



Utility of gambling under p(olynomial)-additive joint receipt and segregation or duplex decomposition[☆]

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ABSTRACT

This article continues our exploration of the utility of gambling, but here under the assumption of a non-additive but p(olynomial)-additive representation of joint receipt. That assumption causes the utility of gambling term to have a multiplicative impact, rather than just an additive one, on what amounts to ordinary weighted utility. Assuming the two rational recursions known as branching and upper gamble decomposition, we investigate separately the rational property of segregation and the non-rational one of duplex decomposition. Under segregation, we show that each pair of disjoint events either exhibits weight complementarity or both support no utility of gambling, a property called UofG-singular. We develop representations for both cases. The former representation is a simple weighted utility with each term multiplied by a function of the event underlying that branch. The latter representation is an ordinary rank-dependent utility with Choquet weights but with no utility of gambling. Under duplex decomposition, we show that weights have the intuitively unacceptable property that there is essentially no dependence upon the events, making duplex decomposition, in this context, of little behavioral interest.

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1. The p-additive representation

1.1. Introduction

In five original articles (Luce & Marley, 2000; Luce, Ng, Marley, & Aczél, 2008a,b; Ng, Luce, & Marley, 2009, in press) and one summary article (Luce, Marley, & Ng, 2009), we have worked with a binary operator \oplus over gambles that represents the idea of having or receiving both gambles. Among the properties assumed were that \oplus has an additive representation U , in the sense that, if f and g are two gambles, then

$$U(f \oplus g) = U(f) + U(g).$$

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A second important property, called the separability of unitary gambles, is the existence of a multiplicative conjoint representation of binary gambles for which there is only one non-trivial (pure consequence, event) pair (defined formally by (20) below).

Luce et al. (2008b) study the case where the structure has an additive representation of \oplus and segregation, (25), holds; the latter says, in effect, that we may “subtract” from the gamble the lesser valued consequence, so as to yield the lesser consequence joint with a binary gamble having only one non-trivial consequence-event branch. Under these conditions, certain of the representations with a utility of gambling term can accommodate the Allais and Ellsberg paradoxes and most, but not all, of the data due to Birnbaum and associates (Marley and Luce (2005) summarize various of these data and fits of earlier models to those data). One purpose of the current study is to discover whether or not a class of non-additive utility representations can accommodate further data. The results show that this is not the case. In fact, the results, as stated in the next paragraph, with forward reference to the later detailed definitions and representations, show that the resulting representations are either a simple weighted utility or ordinary rank-dependent utility, neither of which accommodate data as well as the representations derived previously assuming an additive representation of \oplus . Nonetheless, these results are interesting in their own right.

We pointed out in that series of papers on the utility of gambling that another equally possible separable representation of \oplus exists,

namely the so-called polynomial-additive (somewhat abbreviated as p-additive) representation, i.e.,

$$U(f \oplus g) = U(f) + U(g) + \delta U(f)U(g) \quad (\delta = \pm 1).$$

Here we begin to investigate that case.

As discussed more fully in Luce (in press), there are three major reasons that the p-additive representation is of potential interest. First, the Hölder 1901 axioms (see (Krantz, Luce, Suppes, & Tversky, 1971) that lead to the additive representation into $(\mathbb{R}, \geq, +)$ equally well have the p-additive ones into $(\mathbb{R}, \geq, +, \cdot)$, and such a more general space plays a critical role in the representation of gambles. Second, these representations are bounded from below for $\delta = 1$ and from above for $\delta = -1$. Third, these representation are absolute scales (in a sense similar to probability and relativistic velocity). What is seriously lacking in the literature are attempts to test whether the p-additive representations are more adequate than the additive ones in accommodating data.

This paper has four things in common with our earlier papers in this series. First, we do not suppose that a gamble which has the same consequence attached to each uncertain event is indifferent to that consequence itself, i.e., we do not assume idempotence. However, second, we do partition each gamble into the joint receipt of two parts: an idempotent one, called a kernel equivalent, Definition 3, joint with a gamble that attaches the identity e of \oplus to each subevent of the original gamble, called an element of chance, Definition 2. Third, we impose the “rational” property of segregation, (25), on binary gambles, which was described earlier. Fourth, we assume that the two recursions called branching, (23), and upper gamble decomposition (UGD), (24), are both satisfied. With these assumptions we show that either the weights over events exhibit complementarity or an event exists that satisfies a property called UofG-singular (Definition 18). For the former the general representation takes a weighted utility form with each term modified by an event-dependent factor ((37) of Theorem 20). For the latter, the standard rank-dependent representation arises but with no utility of gambling term, (Theorem 21).

Finally, we replace segregation by the property of duplex decomposition, (46), that led to some significant results in the additive case despite its being “non-rational” in nature. Nonetheless, experiments seem to show that it holds for some respondents. But under both branching and UGD the weights show essentially no dependence upon the events (Proposition 24), a clearly unacceptable property for behavioral applications.

1.2. General background

The background given here is extracted, with only minor revisions, from Luce et al. (2008a, Section 1, pages 3–4), with kind permission of Springer Science + Business Media.

Let X denote a set of pure consequences – ones for which chance or uncertainty plays no role. The adjective “pure” is used to distinguish such consequences from those that are, in fact, gambles; see below. Let X include a distinguished element e that is interpreted as representing no change from the status quo.

Let \mathfrak{B} be a set of events (sets) containing, with any two events, also their union and difference, thus also their intersection and the empty element \emptyset . In mathematical terms we say that \mathfrak{B} is a Boolean ring. Let $\mathfrak{B}^* = \mathfrak{B} \setminus \{\emptyset\}$.

Let \mathbf{E} be a finite “experiment” or “chance phenomenon”. Let (C_1, \dots, C_n) , $C_i \in \mathfrak{B}$, $C_i \cap C_j = \emptyset$ if $i \neq j$, denote the partition defining \mathbf{E} . The “universal” event for the experiment \mathbf{E} is the union $\Omega = \bigcup_{i=1}^n C_i$. We place quotes around “universal” because the event Ω is only universal for the purpose of a particular local chance experiment \mathbf{E} , not in the global sense of a state space (Savage, 1954). Thus, we admit the possibility of many experiments each having its own Ω .

A first-order gamble consists of n , $1 \leq n < \infty$, (pure consequence, event) pairs, each of which is called a branch of the first-order gamble. Thus, replacing the parentheses of the branches by semicolons, we may write such a gamble as

$$g_{[n]} = (x_1, C_1; \dots; x_i, C_i; \dots; x_n, C_n). \tag{1}$$

Whenever we use Ω along with $g_{[n]}$, it is implicit that $\Omega = \bigcup_{i=1}^n C_i$ is the universal event underlying the gamble. We sometimes suppress the subscript and write g in place of $g_{[n]}$.

A second-order gamble consists of (consequence, event) pairs, also called branches, where by a “consequence” we mean either a pure one or a first-order gamble, and at least one consequence is a first-order gamble. From here on, a gamble means either a pure consequence, a first-order gamble, or a second-order gamble, and the unmodified term consequence means either a pure consequence or a first-order gamble in a second-order gamble.

In addition we assume that a person can have the joint receipt of pure consequences and gambles, which operation is denoted by \oplus .

Let \mathcal{G} denote the closure of all gambles under \oplus .

Assume that a decision maker has a preference order \succsim over \mathcal{G} , which we usually denote just as \succsim , and that it is a weak order. As usual, \preccurlyeq denotes the converse of \succsim and \sim denotes the corresponding indifference relation: $\sim := \succsim \cap \preccurlyeq$ and $\succ := \succsim \setminus \sim$.

The gambles $g \succsim e$ are called gains, and those with $g \preccurlyeq e$ are losses.

We will also assume that the joint receipt \oplus is monotonic in the sense that, for all gambles $f, f', g \in \mathcal{G}$,

$$f \succsim f' \Leftrightarrow f \oplus g \succsim f' \oplus g. \tag{2}$$

This was empirically explored in Cho and Fisher (2000) and was not rejected. We assume that X is closed under \oplus , i.e., for any $x, y \in X$, $x \oplus y \in X$; \oplus is associative and commutative; no change from the status quo, e , is an identity of \oplus ; and for each $x \in X$, there is an inverse $x^{-1} \in X$ such that $x \oplus x^{-1} \sim e$. With \oplus defined on the equivalences of \sim , the structure is a commutative group. Note that $(x \oplus y)^{-1} \sim x^{-1} \oplus y^{-1}$.

We assume that, for each gamble $g \in \mathcal{G}$, the set X is sufficiently rich that it contains an element, denoted $CE(g)$, such that $CE(g) \sim g$. This pure consequence is called a certainty equivalent of the gamble. Assuming gambles have certainty equivalents, \oplus is an abelian group operation on the equivalence classes of \mathcal{G} .

We also assume the property of certainty:

$$(x_1, \emptyset; \dots; x_{i-1}, \emptyset; x_i, \Omega; x_{i+1}, \emptyset; \dots; x_n, \emptyset) \sim x_i; \tag{3}$$

and the property of expansibility:

$$(x_1, C_1; \dots; x_{i-1}, C_{i-1}; x_i, \emptyset; x_{i+1}, C_{i+1}; \dots; x_n, C_n) \sim (x_1, C_1; \dots; x_{i-1}, C_{i-1}; x_{i+1}, C_{i+1}; \dots; x_n, C_n). \tag{4}$$

Because of this assumption, from now on when we write a gamble, Eq. (1), or a partition (C_1, C_2, \dots, C_n) , we assume that every event C_i is non-null.

A gamble (1) is ranked when the consequences are numbered in order of preference, i.e.,

$$x_1 \succsim x_2 \succsim \dots \succsim x_n \tag{5}$$

and the corresponding event partition is treated as an ordered n -tuple with that induced order of indices.

We assume that any permutation of the branches yields an indifferent gamble, i.e., for any permutation π ,

$$(x_1, C_1; x_2, C_2; \dots; x_n, C_n) \sim (x_{\pi(1)}, C_{\pi(1)}; x_{\pi(2)}, C_{\pi(2)}; \dots; x_{\pi(n)}, C_{\pi(n)}). \tag{6}$$

Ranking is essential in formulating some properties, such as comonotonic consequence monotonicity below. When we assume the ranked form, we explicitly say so. Otherwise, we do not assume that the consequences are ranked.

1.3. Several definitions and assumptions

The first definition arises because we are dealing with ranked gambles.

Definition 1. Co-monotonic consequence monotonicity is satisfied iff, for every $i \in \{1, 2, \dots, n\}$, when $x_i, x'_i \in X$ have the same rank position among the other consequences, then

$$x'_i \succ x_i \text{ iff } (x_1, C_1; \dots; x'_i, C_i; \dots; x_n, C_n) \succ (x_1, C_1; \dots; x_i, C_i; \dots; x_n, C_n). \tag{7}$$

We assume that co-monotonic consequence monotonicity holds.

A function U from the domain of pure consequences and gambles to the real numbers is called a *utility function* if it is order preserving and maps the status quo element e to 0, i.e.,

$$g \succ h \text{ iff } U(g) \geq U(h), \tag{8}$$

$$U(e) = 0. \tag{9}$$

The existence of certainty equivalents, plus the assumption of co-monotonic consequence monotonicity, justifies using the same notation U over both gambles and pure consequences. Given that \succ has the numerical representation of (8), then \succ must be a weak order. Whenever a function U occurs in the remainder of the paper it is to be interpreted as such a utility function.

