regarding" criteria of equality, in which all identically situated actors (in this case political parties) must be identically treated. Our four criteria for measures of bias and measures of proportionality may seem to be either trivial or tautological. As we shall see, the most common measures of bias/proportionality that have been proposed do not satisfy them.

**Measure 1: Bias as the Simple Discrepancy Between Seats and Votes**

Consider the hypothetical graph of seat share as a function of vote share (Figure 1). Consider an election time + that generates some point \((V_+, S_+)\) on that graph (i.e., a hypothetical election outcome). A seemingly natural definition of the bias in that election outcome is the discrepancy between the observed \((V_+, S_+)\) outcome and that obtained if votes were transformed into seats in a perfectly proportional manner—i.e., the outcome \((V_+, S_+)\).

We may specify this discrepancy, which we shall label \( D_1 \), in terms of a difference measure:

\[
D_1 = V_+ - S_+.
\]  

(4)

This measure will be positive or negative depending on whether party I or party II is favored (i.e., receives a greater seat share than vote share). This measure of bias is the one most commonly used in the political geography literature (see, e.g., Johnston, 1979:58-60). It is also the most common measure of "fairness" of election outcomes in the political science literature on the representation of racial minorities in ward vs. at-large elections (see the literature review in Grofman, 1981).

Imagine that the seats-votes relationship for a given legislature is being generated by the cubic relationship pictured in Figure 1. It is apparent from Figure 2 that bias, as defined by the \( D_1 \) measure, depends on \( V_+ \). If we happen to observe an election (or series of elections) in which \( V_+ \) is close to .5, \( D_1 \) will be near zero. If we observe an election or elections in which \( V_+ \) is around .7 (.3), we will find a very large positive (negative) bias. Some authors have compared \( D_1 \) values in different elections or sets of elections to measure differences in bias at different points in time between two different polities, or
between two different types of election systems (see, e.g., Uslaner and Weber, 1979; Cole, 1974; MacManus, 1978; Taebel, 1978; Rabinowitz and Hamilton, 1980). However, unless the \( V_t \) values across the different sets of elections are nearly identically distributed, such comparisons are not really meaningful, since the identical underlying functional relationship between seats and votes can give rise to very different \( D_t \) values depending on the value(s) of \( V_t \) in the election(s) sampled.\(^4\) Unless \( B_1 = 1 \), if seats and votes are related as in Equation (1), the relationship between \( D_t \) and \( V_t \) will be nonmonotonic.

If we look at a hypothesized direct linear relationship between seat share and vote share

\[
S = B_1 V + \psi, \quad (5)
\]

we see that if \( V = 1 - V = .5 \), then \( S = 1 - S = .5 \) only if \( \psi = .5(1 - B_1) \). If \( \psi = 0 \), this requires \( B_1 = 1 \).\(^5\)

If the relationship between \( S \) and \( V \) is as specified in Equation (5), then

\[
D_t = V_t - S_t = V_t - B_1 V_t - \psi = V_t(1 - B_1) - \psi. \quad (6)
\]

Hence, estimated bias \( (D_t) \) will decrease (and at some point become negative) with increasing \( V_t \) if \( B_1 > 1 \), while it will linearly increase with increasing \( V_t \) if \( B_1 < 1 \).

Hence, if we posit a power relationship [as in Equation (11)], \( D_t \) can be expected to vary with \( V_t \) nonmonotonically; and even if we posit a linear seats-votes relationship [as in Equation (5)], then \( D_t \) becomes a linear function of \( V_t \). In neither case does \( D_t \) offer a desirable measure of bias. Moreover, even if \( B_0 = 0(\psi = 0) \) and there is perfect symmetry in the seats-vote transformation rule for each of the parties, \( D_t \) will still be nonzero. Hence, \( D_t \) fails to satisfy either criterion B2 or criterion P2 and thus is not suitable as a measure of either bias or proportionality.

