Entropy-Related Measures of the Utility of Gambling

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1 Background of Work Reported

1.1 Roles of Peter Fishburn on this topic

The first author has known Peter for a very long time, dating back some 45 years to when we met at a colloquium he gave at the University of Pennsylvania. After that our paths crossed fairly often. For example, in the early 1970s, he spent a year at the Institute for Advanced Study where Luce spent three years until the attempt to establish a program in scientific social science was abandoned for a more literary approach favored by the humanists and, surprisingly, the mathematicians then at the Institute. The second author has learnt a tremendous amount about both substantive and technical issues from Peter's work, beginning with Peter's book "Utility Theory for Decision Making" (Fishburn, 1970), which he reviewed for *Contemporary Psychology* (see Marley, 1972).

Peter's volume on interval orders (Fishburn, 1985) was a marvelous development of various ideas related to the algebra of imperfect discrimination that elaborated the first author's initial work on semiorders (Luce, 1956).

Beginning in 1988, Peter made a major contribution in his integrative book "Nonlinear Preference and Utility Theory." And in the first half of the 1990s, Fishburn and Luce collaborated on three efforts to understand better the rank-dependent generalizations of expected utility that had attracted considerable notice in the 1980s (Fishburn & Luce, 1995; Luce & Fishburn, 1991, 1995). It was here that we first came up with the so-called p-additive form for the utility of joint receipts. All of that played a major role in Luce's (2000) attempt to pull together many of the results about utility, both experimental and theoretical, of the period starting in 1979.

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And joint receipts play a key role in our attempt to incorporate a concept of the utility of gambling into the situation, which is described in this paper.

To our knowledge, Peter directly addressed the issue of the utility of gambling just once (Fishburn, 1980), where he presented the first, but very restricted, formal model of it (Sect. 1.2).

So, in sum, we have learned much from Peter and are still tilling grounds that he was, in very many ways, influential in developing in modern utility theory. This chapter pays tribute to Peter not by commenting directly on his contributions, but by summarizing some generalizations found in several articles cited below.

1.2 Utility of Gambling

The founders of "modern" utility theory, Ramsey (1931), von Neumann and Morgenstern (1947), and, less explicitly, Savage (1954, pp. 13–17) all noted that their theories could not accommodate the existence of utility of gambling (UofG) per se. For example Ramsey (1931, p. 172) contended that the method of establishing beliefs in terms of bets is "…inexact…partly because the person may have a special eagerness or reluctance to bet, because he either enjoys or dislikes excitement…The difficulty is like that of separating two different cooperating forces." This 1931 essay was actually dated 1926. Over two decades later von Neumann and Morgenstern (1947, p. 28) remarked: "Since [our axioms] secure that the necessary construction can be carried out, concepts like a 'specific utility of gambling' cannot be formulated free of contradiction on this level." Adjoined is the footnote: "This may seem to be a paradoxical assertion. But anybody who has seriously tried to axiomatize that elusive concept, will probably concur with it."

Furthermore, the sharp partition in these theories of valueless events and valued consequences is often not the case in reality. Insurance on an airplane trip represents such a separation, but not all of the events that might occur are valueless in their own right – for instance, a crash of your flight.

Most theoretical work has ruled out UofG by incorporating in some fashion a version of idempotence, namely, that attaching the same consequence x to each chance event arising from a chance "experiment" is perceived as indifferent to receiving x with certainty. Savage (1954) called such gambles "constant acts." That indifference means that no utility or disutility accrues either to the events themselves or to the execution of the experiment, as such.

Ignoring both the value of events and the utility associated with uncertainty and/or risk is a major idealization that has only rarely been questioned or addressed. Some formal models of UofG appearing in the utility literature focused on the risky cases[1], and typically involved modifications of the expected utility representation. Conlisk (1993) summarizes them from an economic perspective and Luce and Marley (2000) from a more psychological one, but with important economic

[1] Those for which each of the possible consequences of the gamble occurs with a specified probability.