The next definition arises because we *do not* assume that gambles are in general e -idempotent:

$$(e, C_1; e, C_2; \dots; e, C_n) \sim e. \tag{10}$$

That fact invites the following two concepts:

Definition 2. An **element of chance** is any gamble for which every consequence is no change from the status quo, e , i.e., gambles of the form $(e, C_1; e, C_2; \dots; e, C_n)$.

Definition 3. For any gamble $g_{[n]} = (x_1, C_1; x_2, C_2; \dots; x_n, C_n)$, its **kernel equivalent**, denoted $KE(g_{[n]})$, is defined to be the pure consequence solution, which we assume exists, to the following indifference:

$$g_{[n]} \sim KE(g_{[n]}) \oplus (e, C_1; e, C_2; \dots; e, C_n). \tag{11}$$

1.4. p -additive gambling structures

We next state several definitions that are close to those found in Luce (2000) but because of non-idempotency are modified slightly.

Definition 4. A utility function $U^{(1)}$ is **p-additive (over \oplus)** if there exists constant δ such that, for any gambles f, g ,

$$U^{(1)}(f \oplus g) = U^{(1)}(f) + U^{(1)}(g) + \delta U^{(1)}(f)U^{(1)}(g). \tag{12}$$

When $\delta \neq 0$, a p -additive utility function is equivalent to

$$1 + \delta U^{(1)}(f \oplus g) = [1 + \delta U^{(1)}(f)][1 + \delta U^{(1)}(g)]. \tag{13}$$

Thus, for $\delta \neq 0$, (11) with (13) yields

$$1 + \delta U^{(1)}(g_{[n]}) = [1 + \delta U^{(1)}(KE(g_{[n]}))] \times [1 + \delta U^{(1)}(e, C_1; e, C_2; \dots; e, C_n)]. \tag{14}$$

We introduce the alternative function

$$V^{(1)}(g) := 1 + \delta U^{(1)}(g). \tag{15}$$

Observe that $U^{(1)}$ is p -additive with $\delta \neq 0$ if and only if $V^{(1)} \neq 1$ and

$$V^{(1)}(f \oplus g) = V^{(1)}(f)V^{(1)}(g). \tag{16}$$

Next, we use the following standard terminology: For a function f from X to Y , the *domain* is X , the *codomain* is Y , and the *range* is the set of elements $y \in Y$ for which there exists $x \in X$ such that $f(x) = y$. If the range of f is equal to Y , we say that f maps X onto its codomain Y .

Proposition 5. Suppose that $U^{(1)}$ is p -additive over \oplus .

1. If $\delta = 0$, $U^{(1)}$ is additive and the range of $U^{(1)}$ is unbounded from both above and below.
2. If $\delta > 0$, the range of $U^{(1)}$ is bounded from below by $-\frac{1}{\delta}$ and unbounded from above.
3. If $\delta < 0$, the range of $U^{(1)}$ is unbounded from below and bounded from above by $\frac{1}{|\delta|}$.
4. For cases 2 and 3,

$$\widehat{U}^{(1)}(x) = \text{sgn}(\delta) \ln [1 + \delta U^{(1)}(x)] = \text{sgn}(\delta) \ln V^{(1)}(x)$$

is an additive representation¹ of \oplus with codomain \mathbb{R} .

All proofs are in the Appendix.

In what follows, we always assume axioms sufficient to ensure, in the three cases $\delta = 0$, $\delta > 0$ and $\delta < 0$, that the ranges of $U^{(1)}$ are, respectively, $]-\infty, \infty[$, $]-\frac{1}{\delta}, \infty[$, $]-\infty, \frac{1}{|\delta}|[$. Thus, for $\delta > 0$, $\delta U^{(1)}$ maps onto $] - 1, \infty[$ and for $\delta < 0$, $\delta U^{(1)}$ maps also onto $] - 1, \infty[$. With no loss of generality, we may assume $\delta = 0, 1, -1$.

We use the alternative notations $U^{(1)}$ or $V^{(1)}$ as convenient, referring to them as utility and value, respectively. Note that, by value, we do not implicitly mean money, but rather a particular transformation of $U^{(1)}$. Also note that for $\delta = \pm 1$ the range of $V^{(1)}$ is $]0, \infty[$, that $V^{(1)}$ is order preserving when $\delta = 1$, and that it is order reversing when $\delta = -1$.

Definition 6. A gambling structure is **joint-receipt decomposable over gains** if for each $x \succ e$ and each event $C \subset \Omega$ there exists an event $D = D(x, C) \subset \Omega$ such that, for all $y \succ e$,

$$(x \oplus y, C; e, \Omega \setminus C) \oplus (e, D; e, \Omega \setminus D) \sim (x, C; e, \Omega \setminus C) \oplus (y, D; e, \Omega \setminus D). \tag{17}$$

Parallel definitions hold for losses and for all of X .

Note that (17) is equivalent to

$$KE(x \oplus y, C; e, \Omega \setminus C) \sim KE(x, C; e, \Omega \setminus C) \oplus KE(y, D; e, \Omega \setminus D). \tag{18}$$

Although joint-receipt decomposition may not seem obvious, Luce (2000, p. 154, Theorem 4.4.5) showed that (18) follows from p -additivity together with the next, rather intuitive, property separability (20). Moreover, the resultant form of $S_\Omega(D)$ is explicitly calculated.

Also, the decomposition of a gamble into its kernel equivalent and elements of chance, plus co-monotonic consequence monotonicity, yields

$$x \succ e \Leftrightarrow (x, C; e, \Omega \setminus C) \succ (e, C; e, \Omega \setminus C) \Leftrightarrow KE(x, C; e, \Omega \setminus C) \oplus (e, C; e, \Omega \setminus C) \succ e \oplus (e, C; e, \Omega \setminus C) \Leftrightarrow KE(x, C; e, \Omega \setminus C) \succ e. \tag{19}$$

Remember, by the term “utility function” we mean an order-preserving function over gains (respectively, losses, all of X) satisfying (8) and (9).

¹ This fact does not mean that we may reduce our results to the additive case. The reason is that our assumptions result in $U^{(1)}$ that is both p -additive and part of the separable representation $U^{(1)}S_\Omega^{(1)}$. That is not true for $\widehat{U}^{(1)}$.

Definition 7. Binary gambles are **separable over gains** if, for some utility function $U^{(2)}$ and some weighting function $S_{\Omega}^{(2)}$,

$$U^{(2)}(KE(x, C; e, D)) = U^{(2)}(x)S_{\Omega}^{(2)}(C), \quad (x \succ e). \quad (20)$$

In parallel, **separability over losses**, and **over all of X** , refers to (20) holding for all $x \succ e$, and for all $x \in X$, respectively.

For strict gains, $x \succ e$, the ordinary axioms for conjoint measurement justify this separable representation provided that we add the following stipulation²: The order \succ_{Ω} induced over Ω by \succ over $KE(x, C_1; e, C_2)$ satisfies *monotonicity of event inclusion* if $C \subseteq E$ implies $C \succ_{\Omega} E$, where \succ_{Ω} is defined by, for $x \succ e$,

$$C \succ_{\Omega} E \Leftrightarrow KE(x, C; e, D) \succ KE(x, E; e, F) \\ (E \cup F = C \cup D = \Omega). \quad (21)$$

This property with separability is equivalent to the condition

$$C \succ_{\Omega} E \Leftrightarrow S_{\Omega}^{(2)}(C) \leq S_{\Omega}^{(2)}(E).$$

A parallel condition holds for losses and for all of X .

Observe that the separable representation is invariant under a power transformation in the sense that $(\tilde{U}^{(2)}, \tilde{S}_{\Omega}^{(2)})$, where

$$\tilde{U}^{(2)}(f) = \alpha \operatorname{sgn}(U^{(2)}(f)) |U^{(2)}(f)|^{\beta},$$

$$\tilde{S}_{\Omega}^{(2)}(C) = S_{\Omega}^{(2)}(C)^{\beta}$$

is another separable representation.

Definition 8. A **p-additive gambling structure over gains (over losses) [over X]** is any gambling structure with the following features:

- (i) a weak order relation \succ ;
- (ii) a strictly monotonic joint receipt group operation \oplus ;
- (iii) gambles satisfy co-monotonic consequence monotonicity, (1), and permutation invariance, (6);
- (iv) there exists a utility function $U^{(1)}$ which is p-additive, (12), with $\delta \neq 0$;
- (v) joint-receipt decomposition, (17), is satisfied over gains (over losses) [over X]; and
- (vi) there is a utility function $U^{(2)}$ with the representation $(U^{(2)}, S_{CUD}^{(2)})$ of $KE(x, C; e, D)$ satisfying one of the following conditions that depend, respectively, on joint-receipt decomposition, (17), being over gains (over losses) [over X]:
 - (a) $(U^{(2)}, S_{CUD}^{(2)})$ is separable, (20), over gains and the range of $U^{(2)}$ over gains is $]0, 1[$.
 - (b) $(U^{(2)}, S_{CUD}^{(2)})$ is separable, (20), over losses and the range of $U^{(2)}$ over losses is $] - 1, 0[$.
 - (c) $(U^{(2)}, S_{CUD}^{(2)})$ is separable, (20), over X and the range of $U^{(2)}$ is either $] - \infty, 1[$ or $] - 1, \infty[$.

1.5. Representations satisfying both p-additivity and separability

To arrive at the strong results that we do, we have assumed that the p-additive utility function $U^{(1)}$ and the separable $U^{(2)}$ of Definition 8 are onto specified intervals, but we shall also assume that the weighting function satisfies the event-density restriction that, for each non-null Ω , S_{Ω} is increasing and onto $]0, 1[$.

Theorem 4.4.6 of Luce (2000), which rests upon a functional equation result of Aczél, Maksa, and Páles (1999), yields a restricted version of the following results. That theorem is restricted in two ways: it is proved only for gains and only for the case³ $\delta = -1$ for $U^{(1)}$. The theorems given below extend that result to losses and to

all of X when separability and joint-receipt decomposability hold over all of X without confining δ to -1 for $U^{(1)}$.

Theorem 9. For any p-additive gambling structure over gains (over losses) [over X], corresponding to the three cases (a), (b), and (c) of Definition 8, there exist a utility function U and a weighting function S_{Ω} such that, respectively, (a) U is p-additive with $\delta = -1$ and (U, S_{Ω}) is a separable representation over gains, (b) U is p-additive with $\delta = 1$ and (U, S_{Ω}) is a separable representation over losses, and (c) U is p-additive with $\delta = -1$ (respectively, $\delta = 1$) and (U, S_{Ω}) is a separable representation over X if $U^{(2)}$ of Definition 8 (vi) (c) has range $] - \infty, 1[$ (respectively, $] - 1, \infty[$).

