

Bisymmetry Properties of Luce's (2004) Global Psychophysical Representation

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Abstract

Given the axioms of Luce (2004) that lead to a p-additive representation Ψ , then the present Proposition 1 shows that the property of bisymmetry of the joint presentation pairs is equivalent to a strong averaging restriction on Ψ that involves a strictly increasing psychophysical function ψ . Proposition 2 shows that bisymmetry must hold for the additive case and for the non-trivial p-additive case bisymmetry is equivalent to commutativity holding. Proposition 3 shows that if the structure is bisymmetric, then associativity cannot hold. And Proposition 4 shows that a simple invariance property yields that ψ is a power function. The reported data for loudness and brightness both strongly support bisymmetry and strongly reject commutativity, implying that pure additivity holds and that associativity does not hold.

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Luce (2004) gave behavioral axioms about two interrelated structures modeling the psychophysics of subjective intensity. The first structure has the form $\langle X \times X, \succsim \rangle$ for which X is interpreted to be physical intensities less the threshold intensity (not, e.g., dB) and so are the non-negative real numbers with the unit of intensity measurement. Thus, by the way that intensity is measured, every threshold = 0 when other features of the signal, such as frequency, are varied. Pairs (x, u) are interpreted in psychophysics as presenting signals x and u to, say, the left and right ears (eyes) simultaneously. A respondent orders such pairs by perceived subjective intensity, e.g., loudness in audition or brightness in vision. Denote that order by \succsim , which is assumed to be a weak order and to be monotonic¹ in the sense that

$$(x, u) \succsim (y, u) \Leftrightarrow x \geq y, \quad (1)$$

$$(x, u) \succsim (x, v) \Leftrightarrow u \geq v. \quad (2)$$

Both solvability and Archimedeaness are also assumed to be satisfied.

The second structure involved formalizing the method of magnitude production first introduced into psychophysics in mid 20th century by S.S. Stevens (summarized in Stevens, 1975). Because that structure plays no explicit role in this note, it is not described here.

Luce (2004, errata 2008) introduced testable behavioral invariances for each of the two structures separately and invariances that link the two structures in a fashion somewhat like the classical theory for the measurement of mass: Mass orderings can be generated either by the concatenation of (homogenous) masses on a pan balance or by the trade-offs between volume of homogenous substances and the various substances themselves. And a linkage is assumed between the two structures that permits a proof that the two resulting measures of mass are identical (see, e.g., Luce, Krantz, Suppes, & Tversky, 1990/2007, pp. 312-318; Luce, 2009).

In the psychophysical context it was shown [Eq. (18) of Theorem 1, p. 448, Luce, 2004] that there is a real, order-preserving representation Ψ onto the domain of physical intensities – that is numbers with a fixed unit – such that^{2, 3}

$$\Psi(x, y) = \Psi(x, 0) + \Psi(0, y) + \delta \Psi(x, 0) \Psi(0, y), \quad \delta = 0, 1. \quad (3)$$

This is called a *p-additive representation* because it is the unique polynomial that can be transformed into an additive representation, as is shown below. But proving that it is the only polynomial that can be put in additive form is trickier (see Aczél, 1966, p. 61).

¹Because of the brightness phenomenon called Fechner’s paradox, monotonicity breaks down when one intensity is substantially smaller than the other. However, in perceived contrast, there seems to be no evidence of an analog to Fechner’s paradox.

²In the context of utility theory, the range of Ψ is an interval of the full real numbers, and δ may also = -1. See Luce (2000).

³The decomposition of (3), as well as (4) below, is problematic for brightness for two reasons: First, because binocular rivalry makes it difficult to actualize these signals directly, but they may be estimated. Second, the non-monotonicity called the Fechner Paradox occurs when x or y approaches 0 as of course is the case for $\Psi(x, 0)$ and $\Psi(0, y)$.

It was also shown that for some constant $\gamma > 0$

$$\frac{\Psi(x, 0)}{\Psi(0, x)} = \gamma. \quad (4)$$

A third property capturing magnitude production was also derived but it does not play a role here and so I do not state it.

I first show that the p-additive form can be put in additive form as follows. For $\delta = 0$, Ψ is already purely additive, and for $\delta = 1$ it can be transformed into an additive form by rewriting (3) as

$$1 + \delta\Psi(x, y) = [1 + \delta\Psi(x, 0)][1 + \delta\Psi(0, y)] \quad (5)$$

If we define

$$\Phi(x, y) := \ln [1 + \delta\Psi(x, y)], \quad (6)$$

then (3) transforms to

$$\Phi(x, y) = \Phi(x, 0) + \Phi(0, y). \quad (7)$$

Note that (6) implies that $1 + \delta\Psi(x, y)$, and so $\delta\Psi(x, y)$, must be dimensionless. Thus, there is no loss of generality in setting $\delta = 1$ in (3). Dr. Ng made this observation in joint work we did together with Dr. A. J. Marley concerned with entropy-modified utility measures.