Measure 2: Bias as the Ratio of Seats to Votes

Some authors (e.g., Robinson and Dye, 1978) have looked at

\[
D_t = \frac{S_t}{V_t}. \quad (7)
\]

as their measure of bias. \( D_t \) satisfies criterion B1 and criterion P1. If seats and votes are linearly related as in Equation (5), we have

\[
D_t = \frac{B_1 V_t + \psi}{V_t} = B_1 + \frac{\psi}{V_t}. \quad (8)
\]

Unless \( V_t \) is much larger than \( \psi \), \( D_t \) appears as a linear function of \( V_t \). Hence, in general, if we are comparing two different seats-votes graphs for bias, unless the two graphs have identical values of \( V_t \) with observations similarly distributed around that mean, \( D_t \) will be very misleading for measurement of bias. In particular, \( D_t \) fails to satisfy criterion B2. For analogous reasons, \( D_t \) is also not a good measure of proportionality. For \( V_t \) very large relative to \( \psi \), it is not too bad, since in this case \( D_t = B_1 \), but for values of \( V_t \geq \psi \), \( D_t \) varies with \( V_t \) and hence fails to satisfy criterion P2.

A very similar argument can be constructed to show that \( D_t \) is unsatisfactory as a measure of either bias or proportionality if the seats-votes relationship is as specified in Equation (2). In both cases \( D_t \) increases monotonically with \( V_t \).

Measure 3: Bias as a Function of Vote Share Needed to Gain a Fifty Percent Seat Share

A number of authors (in particular, Tuttle, 1973) have proposed to define electoral bias in two-party elections as the difference between .5 and the vote share a party needs to get a .5 fraction of the seats. Let \( V(.5) \) denote the vote share required to earn a 50 percent seat share. We may define our third measure of bias, \( D_2 \), as

\[
D_2 = V(.5) - \frac{1}{2}. \quad (9)
\]

If seats and votes are linearly related according to Equation (5), then at \( S = .5 \)

\[
.5 = B_1 V(.5) + \psi,
\]

and hence

\[
V(.5) = \frac{.5 - \psi}{B_1}. \quad (10)
\]
Thus, if seats and votes are linearly related as in Equation (5), then

\[ D_2 = \frac{1 - 2\psi - B_1}{2B_1}. \]  

(11)

For example, Dahl (1956) looked at U.S. Congressional elections (1928-1954) and U.S. Senate elections (1928-1952) and found best fitting regression lines of \( S = 2.5V - .70 \) and \( S = 3.02V - .95 \), respectively. Using Equation (9), we find that those give rise to bias measures \( D_2 \) of -.02 for both House and Senate elections (a negative value indicates advantage for the Democrats as Dahl defined his variables).

If the relationship between seats and votes is of the nonlinear form specified in Equation (2), for \( S = .5 \) we have

\[ B_1 \log_e \left( \frac{V_{.5}}{1 - V_{.5}} \right) = \log_e (1 - e). \]  

(12)

Analogously, taking logarithms on both sides of Equation (3), we obtain

\[ B_1 \log_e \left( \frac{V_{.5}}{1 - V_{.5}} \right) = \log_e (1 - e). \]  

(13)

While we could use (12) or (13) to obtain a value for \( \log_e V_{(.5)}/1 - V_{(.5)} \) and then solve for \( V_{(.5)} \), it is easy to use a well-known approximation to \( \log (1 + x) \) (see, e.g., Feller, 1957) to reexpress Equation (13) as

\[ B_1 \log_e \left( \frac{V_{.5}}{1 - V_{.5}} \right) = -e. \]  

(14)

Thus, Equation (2) and Equation (3) have essentially identical approximations. Henceforth, we shall use Equation (2) to estimate our logit model. After taking antilogarithms and performing some simple algebraic manipulations on Equation (12), we obtain a convenient expression for \( V_{(.5)} \),

\[ V_{(.5)} = \frac{1}{e^{B_0B_1} + 1}. \]  

(15)

Hence, where seats and votes are related as in Equation (2), we have

\[ D_2 = \frac{1}{e^{B_0B_1} + 1}. \]  

(16)

\( D_2 \) is in several ways an admirable measure of bias. \( D_2 \) satisfies both criterion B1 and criterion B2 and permits meaningful comparisons. Also, \( D_2 \) focuses attention on the crucial point in a two-party competition, the point at which control of the legislature changes hands. Moreover, the estimates of \( D_2 \) do not appear to be substantially affected by the choice of Equation (2) or Equation (5). Tufte (1973: 546, Table 2) fits the logit model [Equation (2)] to data from Great Britain \( (B_0 = -.02, B_1 = 2.88) \), New Zealand \( (B_0 = -.12, B_1 = 2.31) \), and the U.S. 1868-1970 \( (B_0 = .09, B_1 = 2.52) \). Using these values to estimate \( D_2 \) from Equation (15), we obtain values of .002, .013, and -.009 for Great Britain, New Zealand, and the U.S., respectively. Tufte (1973:543, Table 1) fitted a regression line to the same data. Using the linear model [Equation (5)], he obtained \( D_2 \) values of .002, .014, and -.009, respectively.