2. Branching and upper gamble decomposition

We study two recursive forms that each reduce a gamble of order n to a compound gamble of order less than n and permit us to use the result for binary gambles, Proposition 14. The idea for the first property, called *branching*, is to form a binary sub-gamble consisting of the two branches having the two most preferred consequences. These could be anywhere in the enumeration; however, by permutation invariance, we may always, with no loss of generality, reorder the gamble so that the consequences are ordered from best to worse. Suppose that $g_{[n]}$, $n \geq 2$, is such a ranked gamble, i.e., with $x_1 \succ x_2 \succ \dots \succ x_n$, $C_i \in \mathfrak{B}^*$,

$$g_{[n]} := (x_1, C_1; \dots; x_i, C_i; \dots; x_n, C_n). \quad (22)$$

(See Section 1.2.) Branching then is the recursion in which the first two branches are combined into a single first-order gamble $g_{[2]}$, i.e.,

Definition 10. Branching holds if

$$g_{[n]} \sim (g_{[2]}, C_1 \cup C_2; x_3, C_3; \dots; x_n, C_n) \\ = ((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3; \dots; x_n, C_n). \quad (23)$$

We do not presume that the ordering necessarily satisfies $(x_1, C_1; x_2, C_2) \succ x_3$, which necessitates some care in the statement of certain parts of Lemma 25.

In like manner, upper gamble decomposition (UGD) is the recursion that treats as indifferent a binary gamble consisting of the branch with the best consequence and all of the remaining branches combined as a single first-order gamble, i.e.,

Definition 11. Upper gamble decomposition (UGD) holds if

$$g_{[n]} \sim (x_1, C_1; g_{[n],-1}, \cup_{j \neq 1} C_j) \quad (24)$$

where $g_{[n],-1} := (x_2, C_2; \dots; x_i, C_i; \dots; x_n, C_n)$.

Again, we do not presume the ordering between x_1 and $g_{[n],-1}$, which also necessitates some care in the statement of certain parts of Lemma 25.

3. Results under segregation

Definition 12. Segregation holds if, for $x, y \in X$ with $x \succ y$ and disjoint events C, D ,

$$(x, C; y, D) \sim (x \oplus y^{-1}, C; e, D) \oplus y. \quad (25)$$

Definition 13. For any partition (C_1, C_2, \dots, C_n) , define

$$H(C_1, C_2, \dots, C_n) := U(e, C_1; e, C_2, \dots; e, C_n) \quad (26)$$

$$G(C_1, C_2, \dots, C_n) := 1 + \delta H(C_1, C_2, \dots, C_n) \quad (\delta \neq 0). \quad (27)$$

Note that, under our assumptions, for $\delta = 1$, $G = 1 + U$ maps onto $]0, \infty[$. And for $\delta = -1$, G again maps onto $]0, \infty[$. Therefore, G is always positive. When $G \equiv 1$, there is no utility of gambling, in which case, because $\delta \neq 0$, we have $U(e, C_1; e, C_2, \dots; e, C_n) = 0$, i.e., $(e, C_1; e, C_2, \dots; e, C_n) \sim e$.

² A formal axiomatization was given by Marley and Luce (2002) for gains (losses).
³ δ in the p-additive representation in Theorem 4.4.6 of Luce (2000) corresponds to $-\delta$ in the p-additive representation used in this article.

3.1. Preliminary results

Proposition 14. Assume a p -additive gambling structure over gains (over losses), and segregation. Then there is a p -additive utility function U over all of X with $\delta = -1$ ($\delta = 1$) such that, for $x, y \in X$ with $x \succsim y$,

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x)S_{CUD}(C) + \delta U(y)(1 - S_{CUD}(C))]G(C, D), \tag{28}$$

i.e., $V(x, C; y, D)$

$$= [V(x)S_{CUD}(C) + V(y)(1 - S_{CUD}(C))]G(C, D). \tag{29}$$

The kernel equivalent is, of course,

$$V(KE(x, C; y, D)) = V(x)S_{CUD}(C) + V(y)(1 - S_{CUD}(C)). \tag{30}$$

Definition 15. Complementarity holds for weighting function S if

$$S_{CUD}(C) + S_{CUD}(D) = 1 \tag{31}$$

for all binary partitions (C, D) .

Complementarity is, of course, restrictive in the sense of limiting the weights to having a major property of conditional probabilities.

Notice that, if the **choice property** also holds, i.e., if for all $C \subseteq D \subseteq \Omega$,

$$S_\Omega(C) = S_\Omega(D)S_D(C), \tag{32}$$

then, multiplying (31) by $S_\Omega(C \cup D)$, it follows immediately that the weights S_Ω are finitely additive.

Corollary 16. Let (U, S_Ω) be the representation obtained in Theorem 9 for a p -additive gambling structure over X . Then segregation implies that complementarity holds for S_Ω and that, for $x, y \in X$ with $x \succsim y$,

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x)S_{CUD}(C) + \delta U(y)S_{CUD}(D)]G(C, D). \tag{33}$$

Lemma 17. Assume a p -additive gambling structure over gains (over losses) and segregation. Suppose that branching, (23), and UGD, (24), hold for $n = 3$. Let (C_1, C_2, C_3) be any trinary partition of Ω . Then

$$G(C_1, C_2) \neq 1 \text{ implies } S_\Omega(C_1 \cup C_2) + S_\Omega(C_3) = 1. \tag{34}$$

Definition 18. A non-null event E is **UofG-singular** if, for each binary partition (E_1, E_2) of E , $(e, E_1; e, E_2) \sim e$, i.e., $G(E_1, E_2) = 1$.

Because $\delta \neq 0$, also we have $G(E_1, E_2) = 1$ implies $(e, E_1; e, E_2) \sim e$.

Proposition 19. Assume a p -additive gambling structure over gains (over losses) that satisfies segregation. Suppose that branching, (23), and UGD, (24), hold for $n = 3$. Suppose that no event is UofG-singular, then S satisfies complementarity.

Note that Corollary 16 and Proposition 19 are similar in that both arrive at complementarity and in that they have some common assumptions: a p -additive structure that satisfies segregation. But they do differ: The Corollary assumes the representation of Theorem 9 whereas the Proposition assumes branching, UGD for $n = 3$, and the absence of any UofG-singular events.

3.2. Under complementarity

We first explore complementarity and obtain the following representation theorem. We do not invoke Proposition 19 to replace complementarity in the theorem statement because the representation obtained implies complementarity. However, that does not imply that there are not any UofG singular events (i.e., that the values of G in (36) all differ from 1). Note that Ω occurring in an expression related to a gamble $(x_1, C_1; \dots; x_n, C_n)$ stands for $\bigcup_{i=1}^n C_i$.

Theorem 20. Assume a p -additive gambling structure over gains (over losses) that satisfies segregation, branching, (23), and UGD, (24) for $n = 3$. Suppose that the weights satisfy complementarity, (31). Then:

1. There exists a function $\lambda : \mathfrak{B}^* \rightarrow]0, \infty[$ such that

$$S_{CUD}(C) = \frac{\lambda(C)}{\lambda(C) + \lambda(D)} \tag{35}$$

and

$$G(C, D) = \frac{\lambda(C) + \lambda(D)}{\lambda(C \cup D)}. \tag{36}$$

Furthermore, the ranked binary and trinary gambles have the representation

$$V(x_1, C_1; x_2, C_2; \dots; x_n, C_n) = \sum_{i=1}^n V(x_i)S_\Omega(C_i)G(C_i, \Omega \setminus C_i) \tag{37}$$

for $n = 2, 3$.

2. Assume branching, (23), for all $n \geq 3$. Then the representation (37) extends to all ranked gambles of general length $n \geq 2$.
3. Gambles with the representation (37) in fact satisfy branching and UGD independent of the rank order of the consequences.
4. The representation also holds for a p -additive gambling structure over X that satisfies segregation, branching and UGD.

Observe, first, that, using (35) and (36), we have

$$S_\Omega(C_i)G(C_i, \Omega \setminus C_i) = \frac{\lambda(C_i)}{\lambda(\Omega)}, \tag{38}$$

which may be inserted in (37). The resulting representation is simpler:

$$V(x_1, C_1; x_2, C_2; \dots; x_n, C_n) = \sum_{i=1}^n V(x_i) \frac{\lambda(C_i)}{\lambda(\Omega)}.$$

For the case $\delta = 1$, it is a simple weighted utility representation (Marley & Luce, 2005, p. 81). However, the former form, (37), makes very clear to decision scientists what the separate roles of S_Ω and G are. For example, an important generalization would be to find axioms that led to either $G_i(C_i, \Omega \setminus C_i)$ or $S_{\Omega,i}(C_i)$, which could account for the empirical fact that an event C in the first position is evaluated differently from that event in the n th position.

Second, the expression for the utility of gambling, which is found by setting $x_i = e, i = 1, \dots, n$, is

$$H(C_1, C_2, \dots, C_n) = \frac{1}{\delta} \left[\sum_{i=1}^n S_\Omega(C_i)G(C, \Omega \setminus C_i) - 1 \right].$$

Equivalently,

$$\begin{aligned} G(C_1, C_2, \dots, C_n) &= \sum_{i=1}^n S_\Omega(C_i)G(C_i, \Omega \setminus C_i) \\ &= \frac{1}{\lambda(\Omega)} \sum_{i=1}^n \lambda(C_i). \end{aligned}$$

3.3. When non-maximal events are UofG-singular

We next explore the case of segregation where all non-maximal events are UofG-singular. The interest of this case lies in the conclusion (43) and its equivalent form (44), which is the general rank-dependent form for gains and losses, but without any utility of gambling term.

Theorem 21. Assume a p -additive gambling structure over gains (over losses) that satisfies segregation. Suppose that branching, (23), and UGD, (24), hold for $n = 3$ and that $G(C_1, C_2) = 1$ for all 3-partitions (C_1, C_2, C_3) . Then,

1.

$$G(C, D) = \begin{cases} 1, & \text{if } C \cup D \text{ is not maximal} \\ A, A > 0, & \text{if } C \cup D \text{ is maximal.} \end{cases}$$

2. The weights satisfy the choice property, (32), for which there is

(a) A strictly increasing function $\mu : \mathfrak{B}^* \rightarrow]0, \infty[$ such that

$$S_D(C) = \frac{\mu(C)}{\mu(D)}, \quad (39)$$

(b) And such that μ is p -additive in the sense that, for some constant b with the unit of $1/\mu$,

$$\mu(C \cup D) = \mu(C) + \mu(D) + b\mu(C)\mu(D), \quad (40)$$

where the choice of b is restricted by

$$1 + b\mu(C) > 0. \quad (41)$$

3. And for $A = 1$ with $C(i)$ defined by

$$C(i) = \begin{cases} \emptyset, & i = 0 \\ \bigcup_{j=1}^i C_j, & i = 1, \dots, n \end{cases} \quad (42)$$

then, for all $x_1 \succ x_2 \succ \dots \succ x_n$,

$$V(g_{[n]}) = \sum_{i=1}^n V(x_i) [S_{\Omega}(C(i)) - S_{\Omega}(C(i-1))], \quad (43)$$

for $n = 2, 3$. If we also assume branching, or UGD, for general higher lengths, then Eq. (43) holds for general $n \geq 2$.

Note: Because $\sum_{i=1}^n [S_{\Omega}(C(i)) - S_{\Omega}(C(i-1))] = 1$ and because $V = 1 - U$ or $1 + U$, (43) is equivalent to

$$U(g_{[n]}) = \sum_{i=1}^n U(x_i) [S_{\Omega}(C(i)) - S_{\Omega}(C(i-1))], \quad (44)$$

which is the very well known rank-dependent form extended to both gains and losses.

These forms, (43) and (44), have been a preoccupation of utility theorists for the past 25 years (Luce, 2000). In that literature, however, one was working with the form (44) in which U is additive over \oplus , whereas our current U is p -additive over \oplus , and so is non-additive.