Define

$$\Theta = \begin{cases} \Psi & \text{if } \delta = 0 \text{ in (3)} \\ \Phi & \text{if } \delta = 1 \text{ in (3) and } \Phi \text{ is defined by (6)} \end{cases}. \quad (8)$$

Note that Θ is order-preserving and additive

$$\Theta(x, y) = \Theta(x, 0) + \Theta(0, y), \quad (9)$$

by (3) when $\delta = 0$ and by (7) when $\delta = 1$. Note also that from (9) with $x = y = 0$ yields

$$\begin{aligned} \Theta(0, 0) &= 2\Theta(0, 0) \\ \Rightarrow \Theta(0, 0) &= 0. \end{aligned} \quad (10)$$

The error corrected in Luce (2008), which C. T. Ng pointed out to me, is that I incorrectly claimed that the p-additive representation (3) plus the empirically supported property of bisymmetry described in (13) below implied that $\delta = 0$ (Corollary to Theorem 2 of Luce, 2004). The specific details were not worked out in that errata. The overall purpose of this note is to fill out some of the details about the relations among the concepts of bisymmetry, commutativity, and associativity. Specifically, a simple condition equivalent to bisymmetry is developed in Proposition 1 below.

Four Propositions

Bisymmetric Operations

In the second part of Luce (2004), I worked with the symmetric match

$$(x, u) \sim (z_s, z_s), \quad (11)$$

where $z_s = z_s(x, u)$ is a function⁴ from $X \times X \xrightarrow{\text{onto}} X$ that is strictly increasing in each variable. Because

$$(x, x) \sim (z_s(x, x), z_s(x, x)),$$

monotonicity implies that

$$z_s(x, x) = x. \quad (12)$$

In particular, $z_s(0, 0) = 0$.

Proposition 1 *Consider a structure $\langle X \times X, \succ \rangle$ satisfying the axioms 1-6 of Luce (2004). Then Parts 1 and 2 are equivalent:*

1. *The function z_s is bisymmetric in the sense of satisfying*

$$z_s(z_s(x, y), z_s(u, v)) \sim z_s(z_s(x, u), z_s(y, v)). \quad (13)$$

2. *There exist a strictly increasing function $\psi : X \xrightarrow{\text{onto}} X$ and a constant $1 \geq \mu \geq 0$ such that*

$$\psi(z_s(x, y)) = \mu\psi(x) + (1 - \mu)\psi(y). \quad (14)$$

And Parts 1 \Leftrightarrow 2 imply

3. *For $\rho = \frac{\mu}{1-\mu}$, ψz_s satisfies*

$$\frac{\psi(z_s(x, 0))}{\psi(z_s(0, x))} = \rho. \quad (15)$$

⁴In Luce (2004) I also defined the operation \oplus_s

$$x \oplus_s u := z_s(x, u),$$

and so

$$(x, u) \sim (x \oplus_s u, x \oplus_s u).$$

Similarly, let $z_l \in X$ be the solution to

$$(x, u) = (z_l(x, u), 0).$$

Define \oplus_l as follows:

$$x \oplus_l u := z_l(x, u).$$

The right operation \oplus_r is defined similarly.

Because of the very noticeable perceptual changes encountered with z_l and z_r , I restrict this note to using z_s .

Although the subsequent empirical articles used the operator notation, the functional notation would have worked just as well. In this article it is more natural to use the function notation.

All proofs are in the appendix.

The form of ψ in (14) is discussed in Proposition 4.

Conditions When Bisymmetry is Satisfied

The next result draws upon the paragraph following the Corollary of Theorem 2 of Luce (2004) which asserts that for some γ (4) is satisfied. Proposition 2 replaces the incorrect inference in Luce (2004) that bisymmetry alone forces $\delta = 0$.

Proposition 2 *Under the assumptions of Proposition 1 and that (3) and (4) hold, then*

1. For $\delta = 0$, bisymmetry (13) is satisfied.
2. For $\delta = 1$, bisymmetry is satisfied iff commutativity⁵ is satisfied in the sense that, for $x, y \in X$,

$$(x, y) \sim (y, x). \quad (16)$$

With both loudness and brightness, the data strongly support bisymmetry of the function z_s , (13), and equally strongly reject commutativity, (16) (Steingrímsson, 2009, 2010 in press; Steingrímsson & Luce, 2005a,b). So, we conclude for $\delta = 0$ that bisymmetry is satisfied and that for $\delta = 1$ that it is not empirically sustained.

Associativity

The associativity notion used next is that defined by Aczél (1966) on p. 253.