A slightly different way to approximate Equation (2) fits well for \( S \) and \( V \) values reasonably near .5 (say between .3 and .7), and fits quite well for \( S \) and \( V \) values between .45 and .55. This method can be used to derive a linearized logit-based estimate for \( D_2 \). We use a Taylor expansion around .5 (see Feller, 1957:49) to obtain

\[ \log_e \left( \frac{p}{1 - p} \right) \approx 4p - \frac{1}{2}. \]  

(17)

Hence, from Equation (2)

\[ 4S - \frac{1}{2} = 4B_1 \left( V - \frac{1}{2} \right) + B_0, \]  

(18)

which may be reexpressed as

\[ S = B_1V + \frac{1}{2} - \frac{B_1}{2} + \frac{B_0}{4}. \]  

(19)

This is not a bad approximation. Consider, for example, Tufte's (1973) linear estimate of the data on the British parliament. He found \( S = 2.83V - .921 \) to be the best fitting regression line. His logit estimates for the same data were \( B_1 = 2.88 \) and \( B_0 = 0.02 \). Substituting these values
into Equation (19), we obtain a linear regression estimate of \( S = 2.88 V - .935 \), which matches very closely the result obtained directly from a regression model, especially when we take into account the standard errors of the various parameter estimates. (Of course if \( B_0 = 0 \), Equation (19) reduces to \( S = 3V - 1 \) for \( B_1 = 3 \), and we have a linear version of the "cube law."\)

Substituting the value of \( \psi \) obtained from Equation (5) into Equation (19), we obtain for the linear approximation to the logit, the nice approximation for \( D_2 \)

\[
D_2 = \frac{B_0}{4B_1}.
\]  

(20)

This is a good approximation: we get values of +.002, +.013, and -.009 for the three cases previously considered—estimates of \( D_2 \) virtually identical to those obtained directly from the fitted regression line.\(^9\) Since \( B_1 \approx 2.5 \) for the three cases considered, a rough and ready approximation of \( D_2 \) for these data sets in Tufte (1973) is \( D_2 = B_0/10 \).

Measure 4: Bias as Seat Share Needed to Gain a Fifty Percent Vote Share

By looking at the .5 vote share rather than the .5 seat share, we can define a measure of bias directly analogous to that of \( D_2 \).\(^6\) Let \( S(.5) \) denote the seat share obtained when party 1 gets a 50 percent share of the votes. We define \( D_3 \) as the difference between .5 and the seat share obtained when party 1 gets a .5 fraction of the vote, i.e.,

\[
D_3 = S(.5) - \frac{1}{2}.
\]  

(21)

For the linear model of Equation (5) for \( V = 1/2 \) we have

\[
S(.5) = .5B_1 + \psi.
\]  

(22)

Hence

\[
D_3 = .5(B_1 - 1) + \psi.
\]  

(22)

For the logit model of Equation (2) for \( V = .5 \) we have

\[
\log_e \left( \frac{S(.5)}{1 - S(.5)} \right) = B_0.
\]  

(24)

Hence

\[
S(.5) = \frac{B_0}{1 + e^B_0}.
\]  

(25)

Thus, for the logit model

\[
D_3 = \frac{B_0}{1 + e^B_0} - \frac{1}{2}.
\]  

(26)

It is interesting to see that, for the logit model, \( B_1 \) does not enter into the specification of bias as measured by \( D_3 \). We may reanalyze Tufte's (1973) estimates of linear and logit models for Great Britain, New Zealand, and the U.S. to obtain estimates of \( D_3 \) for those three countries. Using the linear model, we obtain \( D_3 \) estimates of -.006, -.032, and -.022, respectively. Using Tufte's logit model estimates for the same data, we obtain \( D_3 \) estimates of -.005, -.030, and -.023. Using our linear approximation to the logit model (and Tufte's logit estimates), we obtain essentially identical values. Again, as with \( D_2 \), logit and linear estimates of bias (\( D_3 \)) in the three cases are virtually identical.\(^10\)

Clearly \( D_3 \) has much the same strength as \( D_2 \). It satisfies criterion \( C_1 \) and criterion \( C_2 \), permits direct comparisons of bias across different sets of elections, is straightforwardly defined, and focuses on a "natural" point on the seats-vote graph for two-party competition, \( V = .5 \), where both parties receive the same vote shares.