It is also well known that, if S_{Ω} is finitely additive, the weights form a probability measure and the representation (44) reduces to

$$U(g_{[n]}) = \sum_{i=1}^n U(x_i) S_{\Omega}(C_i), \quad (45)$$

which is called a *subjective expected utility* (SEU). Because in this case event complementarity is satisfied, the representation is also (37) with $G(C, D) = 1$ and $\lambda = \mu$.

4. Results under duplex decomposition

Although the following property, unlike segregation, is decidedly non-rational, two very good reasons lead to studying it: (1) Empirical support for some respondents (Cho, Luce, & Truong, 2000; Karabatsos, 2005), and (2) the fact that it led to interesting results in work on the utility of gambling assuming additive joint receipts (Luce et al., 2008a,b). Admittedly, in the non-trivial p -additive case the conclusion is that nothing of interest arises, but we think it is valuable to see that.

Preliminary result

Definition 22. Gambles are said to satisfy **duplex decomposition** over X if, for all $x, y \in X$,

$$(x, C; y, D) \oplus (e, C'; e, D') \sim (x, C; e, D) \oplus (e, C'; y, D'), \quad (46)$$

where (C', D') is an independent realization of (C, D) .

In terms of the “bottom line”, this concept is less rational than is segregation. For example, on the right of (46) two of the possible consequences are $x \oplus y$, which occurs if C and D' both occur, and $e \oplus e$, which occurs if D and C' both occur, whereas on the left neither of these consequences can arise.

Proposition 23. For a p -additive gamble structure over X , duplex decomposition implies

$$1 + \delta U(KE(x, C; y, D)) = [1 + \delta U(x)S_{C \cup D}(C)][1 + \delta U(y)S_{C \cup D}(D)] \quad (47)$$

and

$$1 + \delta U(x, C; y, D) = [1 + \delta U(x)S_{C \cup D}(C)][1 + \delta U(y)S_{C \cup D}(D)]G(C, D). \quad (48)$$

The V notation of (15) does not simplify the duplex decomposition result.

4.1. Results adding branching and UGD

Proposition 24. Assume a (separable) utility function U that is p -additive over X that also satisfies (48). If both branching and upper gamble decomposition are satisfied, then, for all trinary partitions (C_1, C_2, C_3) ,

$$S_{\Omega}(C_1) = S_{C_1 \cup C_2}(C_1). \quad (49)$$

Taking a decision-making perspective, property (49) is intuitively unsatisfactory because changing the event that underlies a gamble should affect the weight attached to any proper subevent.

5. Conclusions

We have worked out results for the p -additive representations over the full domain of consequences, both gains and losses under the following main assumptions: permutation invariance, separability, joint-receipt decomposition, branching, and upper gamble decomposition. The results are as follows. (i) For the rational property of segregation, we develop representations when the weights over events exhibit complementarity. We also develop separately the case of no utility of gambling. In the first case, we found a utility representation of the following form: a simple weighted utility of consequences multiplied by a function of the event underlying each branch. For the case of no utility of gambling, we found in Theorem 21 the usual rank-dependent representation over the full domain X and gambles with the weights satisfying

the choice property with a p-additive representation. (ii) Replacing segregation by the non-rational property of duplex decomposition leads to the weights showing essentially no dependence upon the events (Proposition 24), a clearly unacceptable property for behavioral applications.

The overall conclusion of the paper is that the, quite defensible, property of p-additivity does not mesh at all well with other properties of gambles that, in the additive case, did lead to interesting results.

Appendix

A.1. Proposition 5

Proof. For the proof, we drop the superscript ⁽¹⁾ from U .

1. For any x , define

$$x(1) = x, x(m) = x(m - 1) \oplus x. \tag{50}$$

By induction, $U(x(m)) = mU(x)$, and so the range is unbounded.

In (13) we may, with no loss of generality, select $\delta = \pm 1$.

2. Select $\delta = 1$. For $x > e$, we have from (13) that $1 + U(x \oplus x) = [1 + U(x)]^2$. Thus

$$U(x \oplus x) = 2U(x) + U(x)^2 > 2U(x).$$

More generally, referring to (50), we have $U(x(m)) > mU(x)$ for each $m \geq 2$. So U is unbounded from above.

Let y be any given consequence less than e . Because U is unbounded from above, there exists x such that $x > x \oplus y > e > y$, i.e.,

$$U(x) > U(x \oplus y) > U(e) = 0 > U(y).$$

By (13), $1 + U(x \oplus y) = [1 + U(x)][1 + U(y)]$. As $1 + U(x \oplus y) > 0$ and $1 + U(x) > 0$, we get $1 + U(y) > 0$. So $U(y) > -1$, proving that U is bounded from below by -1 .

3. Select $\delta = -1$. The argument is similar to Case 2. ■

A.2. Theorem 9

Proof. We first make a general observation on the existence of a p-additive utility function $U^{(1)}$ with a $\delta \neq 0$. It was pointed out earlier that there is no loss of generality to assume that $\delta = 1$ or $\delta = -1$. We shall show further that the existence of a $U^{(1)}$ with a $\delta = 1$ is in fact equivalent to the existence of a $U^{(1)}$ with a $\delta = -1$.

Suppose that $U^{(1)}$ is a p-additive utility function with $\delta = 1$ and range $] - 1, \infty[$. Consider

$$\tilde{U}^{(1)} := 1 - \frac{1}{1 + U^{(1)}}.$$

It is easy to check that $\tilde{U}^{(1)}$ is a p-additive utility function with $\delta = -1$ and range $] - \infty, 1[$. Conversely, if $\tilde{U}^{(1)}$ is a p-additive utility function with $\delta = -1$ and range $] - \infty, 1[$, then

$$U^{(1)} := \frac{1}{1 - \tilde{U}^{(1)}} - 1$$

is a p-additive utility function with $\delta = 1$ and range $] - 1, \infty[$.

(a). By assumption, and by the above general observation, we may fix a p-additive $U^{(1)}$ having $\delta = -1$ and so $U^{(1)}$ on gains is onto $[0, 1[$. Also, $U^{(2)}$ on gains is onto $[0, 1[$, and so, by Theorem 4.4.6 of Luce (2000), which rests upon the result in Aczél et al. (1999), there is a $\beta > 0$ such that

$$U_+ := (U^{(2)})^\beta \tag{51}$$

over gains is p-additive with $\delta = -1$ and (U_+, S_{CUD}) , where

$$S_{CUD} := \left(S_{CUD}^{(2)} \right)^\beta, \tag{52}$$

is separable over $KE(x, C; e, D)$, $x > e$.

Thus, over gains, $x > e, y > e$, we have

$$\begin{aligned} U_+(x \oplus y) &= U_+(x) + U_+(y) - U_+(x)U_+(y), \\ \Leftrightarrow V_+(x \oplus y) &= V_+(x)V_+(y), \end{aligned} \tag{53}$$

where $V_+ := 1 - U_+$ is defined over all gains. From the underlying group structure, for any $x \in X$ which is a loss, $x^{-1} \in X$ is a gain. We extend V_+ on gains to V on all of X by

$$V(x) = \begin{cases} V_+(x), & \forall x > e \\ 1, & x \sim e \\ 1/V_+(x^{-1}), & \forall x < e. \end{cases} \tag{54}$$

It is straightforward to check that V indeed extends (53) to a multiplicative function on X :

$$V(x \oplus y) = V(x)V(y), \quad \forall x, y \in X. \tag{55}$$

The range of V is $]0, \infty[$. Let U on all of X be defined by

$$U = 1 - V. \tag{56}$$

It is an extension of U_+ and so (U, S_{CUD}) is separable over gains. The range of U is $] - \infty, 1[$, and (55) is equivalent to the p-additivity of U , with $\delta = -1$. The pair U and V extend (53) to X . It is not difficult to show that they are the unique extensions. Also, given that $U^{(2)}$ is a utility function over X , and using (51) and (54)–(56), it is not difficult to show that U is a utility function over X .

(b). Consider a dual ordering \succsim' which is the transpose of \succsim , i.e., $x \succsim' y$ if and only if $y \succsim x$. If a utility function $U^{(1)}$ is p-additive for the structure \succsim with δ , then $-U^{(1)}$ is a corresponding utility function for \succsim' with $-\delta$. If $(U^{(2)}, S_\Omega)$ is a separable representation over losses for \succsim , then $(-U^{(2)}, S_\Omega)$ is a separable representation over gains for \succsim' . If joint-receipt decomposition holds for \succsim losses, then joint-receipt decomposition holds for \succsim' gains. With this correspondence, the (b) statement, which is for \succsim , is the (a) statement for \succsim' .

(c) **Part 1.** Suppose that there exists a separable representation $(U^{(2)}, S_\Omega)$ over all of X in which the range of $U^{(2)}$ is $] - \infty, 1[$. Fixing such a separable representation, and, by the leading general observation, we may fix a p-additive utility function $U^{(1)}$ having $\delta = -1$. Then $U^{(1)}$ is onto $] - \infty, 1[$. So, $U^{(1)}$ on gains is onto $[0, 1[$. Also, $U^{(2)}$ on gains is onto $[0, 1[$. Then, by (a), there is a utility function U that is p-additive over all of X with $\delta = -1$, and is (U, S_{CUD}) separable over gains, i.e., for unitary gambles with gains,

$$U(KE(x, C; e, D)) = U(x)S_{CUD}(C) \quad (x \succ e). \tag{57}$$

So the remaining issue is what happens to the unitary gambles with losses.

Let $(x, C; e, D)$, $x < e$, be given. We shall proceed to show that

$$U(KE(x, C; e, D)) = U(x)S_{CUD}(C). \tag{58}$$

By joint-receipt decomposition over X ,

$$KE(x \oplus x^{-1}, C; e, D) = KE(x, C; e, D) \oplus KE(x^{-1}, C'; e, D') \tag{59}$$

for some C' and D' , with $C \cup D = C' \cup D' = \Omega$. Observing $KE(x \oplus x^{-1}, C; e, D) = e$ and applying multiplicative $V = 1 - U$, (56), to the above, we get

$$1 = V(KE(x, C; e, D))V(KE(x^{-1}, C'; e, D')).$$

That is,

$$1 = [1 - U(KE(x, C; e, D))][1 - U(KE(x^{-1}, C'; e, D'))]. \tag{60}$$

As x^{-1} is a gain, so is $KE(x^{-1}, C'; e, D')$, and therefore, by the obtained properties on gains, we have

$$U(KE(x^{-1}, C'; e, D')) = U(x^{-1})S_{\Omega}(C'), \tag{61}$$

$$KE(x^{-1} \oplus y, C'; e, D') = KE(x^{-1}, C'; e, D') \oplus KE(y, C''; e, D''), \tag{62}$$

$$S_{\Omega}(C'') = S_{\Omega}(C') \frac{1 - U(x^{-1})}{1 - U(x^{-1})S_{\Omega}(C')} \tag{63}$$

where the first equation comes from (57), the second holds for some (C'', D'') , $C'' \cup D'' = \Omega$, and for all $y \in X$ and is due to joint-receipt decomposition over X , whereas the third relation is known (see Luce (2000, p. 170), penultimate line). Comparing the middle equation at $y = x$ with (59) using the symmetry and strict monotonicity of \oplus , we get

$$KE(x, C; e, D) = KE(x, C''; e, D'').$$

Applying $U^{(2)}$ to both sides and using separability, we get first $U^{(2)}(x)S_{\Omega}^{(2)}(C) = U^{(2)}(x)S_{\Omega}^{(2)}(C'')$. Canceling the common factor $U^{(2)}(x)$, raising to the power β , and referring to (52), we get

$$S_{\Omega}(C) = S_{\Omega}(C''). \tag{64}$$

Putting that into (63) gives

$$S_{\Omega}(C) = S_{\Omega}(C') \frac{1 - U(x^{-1})}{1 - U(x^{-1})S_{\Omega}(C')}. \tag{65}$$