Proposition 3 *Under the assumptions of Proposition 1 and assuming that bisymmetry is also satisfied, then the structure is not associative in the sense that, for $x, y, u \in X$,*

$$((x, y), u) \sim (x, (y, u)) \quad (17)$$

$$\Leftrightarrow z_s([z_s(x, y), z_s(x, y)], u) \sim z_s(x, [z_s(y, u), z_s(y, u)]). \quad (18)$$

Note that commutativity and associativity are both assumed for Hölder's (1901) theorem, which if the mapping is to $\langle X \times X, +, \cdot, \geq \rangle$ yields (9) (see Luce, 2010). In this connection Aczél (1966, p. 278) makes the insightful comment that "...[bisymmetry is] most used in structures without the property associativity – in a certain respect, it has been used as a substitute for associativity and also for commutativity (symmetry).

Estimating ψ and μ

Recall Proposition 1 showed that bisymmetry is equivalent to $z_s(x, y)$ satisfying

$$\psi(z_s(x, y)) = \mu\psi(x) + (1 - \mu)\psi(y). \quad (19)$$

⁵In Luce (2004) I called this property joint-presentation symmetry, but in mathematics it is usually called commutativity.

In principle, were one able to collect sufficient data, one could estimate the unknowns ψ and μ . But in practice, doing that is not really feasible.

Matters are greatly simplified when, for any $\kappa > 0$, the following testable invariance is satisfied: There exists a constant $\beta > 0$ such that

$$z_s(\kappa x, \kappa y) = \kappa^\beta z_s(x, y) \quad x \neq y. \quad (20)$$

Proposition 4 *Under the assumptions of Proposition 1 and assuming that bisymmetry is also satisfied, then (20) is equivalent to*

$$\psi(x) = \alpha x^\beta. \quad (21)$$

This result is proved by Aczél (1966, p. 15). So β is estimated by converting (20) into dB form leading to

$$\begin{aligned} z_{s,dB}(kx, ky) &= \beta k_{dB} + z_{s,dB}(x, y) \\ \Leftrightarrow \beta &= \frac{z_{s,dB}(kx, ky) - z_{s,dB}(x, y)}{k_{dB}}. \end{aligned} \quad (22)$$

Using a number of (x, y) pairs, see the degree to which the right ratio in (22) is constant. The degree to which it is constant yields an estimate of β . Then the estimation problem (19) reduces to finding the one parameter μ in the expression:

$$z_s(x, y)^\beta = \mu x^\beta + (1 - \mu) y^\beta. \quad (23)$$

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Appendices: Proofs

Proposition 1.

Proof. Part 1 \Rightarrow Part 2. Aczél, (1966, p.287) proves that a bisymmetric function z_s is equivalent to its satisfying

$$\psi [z_s(x, y)] = \mu\psi(x) + \nu\psi(y) + \eta,$$

where ψ is strictly increasing: $X \xrightarrow{onto} X$ and μ, ν , and η are non-negative constants. Because $\psi(0) = 0$ and (12) this implies $\eta = 0$, and so

$$\psi [z_s(x, y)] = \mu\psi(x) + \nu\psi(y), \quad \mu \geq 0, \nu \geq 0, \text{ with } \psi [z_s(x, x)] = \psi(x).$$

Setting $x = y$ in the above display yields

$$\mu + \nu = 1,$$

leading to the weighted average representation

$$\psi [z_s(x, y)] = \mu\psi(x) + (1 - \mu)\psi(y), \quad 1 \geq \mu \geq 0, \text{ with } \psi [z_s(x, x)] = \psi(x). \quad (24)$$

Part 2 \Rightarrow Part 1. Consider the left side of bisymmetry,

$$\begin{aligned} \psi (z_s [z_s(x, y), z_s(u, v)]) &= \mu\psi (z_s(x, y)) + (1 - \mu)\psi (z_s(u, v)) \\ &= \mu [\mu\psi (x) + (1 - \mu)\psi (y)] + (1 - \mu) [\mu\psi (u) + (1 - \mu)\psi (v)] \\ &= \mu^2\psi (x) + \mu(1 - \mu) [\psi (y) + \psi (u)] + (1 - \mu)^2\psi (v) \\ &= \mu^2\psi (x) + \mu(1 - \mu) [\psi (u) + \psi (y)] + (1 - \mu)^2\psi (v) \\ &= \psi (z_s (z_s(x, u), z_s(y, v))). \end{aligned}$$

Taking ψ^{-1} proves bisymmetry.