\( D_2 \) and \( D_3 \) are in fact closely related. For the linear model from Equation (10) we have

\[
-D_2 = \frac{.5(B_1 - 1) + \psi}{B_1}.
\]  

(27)
Substituting in Equation (10) we obtain

\[ D_3 = -B_1D_2. \]  

(28)

This relationship has been noted by Tutte (1973:543, n. 4). For the linear approximation of \( D_3 \) derived from a logit estimate, it follows that

\[ D_3 = \frac{B_0}{4} \cdot 11. \]  

(29)

The only real problem with \( D_2 \) and \( D_3 \) appears to be that each focuses exclusively on one point on the seats–votes graph, the point that corresponds to a seat (vote) share of .5. While it is clearly natural to focus on such points (especially the former), it may be that different measures of bias would be generated were we to look elsewhere on the graph. Of course, if we pick a particular estimating technique (say the logit model), then whether we measure bias at \( V(.5) \) (or at some other \( V \) value) might appear arbitrary, as long as we are always consistent in our choice. This is, however, too simplistic a view.

For \( D_2 \), both the logit and the linear model imply (at least in some range around \( V = .5 \)) a consistency in the direction of bias; i.e., if party \( I \) is advantaged (disadvantaged) when \( S = .5 \), it will also be advantaged (disadvantaged) when \( S = .5 \pm d \) (see Figure 1). However, it is, empirically, not true that a districting system (and distribution of partisan strength and differential turnout) that favors one party for certain values of \( V \) will necessarily favor that same party for all values of \( V \), even those close to \( V(.5) \). The same features of a districting system that are advantageous to a party at one level of overall vote strength (e.g., winning a number of districts by bare majorities or having a larger number of "safe" seats than its opponent) may become disadvantageous (relative to the proportionality norm) if its vote strength changes. Exactly analogous remarks apply if we use \( D_3 \). This potential difficulty with \( D_2 \) or \( D_3 \) has led several authors to a somewhat more general approach to measuring bias that looks at bias at points other than \( S = .5 \) or \( V = .5 \).

**Measure 5:** Bias as the Difference Between the Seat Shares Gained by Party I and by Party II When Each Obtains an Identical Share of the Vote

The fifth measure we look at is closely related to (indeed can be thought of as a natural generalization of) \( D_3 \). Let us look at what happens when party I receives a 50 percent vote share. If seats–votes are linearly related as in Equation (5), then party I will receive .5B_1 + \( \psi \) seats, while with a vote share of .5, party II will receive 1 - .5B_1 - \( \psi \) seats. The difference in seats received by the two parties is given by

\[ B_1 + 2\psi - 1 = -2B_1D_2 = 2D_3. \]  

(30)

Consider any other value of \( V \), which we denote \( V(x) \).

Let us define

\[ D_4(x) = \text{seat share of party I if its vote share is } V(x) - \text{seat share of party II if its vote share is } V(x). \]  

(31)

Note that, in this measure, bias is independent of \( V \). Hence, whatever values of \( V \) we actually observe should not affect the amount of bias we "detect," and thus \( D_4(x) \) satisfies criterion B2. It also satisfies criterion B1.

For the logit case of Equation (2), for \( B_1 \) unknown, it is difficult to solve directly for the required expression. Using Equation (19), we find that, for the linear approximation to the logit model,

\[ D_4 = \frac{B_0}{2}. \]  

(32)

This is a rather nice result. Note that for the linearized logit estimates \( D_4 \) is independent of \( B_1 \) and of \( x \), and hence we may drop the \( x \)-subscript.

Just as \( D_4(x) \) is a natural generalization of \( D_3 \), we may readily generate a measure that is a natural generalization of \( D_2 \).
Measure 6: Bias as the Difference Between the Vote Shares Obtained by Party I and by Party II when Each Obtains an Identical Share of the Seats.

We may define $D_5(x)$:

$$D_5(x) = \begin{cases} \text{vote share of party I if its seat share is } S(x), \\ \text{vote share of party II if its seat share is } S(x). \end{cases}$$

(33)

If seats and votes are linearly related as in Equation (5), we have

$$D_5 = \frac{-D_2}{B_1} = \frac{1 - 2\psi - B_1}{B_1} = 2D_2 = \frac{-2D_3}{B_1}. \quad (34)$$

Since exploration of the logit model for this case adds little new, we omit it.