Putting (61) into (60) gives

$$\begin{aligned} U(KE(x, C; e, D)) &= 1 - \frac{1}{1 - U(x^{-1})S_{\Omega}(C')} \\ &= \frac{-U(x^{-1})S_{\Omega}(C')}{1 - U(x^{-1})S_{\Omega}(C')}. \end{aligned} \tag{66}$$

But

$$\begin{aligned} U(x^{-1}) &= 1 - V(x^{-1}) = 1 - \frac{1}{V(x)} = 1 - \frac{1}{1 - U(x)} \\ &= \frac{-U(x)}{1 - U(x)}. \end{aligned} \tag{67}$$

Using that to replace $U(x^{-1})$ in (66), we get

$$U(KE(x, C; e, D)) = \frac{U(x)S_{\Omega}(C')}{1 - U(x) + U(x)S_{\Omega}(C')}. \tag{68}$$

Putting (67) into (65) gives

$$\begin{aligned} S_{\Omega}(C) &= S_{\Omega}(C') \frac{1 - \frac{-U(x)}{1 - U(x)}}{1 - \frac{-U(x)}{1 - U(x)}S_{\Omega}(C')} \\ &= S_{\Omega}(C') \frac{1}{1 - U(x) + U(x)S_{\Omega}(C')}. \end{aligned} \tag{70}$$

Comparing that with (68) we arrive at (58), as claimed. This proves (U, S_{Ω}) separability over X .

(c) Part 2. Suppose that there exists a separable representation $(U^{(2)}, S_{\Omega})$ over all of X in which the range of $U^{(2)}$ is $] - 1, \infty[$.

Consider a dual ordering \succsim' which is the transpose of \succsim , i.e., $x \succsim' y$ if and only if $y \succsim x$. If representation $U^{(1)}$ is p-additive for the structure \succsim with δ , then $-U^{(1)}$ is a corresponding representation for \succsim' with $-\delta$. If $(U^{(2)}, S_{\Omega})$ is a separable representation over all of X for \succsim , then $(-U^{(2)}, S_{\Omega})$ is a separable representation over all of X for \succsim' . With this correspondence, the **(c) Part 2** statement, which is for \succsim , is the **(c) Part 1** statement for \succsim' . ■

A.3. Proposition 14

Proof. Part 1. Assume a p-additive gambling structure over gains. In particular, there exists a p-additive $U^{(1)}$ with $\delta = -1$ and a separable representation $(U^{(2)}, S_{\Omega}^{(2)})$ over gains. Then, by part **(a)** of the proof of Theorem 9, there is a U that is p-additive with $\delta = -1$ over all of X and S_{Ω} such that (U, S_{Ω}) is a separable representation over gains.

Using p-additivity with $\delta = -1$ and the decomposition into kernel equivalents and elements of chance, and then segregation, gives

$$\begin{aligned} 1 - U(x, C; y, D) &= [1 - U(KE(x, C; y, D))][1 - H(C, D)] \\ &= [1 - U(KE(x, C; y, D))]G(C, D) \\ &= [1 - U(KE(x \oplus y^{-1}, C; e, D) \oplus y)]G(C, D). \end{aligned} \tag{71}$$

By p-additivity with $\delta = -1$ and the definition of the inverse y^{-1} , it is a routine calculation to show that

$$U(x \oplus y^{-1}) = \frac{U(x) - U(y)}{1 - U(y)}.$$

Now, since $x \succ y$, we have $x \oplus y^{-1} \succ e$, and so, using separability over gains,

$$\begin{aligned} U(KE(x \oplus y^{-1}, C; e, D) \oplus y) &= U(KE(x \oplus y^{-1}, C; e, D)) [1 - U(y)] + U(y) \\ &= U((x \oplus y^{-1})S_{C \cup D}(C)) [1 - U(y)] + U(y) \\ &= \frac{U(x) - U(y)}{1 - U(y)} S_{C \cup D}(C) [1 - U(y)] + U(y) \\ &= U(x)S_{C \cup D}(C) + U(y) [1 - S_{C \cup D}(C)], \end{aligned}$$

which, with (71), gives (28).

Part 2. Assume a p-additive gambling structure over losses. In particular, there exists a p-additive $U^{(1)}$ with $\delta = 1$ over losses and a separable representation $(U^{(2)}, S_{\Omega}^{(2)})$ over losses. Then, by part **(b)** of the proof of Theorem 9, there is a U that is p-additive with $\delta = 1$ over all of X and S_{Ω} such that (U, S_{Ω}) is a separable representation over losses. Then, for $x \succ y$, we have by segregation that

$$\begin{aligned} (x, C; y, D) &\sim (x \oplus y^{-1}, C; e, D) \oplus y \oplus (x^{-1} \oplus x) \\ &\sim \left((y \oplus x^{-1})^{-1}, C; e, D \right) \oplus (y \oplus x^{-1}) \oplus x \\ &\sim (e, C; y \oplus x^{-1}, D) \oplus x. \end{aligned}$$

Using this equivalence, separability over losses for $e \succ y \oplus x^{-1}$, and with $\delta = -1$ replaced with $\delta = 1$, the proof of (28) parallels that in **Part 1**. ■

A.4. Corollary 16

Proof. By Theorem 9 for a p-additive gambling structure over X , we have (U, S_{Ω}) , where U is p-additive with $\delta = \pm 1$ over all of X and the representation is separable over all of X . In particular, for $y \lesssim e$,

$$U(KE(e, C; y, D)) = U(y)S_{C \cup D}(D). \tag{72}$$

A p-additive gambling structure over X is clearly a p-additive structure over gains, so, by Proposition 14, (28) holds and (30) yields, in particular, for $y \lesssim e$, that

$$U(KE(e, C; y, D)) = U(y)[1 - S_{C \cup D}(C)].$$

Comparing it with (72) gives $S_{C \cup D}(D) = 1 - S_{C \cup D}(C)$, i.e., complementarity follows. With complementarity, (28) becomes (33). ■

A.5. Two lemmas

Lemma 25. Assume a p -additive gambling structure over gains (over losses) and segregation.

1. Suppose that branching, (23), holds for $n = 3$. If

$$(x_1, C_1; x_2, C_2) \succsim x_3, \tag{73}$$

then

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\ &= 1 + S_{\Omega}(C_1 \cup C_2) (G(C_1, C_2) - 1) \\ & \quad + \delta U(x_1)S_{C_1 \cup C_2}(C_1)S_{\Omega}(C_1 \cup C_2)G(C_1, C_2) \\ & \quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)]S_{\Omega}(C_1 \cup C_2)G(C_1, C_2) \\ & \quad + \delta U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)]. \end{aligned} \tag{74}$$

If

$$x_3 \succ (x_1, C_1; x_2, C_2), \tag{75}$$

then

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\ &= 1 + [1 - S_{\Omega}(C_3)][G(C_1, C_2) - 1] \\ & \quad + \delta U(x_1)S_{C_1 \cup C_2}(C_1)G(C_1, C_2)[1 - S_{\Omega}(C_3)] \\ & \quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)]G(C_1, C_2)[1 - S_{\Omega}(C_3)] \\ & \quad + \delta U(x_3)S_{\Omega}(C_3). \end{aligned} \tag{76}$$

2. Suppose that UGD, (24), holds for $n = 3$. If

$$x_1 \succsim (x_2, C_2; x_3, C_3) \tag{77}$$

holds, then

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\ &= 1 + [1 - S_{\Omega}(C_1)][G(C_2, C_3) - 1] \\ & \quad + \delta U(x_1)S_{\Omega}(C_1) \\ & \quad + \delta U(x_2)S_{C_2 \cup C_3}(C_2) [1 - S_{\Omega}(C_1)]G(C_2, C_3) \\ & \quad + \delta U(x_3) [1 - S_{C_2 \cup C_3}(C_2)] [1 - S_{\Omega}(C_1)]G(C_2, C_3). \end{aligned} \tag{78}$$

If

$$(x_2, C_2; x_3, C_3) \succ x_1 \tag{79}$$

holds, then

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\ &= 1 + S_{\Omega}(C_2 \cup C_3) [G(C_2, C_3) - 1] \\ & \quad + \delta U(x_1)[1 - S_{\Omega}(C_2 \cup C_3)] \\ & \quad + \delta U(x_2)S_{C_2 \cup C_3}(C_2)S_{\Omega}(C_2 \cup C_3)G(C_2, C_3) \\ & \quad + \delta U(x_3)[1 - S_{C_2 \cup C_3}(C_2)]S_{\Omega}(C_2 \cup C_3)G(C_2, C_3). \end{aligned} \tag{80}$$

Proof. 1. Using branching for $n = 3$ and (28), we get, for all trinary gambles satisfying (73),

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\ &= \frac{1 + \delta U((x_1, C_1; x_2, C_2), C_1 \cup C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\ &= 1 + \delta U(x_1, C_1; x_2, C_2)S_{\Omega}(C_1 \cup C_2) \\ & \quad + \delta U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)] \\ &= 1 + \{ [1 + \delta U(x_1)S_{C_1 \cup C_2}(C_1) + \delta U(x_2) \\ & \quad \times [1 - S_{C_1 \cup C_2}(C_1)]]G(C_1, C_2) - 1 \} \end{aligned}$$

$$\begin{aligned} & \times S_{\Omega}(C_1 \cup C_2) + \delta U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)] \\ &= 1 + S_{\Omega}(C_1 \cup C_2) [G(C_1, C_2) - 1] \\ & \quad + \delta U(x_1)S_{C_1 \cup C_2}(C_1)G(C_1, C_2)S_{\Omega}(C_1 \cup C_2) \\ & \quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)]G(C_1, C_2)S_{\Omega}(C_1 \cup C_2) \\ & \quad + \delta U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)], \end{aligned}$$

which proves (74).

In the case of (75), again with (28), we have

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\ &= \frac{1 + \delta U(x_3, C_3; (x_1, C_1; x_2, C_2), C_1 \cup C_2)}{G(C_3, C_1 \cup C_2)} \\ &= 1 + \delta U(x_1, C_1; x_2, C_2)[1 - S_{\Omega}(C_3)] + \delta U(x_3)S_{\Omega}(C_3) \\ &= 1 + \{ [1 + \delta U(x_1)S_{C_1 \cup C_2}(C_1) + \delta U(x_2) \\ & \quad \times [1 - S_{C_1 \cup C_2}(C_1)]]G(C_1, C_2) - 1 \} \\ & \quad \times [1 - S_{\Omega}(C_3)] + \delta U(x_3)S_{\Omega}(C_3) \\ &= 1 + [1 - S_{\Omega}(C_3)][G(C_1, C_2) - 1] \\ & \quad + \delta U(x_1)S_{C_1 \cup C_2}(C_1)G(C_1, C_2)[1 - S_{\Omega}(C_3)] \\ & \quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)]G(C_1, C_2)[1 - S_{\Omega}(C_3)] \\ & \quad + \delta U(x_3)S_{\Omega}(C_3), \end{aligned}$$

which proves (76).