Part 3. Given (24), observe that

$$\begin{aligned}\psi(z_s(x, 0)) &= \mu\psi(x) \\ \psi(z_s(0, y)) &= (1 - \mu)\psi(y),\end{aligned}$$

so

$$\frac{\psi(z_s(x, 0))}{\psi(z_s(0, x))} = \frac{\mu\psi(x)}{(1 - \mu)\psi(x)} = \frac{\mu}{1 - \mu} = \rho.$$

■

Because we know empirically that commutativity fails, $\rho \neq 1$ and $\mu \neq \frac{1}{2}$.

Proposition 2

Proof. Part 1. Assume $\delta = 0$. By Proposition 1 above with $\Theta = \Psi$, and because $\Psi(x, u)$ is additive as is $\psi(z_s(x, u))$ so they are proportionate and so $\Psi(x, u)$ satisfies bisymmetry and $\rho = \gamma$.

Part 2. Assume $\delta = 1$. By the same argument as in Part 1, we know that $\Phi(x, u)$ is proportionate to $\psi(z_s(x, u))$. Then from (4), (6), and (15) for all $x \in X$

$$\begin{aligned}\rho &= \frac{\psi(z_s(x, 0))}{\psi(z_s(0, x))} \\ &= \frac{\ln[1 + \Psi(x, 0)]}{\ln[1 + \Psi(0, x)]} \\ &= \frac{\ln[1 + \gamma\Psi(0, x)]}{\ln[1 + \Psi(0, x)]}.\end{aligned}$$

Setting $Z := \Psi(0, x)$, this is equivalent to, for all Z ,

$$\begin{aligned}\rho &= \frac{\ln(1 + \gamma Z)}{\ln(1 + Z)} \\ \Leftrightarrow \rho \ln(1 + Z) &= \ln(1 + \gamma Z) \\ \Leftrightarrow \ln(1 + \gamma Z)^\rho &= \ln(1 + Z) \\ \Leftrightarrow (1 + Z)^\rho &= 1 + \gamma Z \\ \Leftrightarrow \rho = \gamma &= 1,\end{aligned}$$

which implies $\mu = \frac{1}{2}$. So by (24), this is equivalent to

$$z_s(x, y) = z_s(y, x) \Leftrightarrow (x, y) \sim (y, x),$$

i.e. commutativity. ■

Proposition 3.

Proof. 1 \Rightarrow 2. Because Proposition 1 is satisfied, repeated use of (13) yields

$$\begin{aligned}
& (x, (y, z)) \sim ((x, y), z) \\
& \Leftrightarrow z_s [x, z_s(y, z)] = z_s [z_s(x, y), z] \\
& \Leftrightarrow \psi (z_s [x, z_s(y, z)]) = \psi (z_s [(z_s(x, y), z)]) \\
& \Leftrightarrow \psi (\psi^{-1} [\mu\psi(x) + (1 - \mu)\psi(z_s(y, z))]) = \psi (\psi^{-1} [\mu\psi(z_s(x, y)) + (1 - \mu)\psi(z)]) \\
& \Leftrightarrow \mu\psi(x) + (1 - \mu)\psi(z_s(y, z)) \sim \mu\psi(z_s(x, y)) + (1 - \mu)\psi(z) \\
& \Leftrightarrow \mu\psi(x) + (1 - \mu)\psi(\psi^{-1} [\mu\psi(y) + (1 - \mu)\psi(z)]) = \mu\psi(\psi^{-1} [\mu\psi(x) + (1 - \mu)\psi(y)]) + (1 - \mu)\psi(z) \\
& \Leftrightarrow \mu\psi(x) + (1 - \mu)\mu\psi(y) + (1 - \mu)^2\psi(z) = \mu^2\psi(x) + \mu(1 - \mu)\psi(y) + (1 - \mu)\psi(z)
\end{aligned}$$

which is clearly equivalent to $\mu = \mu^2 \Leftrightarrow \mu = 1$ and $(1 - \mu) = (1 - \mu)^2 \Leftrightarrow \mu = 0$, a contradiction. ■

Proposition 4.

Proof. Given $x \neq y$,

$$\psi [z_s(x, y)] = \mu\psi(x) + (1 - \mu)\psi(y),$$

and (20), i.e.,

$$z_s(\kappa x, \kappa y) = \kappa^\beta z_s(x, y), \quad \kappa > 0,$$

we have

$$\begin{aligned}
& \mu\psi(\kappa x) + (1 - \mu)\psi(\kappa y) \\
& = \psi [z_s(\kappa x, \kappa y)] \\
& = \kappa^\beta \psi [z_s(x, y)] \\
& = \kappa^\beta [\mu\psi(x) + (1 - \mu)\psi(y)].
\end{aligned}$$

Setting $y = 0$, this is equivalent to

$$\psi(\kappa x) = \kappa^\beta \psi(x),$$

which for ψ strictly increasing and onto is known from Aczél (1966, p. 15) to be equivalent to ψ being a power function (21). ■