We shall not deal with the properties of $D_5$ since we wish to turn to a still further generalization of $D_5$ (and $D_2$).

Measure 7: Bias as a Gini-Index-Like Measure of the Area Under Seats-Votes Discrepancy Curves.

If party I receives a given share of the vote, there is a minimum share of the seats that it could win. This minimum share would occur if party II received a majority of the votes in as many districts as possible; and party I as far as possible had its votes concentrated into a handful of districts that it carried unanimously. We denote this minimum as $S_{\min}$. Similarly, if party I receives a vote share of $x$, there is maximum seat share it could win. This maximum share would occur if its votes were spread so as to give it a bare majority in as many districts as possible. We denote this maximum as $S_{\max}$. Figure 2 shows minimum and maximum seats curves for a legislature with a very large number of districts, all of which are contested and all of which are of equal size.

Until now, we have implicitly assumed that data on the seats-votes relationship were generated across a series of elections, with each election specifying one point on the seats-votes graph. If district level data are available, an alternative method exists for generating a seats-votes curve. As far as we are aware, Butler (1951) was the first to suggest this procedure. Tufte (1973) also makes use of

Figure 2. Graph of Theoretical Maximum/Minimum Seat Shares in a Two-Party, Single-Member District Election (Source: Grafman, 1975:318, Figure 6).
### TABLE 1

**DEMOCRATIC AND REPUBLICAN SEAT SHARES IN THE CONNECTICUT ASSEMBLY**

**AT SELECTED PROPORTIONS OF THE STATEWIDE (TWO-PARTY) VOTE IN 1970 AND 1972**

**FOR HYPOTHETICAL ELECTIONS BASED ON UNIFORM SWINGS ACROSS ALL DISTRICTS FROM THE OBSERVED SEAT-VOTE VALUE IN THAT YEAR**

<table>
<thead>
<tr>
<th>Proportion of State-wide Vote</th>
<th>45±</th>
<th>46±</th>
<th>47±</th>
<th>48±</th>
<th>49±</th>
<th>50±</th>
<th>51±</th>
<th>52±</th>
<th>53±</th>
<th>54±</th>
<th>55±</th>
<th>Swing Ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Dem 1970</strong></td>
<td>42.9</td>
<td>45.8</td>
<td>48.0</td>
<td>49.7</td>
<td>52.0</td>
<td>55.9</td>
<td>55.9</td>
<td>59.3</td>
<td>62.1</td>
<td>64.4</td>
<td>65.5</td>
<td></td>
</tr>
<tr>
<td><strong>Rep 1970</strong></td>
<td>34.5</td>
<td>35.6</td>
<td>37.9</td>
<td>40.7</td>
<td>44.1</td>
<td>44.1</td>
<td>48.0</td>
<td>50.3</td>
<td>52.0</td>
<td>54.2</td>
<td>57.1</td>
<td>2.26</td>
</tr>
<tr>
<td><strong>Bias</strong></td>
<td>+8.4</td>
<td>+10.2</td>
<td>+10.1</td>
<td>+9.0</td>
<td>+7.9</td>
<td>+11.8</td>
<td>+7.9</td>
<td>+9.0</td>
<td>+10.1</td>
<td>+10.2</td>
<td>+8.4</td>
<td>2.26</td>
</tr>
<tr>
<td><strong>Dem 1972</strong></td>
<td>36.4</td>
<td>38.4</td>
<td>40.4+</td>
<td>41.1</td>
<td>42.4</td>
<td>48.3</td>
<td>51.0</td>
<td>55.0</td>
<td>58.3</td>
<td>62.9</td>
<td>67.5</td>
<td>3.11</td>
</tr>
<tr>
<td><strong>Rep 1972</strong></td>
<td>32.5</td>
<td>37.1</td>
<td>41.7</td>
<td>45.0</td>
<td>49.0</td>
<td>51.7</td>
<td>57.6</td>
<td>58.9</td>
<td>59.6+</td>
<td>61.6</td>
<td>63.6</td>
<td></td>
</tr>
<tr>
<td><strong>Bias</strong></td>
<td>+3.9</td>
<td>+1.3</td>
<td>-1.3</td>
<td>-3.9</td>
<td>-6.6</td>
<td>-3.4</td>
<td>-6.6</td>
<td>-3.9</td>
<td>-1.3</td>
<td>+1.3</td>
<td>+3.9</td>
<td>3.11</td>
</tr>
</tbody>
</table>