2. Now, assuming UGD, (24), for $n = 3$ and using (28) and the assumption (77), we have

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\ &= \frac{1 + \delta U(x_1, C_1; (x_2, C_2; x_3, C_3), C_2 \cup C_3)}{G(C_1, C_2 \cup C_3)} \\ &= 1 + \delta U(x_1)S_{\Omega}(C_1) + \delta U(x_2, C_2; x_3, C_3) [1 - S_{\Omega}(C_1)] \\ &= 1 + \delta U(x_1)S_{\Omega}(C_1) \\ & \quad + \{ [1 + \delta U(x_2)S_{C_2 \cup C_3}(C_2) + \delta U(x_3) \\ & \quad \times [1 - S_{C_2 \cup C_3}(C_2)]]G(C_2, C_3) - 1 \} [1 - S_{\Omega}(C_1)] \\ &= 1 + [1 - S_{\Omega}(C_1)][G(C_2, C_3) - 1] + \delta U(x_1)S_{\Omega}(C_1) \\ & \quad + \delta U(x_2)S_{C_2 \cup C_3}(C_2) [1 - S_{\Omega}(C_1)]G(C_2, C_3) \\ & \quad + \delta U(x_3)[1 - S_{C_2 \cup C_3}(C_2)] [1 - S_{\Omega}(C_1)]G(C_2, C_3), \end{aligned}$$

which proves (78).

Assuming (79), we get

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1, C_2 \cup C_3)} \\ &= \frac{1 + \delta U((x_2, C_2; x_3, C_3), C_2 \cup C_3; x_1, C_1)}{G(C_2 \cup C_3, C_1)} \\ &= 1 + \delta U(x_1)[1 - S_{\Omega}(C_2 \cup C_3)] \\ & \quad + \delta U(x_2, C_2; x_3, C_3)S_{\Omega}(C_2 \cup C_3) \\ &= 1 + \delta U(x_1)[1 - S_{\Omega}(C_2 \cup C_3)] \\ & \quad + \{ [1 + \delta U(x_2)S_{C_2 \cup C_3}(C_2) + \delta U(x_3) \\ & \quad \times [1 - S_{C_2 \cup C_3}(C_2)]]G(C_2, C_3) - 1 \} S_{\Omega}(C_2 \cup C_3) \\ &= 1 + S_{\Omega}(C_2 \cup C_3) [G(C_2, C_3) - 1] + \delta U(x_1)[1 - S_{\Omega}(C_2 \cup C_3)] \\ & \quad + \delta U(x_2)S_{C_2 \cup C_3}(C_2)S_{\Omega}(C_2 \cup C_3)G(C_2, C_3) \\ & \quad + \delta U(x_3)[1 - S_{C_2 \cup C_3}(C_2)]S_{\Omega}(C_2 \cup C_3)G(C_2, C_3), \end{aligned}$$

which proves (80). ■

Lemma 26. Assume a p -additive gambling structure over gains (over losses) and segregation. Suppose that branching, (23), and UGD, (24),

holds for $n = 3$. Then the following relations necessarily hold:

$$\frac{S_{C_1 \cup C_2}(C_1)S_{\Omega}(C_1 \cup C_2)G(C_1, C_2)}{G(C_1, C_2 \cup C_3)} = \frac{S_{\Omega}(C_1)}{G(C_1 \cup C_2, C_3)} \tag{81}$$

$$\frac{(1 - S_{C_1 \cup C_2}(C_1)) S_{\Omega}(C_1 \cup C_2)G(C_1, C_2)}{G(C_1, C_2 \cup C_3)} = \frac{S_{C_2 \cup C_3}(C_2) (1 - S_{\Omega}(C_1)) G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{82}$$

$$\frac{1 - S_{\Omega}(C_1 \cup C_2)}{G(C_1, C_2 \cup C_3)} = \frac{(1 - S_{C_2 \cup C_3}(C_2)) (1 - S_{\Omega}(C_1)) G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{83}$$

$$\frac{1 + S_{\Omega}(C_1 \cup C_2) [G(C_1, C_2) - 1]}{G(C_1, C_2 \cup C_3)} = \frac{1 + [1 - S_{\Omega}(C_1)] [G(C_2, C_3) - 1]}{G(C_1 \cup C_2, C_3)} \tag{84}$$

for all ordered partitions (C_1, C_2, C_3) , where $\Omega = C_1 \cup C_2 \cup C_3$.

Proof. Let the ordered partition (C_1, C_2, C_3) be arbitrarily given.

1. We shall first show that there exist ranked consequences $y_1 \succ y_2 \succ y_3$ such that the gamble $(y_1, C_1; y_2, C_2; y_3, C_3)$ satisfies

$$(y_1, C_1; y_2, C_2) \succ y_3 \tag{85}$$

and

$$y_1 \succ (y_2, C_2; y_3, C_3). \tag{86}$$

Let (C_1, C_2, C_3) be given. There are two cases to consider.

Case 1. Suppose that $\delta > 0$. Using (29), for ranked $y_1 \succ y_2 \succ y_3$, (85) is equivalent to

$$\begin{aligned} & [V(y_1)S_{C_1 \cup C_2}(C_1) + V(y_2) (1 - S_{C_1 \cup C_2}(C_1))] \\ & \times G(C_1, C_2) > V(y_3) \end{aligned} \tag{87}$$

and (86) is equivalent to

$$\begin{aligned} & V(y_1) \\ & > [V(y_2)S_{C_2 \cup C_3}(C_2) + V(y_3) (1 - S_{C_2 \cup C_3}(C_2))] G(C_2, C_3). \end{aligned} \tag{88}$$

We may arbitrarily fix $y_2 \succ y_3$. We see that inequalities (87) and (88) hold for $V(y_1)$ sufficiently large. We may choose y_1 accordingly.

Case 2. Suppose that $\delta < 0$. For ranked $y_1 \succ y_2 \succ y_3$ and the fact that V is order reversing, (85) is equivalent to

$$\begin{aligned} & [V(y_1)S_{C_1 \cup C_2}(C_1) + V(y_2) (1 - S_{C_1 \cup C_2}(C_1))] G(C_1, C_2) \\ & < V(y_3), \end{aligned} \tag{89}$$

and (86) is equivalent to

$$\begin{aligned} & V(y_1) \\ & < [V(y_2)S_{C_2 \cup C_3}(C_2) + V(y_3) (1 - S_{C_2 \cup C_3}(C_2))] G(C_2, C_3). \end{aligned} \tag{90}$$

We may first arbitrarily fix y_3 . Because V is order reversing with range $]0, \infty[$, we may then choose y_1 and y_2 with $y_1 \succ y_2 \succ y_3$, and so $V(y_1) < V(y_2) < V(y_3)$, with $V(y_1)$ and $V(y_2)$ sufficiently small so that (89) holds and, in turn, $V(y_1)$ sufficiently smaller than $V(y_2)$ that (90) holds.

2. For ranked consequences $x_1 \succsim x_2 \succsim x_3$ with $U(x_i)$ sufficiently near $U(y_i)$, (73) and (77) hold. By Lemma 25, we have both (74) and (78). Then, the coefficients of the $U(x_i)$ must be equal in the two equations, leading to (81)–(84). ■

A.6. Lemma 17

Proof. Because $G(C_1, C_2) = V(e, C_1; e, C_2)$ and $\delta \neq 0$, the assumption that $G(C_1, C_2) \neq 1$ is equivalent to $(e, C_1; e, C_2) \approx e$. So either $(e, C_1; e, C_2) < e$ or $(e, C_1; e, C_2) > e$.

1. Suppose that

$$(e, C_1; e, C_2) < e.$$

By choosing x_1, x_2 and x_3 , with $U(x_1), U(x_2)$ and $U(x_3)$ sufficiently near $U(e) = 0$, we see that there exist consequences x_1, x_2 and x_3 , with $x_1 \succ x_2 \succ x_3$, such that

$$(x_1, C_1; x_2, C_2) < x_3$$

holds. This is (75), so (76) holds. For such chosen $x_1 \succ x_2 \succ x_3$, we consider two cases:

Case 1.1. Suppose that some gamble satisfies $x_1 \succ (x_2, C_2; x_3, C_3)$. This is (77) and so (78) holds. Thus, both (76) and (78) hold. Equating the coefficients we obtain the following relations:

$$\frac{S_{C_1 \cup C_2}(C_1)G(C_1, C_2)[1 - S_{\Omega}(C_3)]}{G(C_1, C_2 \cup C_3)} = \frac{S_{\Omega}(C_1)}{G(C_1 \cup C_2, C_3)} \tag{91}$$

$$\frac{[1 - S_{C_1 \cup C_2}(C_1)]G(C_1, C_2)[1 - S_{\Omega}(C_3)]}{G(C_1, C_2 \cup C_3)} \tag{92}$$

$$= \frac{S_{C_2 \cup C_3}(C_2) [1 - S_{\Omega}(C_1)] G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{93}$$

$$\frac{S_{\Omega}(C_3)}{G(C_1, C_2 \cup C_3)} = \frac{[1 - S_{C_2 \cup C_3}(C_2)] [1 - S_{\Omega}(C_1)] G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{94}$$

$$\begin{aligned} & \frac{1 + [1 - S_{\Omega}(C_3)] [G(C_1, C_2) - 1]}{G(C_1, C_2 \cup C_3)} \\ & = \frac{1 + [1 - S_{\Omega}(C_1)] [G(C_2, C_3) - 1]}{G(C_1 \cup C_2, C_3)}. \end{aligned} \tag{95}$$

Equating (91) to (81) from Lemma 26 yields (34):

$$S_{\Omega}(C_1 \cup C_2) + S_{\Omega}(C_3) = 1. \tag{96}$$

Case 1.2. Suppose that some gamble satisfies $x_1 < (x_2, C_2; x_3, C_3)$. This is (79) and so (80) holds. Thus, both (76) and (80) hold. Equating the coefficients, we get

$$\frac{S_{C_1 \cup C_2}(C_1)G(C_1, C_2)[1 - S_{\Omega}(C_3)]}{G(C_1, C_2 \cup C_3)} = \frac{1 - S_{\Omega}(C_2 \cup C_3)}{G(C_1 \cup C_2, C_3)} \tag{97}$$

$$\begin{aligned} & \frac{[1 - S_{C_1 \cup C_2}(C_1)]G(C_1, C_2)[1 - S_{\Omega}(C_3)]}{G(C_1, C_2 \cup C_3)} \\ & = \frac{S_{C_2 \cup C_3}(C_2)S_{\Omega}(C_2 \cup C_3)G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \end{aligned} \tag{98}$$

$$\frac{S_{\Omega}(C_3)}{G(C_1, C_2 \cup C_3)} = \frac{[1 - S_{C_2 \cup C_3}(C_2)]S_{\Omega}(C_2 \cup C_3)G(C_2, C_3)}{G(C_1 \cup C_2, C_3)} \tag{99}$$

$$\begin{aligned} & \frac{1 + [1 - S_{\Omega}(C_3)] [G(C_1, C_2) - 1]}{G(C_1, C_2 \cup C_3)} \\ & = \frac{1 + S_{\Omega}(C_2 \cup C_3) [G(C_2, C_3) - 1]}{G(C_1 \cup C_2, C_3)}. \end{aligned} \tag{100}$$