**Source:** Scarrow (1981, Table III). Cell entries indicate seat percentages that a party would have achieved at the (column) specified vote share. Arrows indicate actual election outcomes. Boxed outcomes represent situations where a party with a vote share less (more) than .5 would achieve a projected seat share greater (less) than .5.
Figure 3A. Graph of Projected Seats Votes Discrepancies in the Connecticut Assembly, 1970 (Data source: [Source])

Figure 3B. Graph of Projected Seats Votes Discrepancies in the Connecticut Assembly, 1972 (Data source: [Source])
Let $S_1(x)$ be the seat share for party 1 corresponding to a vote share of $x$, and similarly define $S_{11}(x)$ for party 11. We define $D_6$ as follows.

$$D_6 = \int_{.5}^{X} S_1(x) - S_{11}(x)$$

$$= \int_{.5}^{X} D_4(x). \tag{35}$$

Hence, if $S$ and $V$ are linearly related as in Equation (5), we have

$$D_6 = \int_{.5}^{X} (B_1x + \psi) - (1 - B_1(1 - x) + \psi)$$

$$= -\int_{.5}^{X} 1 - B_1 - 2\psi \int_{.5}^{X} D_4$$

$$= (.5 - x)(1 - B_1 - 2\psi). \tag{36}$$

We shall not bother to work out the implications for $D_6$ of a nonlinear seats-votes relationship such as in Equation (2).

**Measure B: A Normalized Measure of Bias in the Interval (.5, X)**

Grofman (1975) has proposed to measure the maximum possible bias in seats votes relationships (over the vote range [0, 1]) for different types of election systems by looking at

$$D_{\text{max-min}} = \int_{0}^{1} S_{\text{max}} - S_{\text{min}}. \tag{37}$$

For two-party single-member-district elections under plurality, $D_{\text{max-min}} = 1/2$. We can generate a value of $D_{\text{max-min}}$ for the vote range (.5, X) by defining

$$D_{\text{max-min}}(x) = \int_{.5}^{X} S_{\text{max}}(x) - S_{\text{min}}(x). \tag{38}$$

For two-party single-member-district plurality contests,

$$D_{\text{max-min}} = x - \frac{1}{2} - \int_{.5}^{X} 2x - 1 = 2x - x^2 - \frac{3}{4}. \tag{39}$$

It would be desirable to have $D_6$ range between -1 and +1 ($D_6$ through $D_5$ vary over that range). We may accomplish this by normalizing $D_6(x)$ by $D_{\text{max-min}}(x)$; i.e., we look at

$$D_6'(x) = \frac{D_6(x)}{2x - x^2 - \frac{3}{4}}. \tag{40}$$

For $x = .55$, the value used by Scarrow, $D_{\text{max-min}}(x) = .0475$. For the 1970 and 1972 Connecticut Assembly elections we show values for $D_1$, $D_2$, $D_3$, $D_4(.50)$, $D_5(.50)$, $D_6(.55)$ and $D_6'(.55)$ in Table 2.

If we compare the 1970 and 1972 elections according to our various measures, we find no agreement among them (although disparities among $D_2$ and $D_3$; $D_4(.5)$ and $D_5(.5)$; and $D_4(.55)$ and $D_5(.55)$ are more apparent than real, since these measures are functionally related to another). In 1970 $V_t$ was close to $S_t$ (.49 v. .52), and we obtain a value of $D_1$ slightly over 1. In 1972 $V_t$ was reasonably close to $S_t$ (.47 v. .40) but now smaller than $S_t$ rather than larger. This gives rise to an anti-Democratic bias, as shown by a $D_1$ value of less than one (.85). Both $D_1$ and $D_2$ are quite misleading, as visual inspection of Figures 3A and 3B suggest. Both $D_1$ and $D_2$ show that 1972 has more bias than 1970. This is not what we observe in Figure 3A. The error arises because in 1970 we had $S_t$ reasonably close to $V_t$, but yet the result turned less than a majority of the votes into a majority of the seats; in 1972 $S_t$ was further from $V_t$ but the differences might reasonably be expected, given a nonlinear (and roughly symmetric) transformation of votes into seats. $D_1$ not only gets the direction wrong but also exaggerates the magnitude of the differences between 1970 and 1972, in that $D_1$ for the latter year is more than twice $D_1$ for the former year in absolute value (.030 v. .066). Measures $D_3$ through $D_5$ get the directionality right but overestimate, in our view, the magnitude of the bias in 1972 relative to that in 1970 by looking only at bias at the point $V = .5$. As can be seen from Figures 3A and 3B, at $V = .5$ the difference between $S_t$ and $S_{11}$ is about .12 for 1970 and about .04 for 1972, a ratio of about 4 to 1. This ratio squares roughly with what we find in comparing the $D_2$ through $D_5$ values for the two years. However, in
1972, while bias stays roughly constant from \( V = .50 \) through \( V = .52 \), at \( V = .53 \) there is a bias reversal and, thus, over the entire range (.5, .55) the net bias in favor of the Republican gets significantly reduced from its value at \( V = .5 \).