Comparing these with the basic relations of Lemma 26 leads to

$$\frac{1 - S_{\Omega}(C_3)}{S_{\Omega}(C_1 \cup C_2)} = \frac{1 - S_{\Omega}(C_2 \cup C_3)}{S_{\Omega}(C_1)} \tag{101}$$

$$\frac{1 - S_{\Omega}(C_3)}{S_{\Omega}(C_1 \cup C_2)} = \frac{S_{\Omega}(C_2 \cup C_3)}{1 - S_{\Omega}(C_1)} \tag{102}$$

$$\frac{S_{\Omega}(C_3)}{1 - S_{\Omega}(C_1 \cup C_2)} = \frac{S_{\Omega}(C_2 \cup C_3)}{1 - S_{\Omega}(C_1)} \tag{103}$$

Multiplying (31) by $G(C_1, C_2)$ yields

$$S_{C_1 \cup C_2}^*(C_1) + S_{C_1 \cup C_2}^*(C_2) = G(C_1, C_2). \tag{123}$$

Using (123), (122) is equivalent to

$$[S_{C_1 \cup C_2}^*(C_1) + S_{C_1 \cup C_2}^*(C_2)]S_{\Omega}^*(C_1 \cup C_2) + S_{\Omega}^*(C_3) \\ = [S_{C_2 \cup C_3}^*(C_2) + S_{C_2 \cup C_3}^*(C_3)]S_{\Omega}^*(C_2 \cup C_3) + S_{\Omega}^*(C_1). \tag{124}$$

Using (121) on each side of (124) yields the symmetric expression

$$\sum_{i=1}^3 S_{\Omega}^*(C_i),$$

and so (121) implies (124). A parallel argument based on the event pair (C_2, C_3) shows that (119) is implied by (121) and so can be dismissed. We are now left with (121), whose solution⁴ is given by

$$S_{C_1 \cup C_2}^*(C_1) = \frac{\lambda(C_1)}{\lambda(C_1 \cup C_2)} \tag{125}$$

for some $\lambda : \mathfrak{B}^* \rightarrow]0, \infty[$. Putting (125) into (123), we get

$$G(C_1, C_2) = \frac{\lambda(C_1) + \lambda(C_2)}{\lambda(C_1 \cup C_2)}, \tag{126}$$

which gives (36). Using (125) and (126) in (120), for any disjoint C and D ,

$$S_{C \cup D}(C) = \frac{S_{C \cup D}^*(C)}{G(C, D)} = \frac{\lambda(C)}{\lambda(C) + \lambda(D)},$$

which gives (35). Clearly, (125) then becomes

$$S_{C \cup D}(C)G(C, D) = \frac{\lambda(C)}{\lambda(C \cup D)},$$

which for $C \cup D = \Omega$ gives (38). Putting the resulting forms of S and G given by (35) and (36) respectively into (114) and (115), and using (38), we obtain (37) for $n = 2, 3$.

Part 2 is a straightforward induction. Part 3 is an observation based on the fact that the right-hand side of (37), for general n , is symmetric. Finally, Corollary 16 holds under the assumptions of Part 4. Therefore, (33), which is (114), holds and the proof of the representation is as in Part 1. ■

A.9. Theorem 21

Proof. The assumptions include those of Lemma 25; hence (81)–(84) hold.

1. The assumption that $G(C_1, C_2) = 1$ for all 3-partitions (C_1, C_2, C_3) means that

$$G(C, D) = 1, \quad C \cup D \text{ not maximal.} \tag{127}$$

If the algebra has no maximal event, then $G = 1$ follows, which is the ordinary default meaning of “no utility of gambling”. So, with (127), (84) is reduced to

$$G(C_1, C_2 \cup C_3) = G(C_1 \cup C_2, C_3). \tag{128}$$

One can show that (128) leads to

$$G(C, D) = A, \quad C \cup D \text{ is maximal} \tag{129}$$

for some constant $A > 0$. Together (127) and (129) are the full meaning of the terminology “no utility of gambling” in the subsequent discussion, albeit A need not be 1.

⁴ Luce (1957) gave it for the finitely additive case. Luce et al. (2008a, 2008b, Theorem 6) extended it to the non-additive case using a result in Ebanks, Kannappan, and Ng (1988).

2. Using the results of Part 1, with (81), we have

$$S_{C_1 \cup C_2}(C_1)S_{\Omega}(C_1 \cup C_2) = S_{\Omega}(C_1),$$

which is the choice property (CP), (32). As noted earlier, the choice property has the representation (39).

With no utility of gambling and the choice property, (82) becomes

$$S_{\Omega}(C_1 \cup C_2) - S_{\Omega}(C_1) = S_{C_2 \cup C_3}(C_2) - S_{C_2 \cup C_3}(C_2)S_{\Omega}(C_1). \tag{130}$$

Substituting (39), (130) becomes

$$\frac{\mu(C_1 \cup C_2)}{\mu(C_1 \cup C_2 \cup C_3)} - \frac{\mu(C_1)}{\mu(C_1 \cup C_2 \cup C_3)} \\ = \frac{\mu(C_2)}{\mu(C_2 \cup C_3)} - \frac{\mu(C_2)}{\mu(C_2 \cup C_3)} \frac{\mu(C_1)}{\mu(C_1 \cup C_2 \cup C_3)}.$$

Rearranging,

$$\frac{\mu(C_1 \cup C_2) - \mu(C_1)}{\mu(C_2)} = \frac{\mu(C_1 \cup C_2 \cup C_3) - \mu(C_1)}{\mu(C_2 \cup C_3)}. \tag{131}$$

We next show, for each non-null C and for all non-null D and \tilde{D} which are disjoint from C , that (131) gives

$$\frac{\mu(C \cup D) - \mu(C)}{\mu(D)} = \frac{\mu(C \cup \tilde{D}) - \mu(C)}{\mu(\tilde{D})}. \tag{132}$$

To that end, we examine four cases:

Case 1. Suppose $D = \tilde{D}$. Then (132) holds trivially.

Case 2. Suppose $D \neq \tilde{D}$ and $D \subset \tilde{D}$. Then, letting $C_1 = C, C_2 = D$ and $C_3 = \tilde{D} \setminus D$ in (131), we arrive at (132).

Case 3. Suppose $D \neq \tilde{D}$ and $\tilde{D} \subset D$. The argument for (132) is similar to that of Case 2.

Case 4. Suppose that $D \not\subset \tilde{D}$ and $\tilde{D} \not\subset D$. First, let $C_1 = C, C_2 = D, C_3 = \tilde{D} \setminus D$ in (131); then

$$\frac{\mu(C \cup D) - \mu(C)}{\mu(D)} = \frac{\mu(C \cup D \cup \tilde{D}) - \mu(C)}{\mu(D \cup \tilde{D})}. \tag{133}$$

Second, setting $C_1 = C, C_2 = \tilde{D}, C_3 = D \setminus \tilde{D}$ in (131) yields

$$\frac{\mu(C \cup \tilde{D}) - \mu(C)}{\mu(\tilde{D})} = \frac{\mu(C \cup D \cup \tilde{D}) - \mu(C)}{\mu(D \cup \tilde{D})}. \tag{134}$$

By (133) and (134), (132) follows.

Thus, we have (132) in all cases.

Hence, for each given non-null C there exists a constant, denoted by $f(C)$, say, such that

$$\frac{\mu(C \cup D) - \mu(C)}{\mu(D)} = f(C) \tag{135}$$

holds for all non-null D disjoint from C . Because μ is strictly increasing, we have

$$f(C) > 0 \tag{136}$$

for all non-null non-maximal C . We rewrite (135) as

$$\mu(C \cup D) = \mu(C) + \mu(D)f(C). \tag{137}$$

By the symmetry of the left-hand side in C and D , we get

$$\mu(D) + \mu(C)f(D) = \mu(C) + \mu(D)f(C).$$

Therefore,

$$\frac{f(C) - 1}{\mu(C)} = \frac{f(D) - 1}{\mu(D)},$$

for all disjoint non-null C, D . So

$$\frac{f(C) - 1}{\mu(C)} = b$$

where b is a constant, leading to $f(C) = 1 + b\mu(C)$. In view of (136), b is restricted by (41).

Substituting f into (137) gives the p-additivity of μ which is holding for all disjoint non-null pairs C, D . Conversely, it is simple to check that μ satisfying (40) is a solution of (131). In turn, we have solved (130).

By no utility of gambling, (83) also reduces to

$$(1 - S_{C_2 \cup C_3}(C_2))(1 - S_{\Omega}(C_1)) = 1 - S_{\Omega}(C_1 \cup C_2),$$

which coincides with (130).

So, the system of (81)–(84) in the case of no utility of gambling is solved by (39) where μ is p-additive, (40).

3. Putting the above special value of G in (74), namely,

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} \\ &= 1 + S_{\Omega}(C_1 \cup C_2)(G(C_1, C_2) - 1) \\ & \quad + \delta U(x_1)S_{C_1 \cup C_2}(C_1)S_{\Omega}(C_1 \cup C_2)G(C_1, C_2) \\ & \quad + \delta U(x_2)[1 - S_{C_1 \cup C_2}(C_1)]S_{\Omega}(C_1 \cup C_2)G(C_1, C_2) \\ & \quad + \delta U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)] \end{aligned}$$

and using the properties of S , we get the representation of the no utility of gambling case:

$$\begin{aligned} & \frac{1 + \delta U(x_1, C_1; x_2, C_2; x_3, C_3)}{G(C_1 \cup C_2, C_3)} = 1 + \delta U(x_1)S_{\Omega}(C_1) \\ & \quad + \delta U(x_2)[S_{\Omega}(C_1 \cup C_2) - S_{\Omega}(C_1)] \\ & \quad + \delta U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)], \end{aligned} \tag{138}$$

where $G(C_1 \cup C_2, C_3) = 1$ except for the case $C_1 \cup C_2 \cup C_3$ is maximal, in which case $G(C_1 \cup C_2, C_3) = A$.

We now restrict our attention to the case $A = 1$ only. Then (138) simplifies to

$$\begin{aligned} U(x_1, C_1; x_2, C_2; x_3, C_3) &= U(x_1)S_{\Omega}(C_1) \\ & \quad + U(x_2)[S_{\Omega}(C_1 \cup C_2) - S_{\Omega}(C_1)] \\ & \quad + U(x_3)[1 - S_{\Omega}(C_1 \cup C_2)]. \end{aligned} \tag{139}$$

Now we develop the representation for gambles of length 4 using branching, UGD and the choice property (CP), and then develop the representation for higher lengths using induction.