### III. CONCLUSIONS

Our concern has been with developing appropriate measures of bias and proportionality. We regard \( B_1 \) as the most appropriate measure of degree of proportionality, with \([B_1 - 1]\) indicating deviation from proportionality. We have demonstrated (a) that the two most common measures of bias (\( D_3 \) and \( D_4 \)) are inappropriate and (b) that most of the remaining measures previously proposed in the literature are, in fact, simple transformations of one another.16

Although \( D_2 \) through \( D_5 \) are reasonable measures, and \( D_2 \) and \( D_3 \), in particular, have the advantages both of ease of calculation and interpretation, the measure of bias best able to deal with properties of the seats-votes relationship over the entire range of \( V \) is \( D_6 \).17 Once we opt for \( D_6 \), however, it makes sense to use \( D_6 \), since the normalization used gives us a measure that will range between -1 and 1.

### NOTES


2. In any actual election system, the value of \( B_1 \) (and of \( B_6 \)) will depend on the spatial distribution of partisan/group support across districts. Very roughly speaking, the more the distribution of partisan/group strength is similar
In all districts, the higher will be \( B_1 \). In general, we would expect \( B_1 > 1 \). See Tufte, 1973; Linehan and Schrot, 1978; Wildgen and Engravron, 1980; Musgrove, 1973; Niaml and Daegan, 1978; Johnston, 1976a, 1979; and Gudgin and Taylor, 1979; and Schrot, 1981, for more on this point. Schrot (1981) has shown that in Equation (2), parameter estimates are sensitive to which party is treated as party I and which as party II and has reestimated some of the equations in Tufte (1973), finding values considerably closer to 3.

3. An S-shaped curve will also be generated under various other plausible assumptions about the underlying functional relationship between seats and votes. See Gudgin and Taylor, 1979:18-19, and Owen and Grofman, 1981. Values of \( S \) for specified values of \( V \) and \( B_1 \) for the seat-vote relationship defined by Equation (1) are given in Table 1 in Grofman (1982b).

Many politicians and lawyers falsely assume that "fair" single-member districting yields proportionality between seats and votes and that absence of proportionality is proof of bias; e.g., "A(n) ... indicator of racial discrimination in the drawing of congressional district lines is that the percentage of Blacks in a state's congressional districts is usually much less than the percentage of Blacks in the state" (Smith, 1975). In fairness to Smith, he also lists other indicators of discrimination including "the division of substantial minorities of Blacks into several contiguous districts so that they are unable to elect a Black in one or more of those districts" (Smith, 1975:671).

4. Even identical \( V \) values in the sets of elections being compared would not alleviate the problem. The relationship specified in Equation (1) (or Equations (2) or (3)) between \( S \) and \( V \) is nonlinear and thus is not expectation-preserving. It should be apparent that the same problem will manifest itself whatever values of \( B_1 \) we pick, although it will be less severe if \( B_1 \) is close to 1.

5. We are deliberately using the same symbol, \( B_1 \), in Equation (5) as in Equation (2), since in both cases \( B_1 \) is taken to be a measure of the swing ratio, even though the value of \( B_1 \) estimated from a linear function as in Equation (5) is unlikely to be identical to that obtained by fitting the power function of Equation (1). March (1957) has shown can be approximated by the straight line \( S = 2.808 V - .904 \). We show below that in the range \( V = .45 \) to .55, Equation (1) can be well approximated by the straight line \( S = 3V - 1 \).

6. We are indebted to Scott Feld for calling this approximation to our attention (cf. March, 1957; Thall, 1959; also see Feld and Grofman, 1980).

7. In these three cases, the differences between linear and logit estimates are minimal. Tufte (1973:543, n. 4), who looks at several other cases in addition to the three we reported, remarks that the linear and the logit method (and other two methods he discusses) "revealed small differences in most estimates (of \( D_2 \)) when the bias was less than 5 percent and the correspondence between seats and votes was fairly high (usually the case); otherwise the estimates diverged."

8. As with \( D_2 \), this need not be true when bias is large. A glance at Figure 1 reveals that linear and logit models are unlikely to yield similar estimates of bias (defined here as the difference between the point on the estimated seat-vote graph and the corresponding point on the proportionality line) if we look at \( S \) values (\( V \) values) away from 0.5.

9. We might also note that if \( S = 1/2 \), then \( D_2 = D_1 \); if \( V = 1/2 \), then \( D_3 = D_1 \).

10. The relationship between \( D_2 \) and \( D_3 \) is considerably more complex when each is estimated from the logit model of Equation (2). Combining Equations (15) and (21) we have

\[
D_2 = D_3 + \frac{1}{B_0/B_1} \left( \frac{B_0}{1 + e^{B_0/B_1}} - \frac{B_0}{1 + e^{-B_0}} \right)
\]

I.e.,

\[
D_2 = D_3 + \left( \frac{-(B_0/B_1)}{1 + e^{-(B_0/B_1)}} \right) \left( \frac{B_0}{1 + e^{B_0}} \right)
\]

I do not find this expression especially enlightening.

11. Even when the regression estimate obtained by substituting the logit estimate into Equation (28) does not correspond perfectly to the best linear fit, it is likely to give a regression line nearly as good (in terms of \( R^2 \)).
12. A measure of bias that is a variant of $D_4$ has been offered by Brookes (1959). We shall not, however, discuss this measure since it adds little or nothing new.

13. If we neglect differences in constituency size and constituency turnout, we are, in effect, looking at what happens when party II's aggregate vote share goes down one percentage point, with the decrease uniform across districts.

14. It might appear that it ought to be harder for party II to gain strength in a district in which it was already strong than in one where it was weak. Except for extreme cases (e.g., districts that are nearly unanimous for a given party), the available statistical evidence seems to support the notion of a swing across districts based on changes in "percentage points and not percentages." According to Tuft (1973:545) "percentages swings are relatively independent of the starting point and are therefore best assessed in terms of untransformed percentages differences." This has been called the "paradox of swing." (See Butler, 1953, for a full discussion of this point; see also Scarrow, 1981, and Taylor and Johnston, 1979, especially Chapter 3.)

15. The pro-Democratic bias appears to be decreasing with increasing $V$ in the 1974 Assembly election, but the effect is slight.

16. We have not attempted to identify the source of bias. Roughly speaking, bias arises when the mean value of overall party strength does not coincide with the median value of party strength across districts (see Soper and Rydon, 1958:97; Johnston, 1979:63-67). Such a discrepancy can occur for a number of reasons. Using a linear approach to estimation, a number of geographers (e.g., Brookes, 1959, 1960; Soper and Rydon, 1958; Gudgin and Taylor, 1979; Taylor and Johnston, 1979) have looked at how $D_2$ (or $D_3$) might be decomposed into components reflecting (a) inequality in the number of voters in the seats won by each of the parties (which in turn can be divided into inequality caused by unequal district size and inequality caused by differential turnout of partisan supporters); (b) differential geographic concentration of partisan support across districts (which in turn can be divided into "natural" differential concentration and that aggravated by the way in which the district lines were drawn: i.e., intentional or unintentional gerrymandering); and (c) the differential impact of minor parties and the distribution of their vote strength. We shall not, however, pursue these issues further here.

Unfortunately, the work on the political geography of electoral relationships done by geographers (primarily British ones) is not familiar to most American political scientists. This work is of very high methodological sophistication and deserves to be far better known. We would especially like to call to the attention of American political scientists Gudgin and Taylor (1979), Johnston (1979), and Taylor and Johnston (1979).

17. A limitation of measures that look only at $V = .5$ is that they are plausible only if elections are fully competitive, with outcomes consistently near an equal vote division.

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