Using branching and CP, we get

$$\begin{aligned} U(x_1, C_1; x_2, C_2; x_3, C_3; x_4, C_4) &= U((x_1, C_1; x_2, C_2; x_3, C_3); C_1 \cup C_2 \cup C_3; x_4, C_4) \\ &= U(x_1)S_{C_1 \cup C_2 \cup C_3}(C_1)S_{\Omega}(C_1 \cup C_2 \cup C_3) \\ & \quad + U(x_2)[S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2) - S_{C_1 \cup C_2 \cup C_3}(C_1)] \\ & \quad \times S_{\Omega}(C_1 \cup C_2 \cup C_3) + U(x_3)[1 - S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2)] \\ & \quad \times S_{\Omega}(C_1 \cup C_2 \cup C_3) + U(x_4)[1 - S_{\Omega}(C_1 \cup C_2 \cup C_3)] \\ &= U(x_1)S_{\Omega}(C_1) + U(x_2)[S_{\Omega}(C_1 \cup C_2) - S_{\Omega}(C_1)] \\ & \quad + U(x_3)[S_{\Omega}(C_1 \cup C_2 \cup C_3) - S_{\Omega}(C_1 \cup C_2)] \\ & \quad + U(x_4)[1 - S_{\Omega}(C_1 \cup C_2 \cup C_3)]. \end{aligned} \tag{140}$$

Using UGD and CP, we get

$$\begin{aligned} U(x_1, C_1; x_2, C_2; x_3, C_3; x_4, C_4) &= U(x_1, C_1; (x_2, C_2; x_3, C_3; x_4, C_4); C_2 \cup C_3 \cup C_4) \\ &= U(x_1)S_{\Omega}(C_1) + U(x_2)S_{C_2 \cup C_3 \cup C_4}(C_2)[1 - S_{\Omega}(C_1)] \\ & \quad + U(x_3)[S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3) - S_{C_2 \cup C_3 \cup C_4}(C_2)] \\ & \quad \times [1 - S_{\Omega}(C_1)] \\ & \quad + U(x_4)[1 - S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3)][1 - S_{\Omega}(C_1)]. \end{aligned} \tag{141}$$

Equating the above two forms yields

$$S_{\Omega}(C_1 \cup C_2) - S_{\Omega}(C_1) = S_{C_2 \cup C_3 \cup C_4}(C_2)[1 - S_{\Omega}(C_1)] \tag{142}$$

$$\begin{aligned} S_{\Omega}(C_1 \cup C_2 \cup C_3) - S_{\Omega}(C_1 \cup C_2) &= [S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3) - S_{C_2 \cup C_3 \cup C_4}(C_2)] \\ & \quad \times [1 - S_{\Omega}(C_1)] \end{aligned} \tag{143}$$

$$\begin{aligned} 1 - S_{\Omega}(C_1 \cup C_2 \cup C_3) &= [1 - S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3)] \\ & \quad \times [1 - S_{\Omega}(C_1)]. \end{aligned} \tag{144}$$

But adding these three equations yields a trivial identity, and so (144) follows from (142) and (143). Writing out (142), (143) and (130) with explicit $\Omega = C_1 \cup C_2 \cup C_3 \cup C_4$, we have the system

$$\begin{aligned} S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2) - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1) &= S_{C_2 \cup C_3 \cup C_4}(C_2)[1 - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1)] \end{aligned} \tag{145}$$

$$\begin{aligned} S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2 \cup C_3) - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2) &= [S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3) - S_{C_2 \cup C_3 \cup C_4}(C_2)] \\ & \quad \times [1 - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1)] \end{aligned} \tag{146}$$

$$\begin{aligned} S_{C_1 \cup C_2 \cup C_3}(C_1 \cup C_2) - S_{C_1 \cup C_2 \cup C_3}(C_1) &= S_{C_2 \cup C_3}(C_2)[1 - S_{C_1 \cup C_2 \cup C_3}(C_1)]. \end{aligned} \tag{147}$$

If we replace C_3 by $C_3 \cup C_4$ in (147), we get (145).

If we replace C_2 by $C_2 \cup C_3$ and C_3 by C_4 in (147), we get

$$\begin{aligned} S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1 \cup C_2 \cup C_3) - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1) &= S_{C_2 \cup C_3 \cup C_4}(C_2 \cup C_3)[1 - S_{C_1 \cup C_2 \cup C_3 \cup C_4}(C_1)]. \end{aligned}$$

Subtracting (145) from the above, we get (146). Therefore, each equation in (142)–(144) is a consequence of (130). So we have shown that when we go from $n = 3$ gambles to $n = 4$ gambles, there are no additional conditions imposed on S_{Ω} .

By simple induction, for the case $A = 1$, we can show that (139) and (140) yields the representation for general length n , (43), where S satisfies CP and (130). ■

A.10. Proposition 23

Proof. Apply the p-additive form to (46):

$$\begin{aligned} & 1 + \delta U[(x, C; y, D) \oplus (e, C'; e, D')] \\ &= 1 + \delta U[(x, C; e, D) \oplus (e, C'; y, D')] \\ &\Leftrightarrow [1 + \delta U(x, C; y, D)]G(C', D') \\ &= [1 + \delta U(x, C; e, D)][1 + \delta U(e, C'; y, D')] \\ &\Leftrightarrow [1 + \delta U(KE(x, C; y, D))]G(C, D)G(C', D') \\ &= [1 + \delta U(KE(x, C; e, D))]G(C, D) \\ & \quad \times [1 + \delta U(KE(e, C'; y, D'))]G(C', D') \\ &\Leftrightarrow 1 + \delta U(KE(x, C; y, D)) \\ &= [1 + \delta U(x)S_{C \cup D}(C)][1 + \delta U(y)S_{C' \cup D'}(D')] \\ &= [1 + \delta U(x)S_{C \cup D}(C)][1 + \delta U(y)S_{C \cup D}(D)], \end{aligned}$$

which is (47), and (48) follows immediately. ■

A.11. Proposition 24

Proof. Rewrite (48) in the form

$$\begin{aligned} U(x, C; y, D) &= U(x)S_{C \cup D}(C)G(C, D) \\ & \quad + U(y)S_{C \cup D}(D)G(C, D) \\ & \quad + \delta U(x)S_{C \cup D}(C)U(y)S_{C \cup D}(D)G(C, D) + H(C, D). \end{aligned} \tag{148}$$

Because of (6), H and thus G are symmetric functions. We shall be using the symmetry of H and G without further mention.

Recall that we defined

$$S_{CUD}^*(C) := S_{CUD}(C)G(C, D), \quad (149)$$

and note that, by definition, $\delta H = G - 1$. So (148) becomes

$$U(x, C; y, D) = U(x)S_{CUD}^*(C) + U(y)S_{CUD}^*(D) + \frac{\delta U(x)U(y)}{G(C, D)} S_{CUD}^*(C)S_{CUD}^*(D) + H(C, D). \quad (150)$$

Because of the symmetry of (150), there is no need in the following to concern ourselves with the rank order between x_3 and $(x_1, C_1; x_2, C_2)$.

Using branching and then (150), we get

$$\begin{aligned} U(x_1, C_1; x_2, C_2; x_3, C_3) &= U(x_1)S_{C_1UC_2}^*(C_1)S_{\Omega}^*(C_1 \cup C_2) \\ &+ U(x_2)S_{C_1UC_2}^*(C_2)S_{\Omega}^*(C_1 \cup C_2) \\ &+ U(x_3) \left[S_{\Omega}^*(C_3) + \frac{\delta}{G(C_1 \cup C_2, C_3)} \right. \\ &\quad \left. \times H(C_1, C_2)S_{\Omega}^*(C_1 \cup C_2)S_{\Omega}^*(C_3) \right] \\ &+ \frac{\delta U(x_1)U(x_2)}{G(C_1, C_2)} S_{C_1UC_2}^*(C_1)S_{C_1UC_2}^*(C_2)S_{\Omega}^*(C_1 \cup C_2) \\ &+ \frac{\delta U(x_1)U(x_3)}{G(C_1 \cup C_2, C_3)} S_{C_1UC_2}^*(C_1)S_{\Omega}^*(C_1 \cup C_2)S_{\Omega}^*(C_3) \\ &+ \frac{\delta U(x_2)U(x_3)}{G(C_1 \cup C_2, C_3)} S_{C_1UC_2}^*(C_2)S_{\Omega}^*(C_1 \cup C_2)S_{\Omega}^*(C_3) \\ &+ \frac{\delta^2 U(x_1)U(x_2)U(x_3)}{G(C_1, C_2)G(C_1 \cup C_2, C_3)} \\ &\quad \times S_{C_1UC_2}^*(C_1)S_{C_1UC_2}^*(C_2)S_{\Omega}^*(C_1 \cup C_2)S_{\Omega}^*(C_3) \\ &+ H(C_1, C_2)S_{\Omega}^*(C_1 \cup C_2) + H(C_1 \cup C_2, C_3). \end{aligned} \quad (151)$$

Using UGD and then (150), we get

$$\begin{aligned} U(x_1, C_1; x_2, C_2; x_3, C_3) &= U(x_1) \left[S_{\Omega}^*(C_1) + \frac{\delta}{G(C_1, C_2 \cup C_3)} H(C_2, C_3)S_{\Omega}^*(C_1)S_{\Omega}^*(C_2 \cup C_3) \right] \\ &+ U(x_2)S_{C_2UC_3}^*(C_2)S_{\Omega}^*(C_2 \cup C_3) \\ &+ U(x_3)S_{C_2UC_3}^*(C_3)S_{\Omega}^*(C_2 \cup C_3) \\ &+ \frac{\delta U(x_1)U(x_2)}{G(C_1, C_2 \cup C_3)} S_{\Omega}^*(C_1)S_{C_2UC_3}^*(C_2)S_{\Omega}^*(C_2 \cup C_3) \\ &+ \frac{\delta U(x_1)U(x_3)}{G(C_1, C_2 \cup C_3)} S_{\Omega}^*(C_1)S_{C_2UC_3}^*(C_3)S_{\Omega}^*(C_2 \cup C_3) \\ &+ \frac{\delta U(x_2)U(x_3)}{G(C_2, C_3)} S_{C_2UC_3}^*(C_2)S_{C_2UC_3}^*(C_3)S_{\Omega}^*(C_2 \cup C_3) \\ &+ \frac{\delta^2 U(x_1)U(x_2)U(x_3)}{G(C_1, C_2 \cup C_3)G(C_2, C_3)} \\ &\quad \times S_{\Omega}^*(C_1)S_{C_2UC_3}^*(C_2)S_{C_2UC_3}^*(C_3)S_{\Omega}^*(C_2 \cup C_3) \\ &+ H(C_2, C_3)S_{\Omega}^*(C_2 \cup C_3) + H(C_1, C_2 \cup C_3). \end{aligned} \quad (152)$$

Because we may vary the x_i independently, the coefficients of the trinomials in the (151) and (152) must be equal. In particular, two equations hold for all partitions (C_1, C_2, C_3) :

$$S_{C_2UC_3}^*(C_2)S_{\Omega}^*(C_2 \cup C_3) = S_{C_1UC_2}^*(C_2)S_{\Omega}^*(C_1 \cup C_2), \quad (153)$$

and

$$\begin{aligned} \frac{S_{\Omega}^*(C_1)S_{C_2UC_3}^*(C_2)S_{\Omega}^*(C_2 \cup C_3)}{G(C_1, C_2 \cup C_3)} \\ = \frac{S_{C_1UC_2}^*(C_1)S_{C_1UC_2}^*(C_2)S_{\Omega}^*(C_1 \cup C_2)}{G(C_1, C_2)}. \end{aligned} \quad (154)$$

Dividing (154) by (153) side by side, we get

$$\frac{1}{G(C_1, C_2 \cup C_3)} S_{\Omega}^*(C_1) = \frac{1}{G(C_1, C_2)} S_{C_1UC_2}^*(C_1).$$

By the definition of S^* , it is equivalent to

$$S_{\Omega}(C_1) = S_{C_1UC_2}(C_1).$$

This proves (49). ■

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