influences. Fishburn (1980) gave the first formal model in which he appended a UofG term to the expected utility of a risky gamble in such a way that this term affects the choice between a pure consequence (sure-thing) and a risky gamble, but does not influence the choice between two risky gambles. He also axiomatized several possible forms for the UofG term, including the case where it is constant for all gambles. Diecidue, Schmidt and Wakker (2004) generalized Fishburn's formulation, but, in the main, they continued to assume that preferences between risky gambles agree with expected utility. Bleichrodt and Schmidt (2002) present a related model, with preferences between risky gambles again agreeing with expected utility but with different utility functions depending upon whether or not one of the alternatives is a pure consequence. Luce and Marley (2000) considered uncertain[2] gambles, with a UofG term that depends on the events, but not the consequences. In that model the UofG term can affect the choice between two gambles when they are based on different events. They also motivate, but do not axiomatize, several possible forms for the UofG term for binary gambles. Le Menestrel (2001) and Pope (1995 and earlier papers) offer process models for the utility of gambling. So far as we know, no one before our work has dealt explicitly with valued uncertain events, often because the underlying structure has been one of risk.

Meginniss (1976) seems to have been the first author to arrive at, in the context of risk, a sensible theory incorporating UofG. Until quite recently, his result appears to have been unknown, ignored, and/or forgotten by utility theorists[3], and its ability to account for many anomalies has not been widely recognized. His result is that the overall utility of a risky gamble is given by a linear weighted utility term plus an (information-theoretic) entropy (Shannon, 1948) term dependent only on the probabilities. His clever proof of the result rested on quite special, unaxiomatized, representational assumptions. Unaware of Meginniss' article, Yang and Qiu (2005) proposed a closely related nonaxiomatized representation involving Shannon's entropy, explored some of its properties, and applied it to some of the well known anomalies. We summarize similar explanations of several such anomalies in Sect. 5.

Ng, Luce, and Marley (2008a) generalized Meginniss' approach in several ways, fundamentally following his general ideas, whereas Luce, Ng, Marley and Aczél (2008 a,b) and Ng, Luce, and Marley (2008b) take an axiomatic approach. Specifically, Luce et al. (2008a), summarized in Sect. 2, treat uncertain gambles and Luce et al. (2008 b), summarized in Sect. 3, extend those results to risky gambles. Ng et al. (2008b), summarized in Sect. 4, further extend the results to obtain representations of the UofG term that involve a weighted value function over events, plus an entropy term involving the same weights. The resulting representations include the "rational" expected utility (EU) and subjective expected utility (SEU) representations as very special cases, with no UofG term.

Section 5 applies a special case of these representations to several sets of data. And Sect. 6 summarizes the results reported in this paper and states four major open problems.

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[2] Those where the events have no readily agreed upon probabilities.

[3] It was brought to our attention in 2004 by our collaborator János Aczél.

1.3 Formulation of Gambles and Utility Representations

We begin with the general concept of uncertain gambles, then extend the results to risky gambles and gambles involving valued events. Because we anticipate that most of our readers are already familiar with standard notations in this domain and need no more than reminders, we are not fully formal – that can be found in Luce et al. (2008 a,b).

The set of pure consequences – no risk or uncertainty – is denoted X. Included in X is a singular element e, called *no change from the status quo*, whose special properties are given below. The set of pure consequences is assumed to be closed under the binary[4] commutative and associative operation of joint receipt, $\oplus$. We postulate a (preference) ordering, $\succsim$, over $\langle X, \oplus \rangle$ that is assumed to be a weak order that is strictly increasing in each argument of $\oplus$. As usual, strict preference is denoted by $\succ$ and indifference by $\sim$. The latter is an equivalence relation. We assume that e is an identity of $\oplus$: for all $x \in X$, $x \oplus e \sim e \oplus x \sim x$. Moreover, X is assumed to satisfy the structural restriction of solvability, namely, for each x, y, there exists z such that $x \sim y \oplus z$. We define $x \ominus y := z$.

Assume that the axioms of the theory of extensive measurement are satisfied (Krantz et al., 1971, Chap. 3) leading to a mapping $U : X \to \mathbb{R}$ such that:

$$x \succsim y \Leftrightarrow U(x) \geq U(y), \tag{1}$$

$$U(x \oplus y) = U(x) + U(y). \tag{2}$$

It follows immediately that $U(e) = 0$ and that $U(x \ominus y) = U(x) - U(y)$.

Let Ω denote a state space of the chance outcomes from some chance "experiment." Let $(C_1, \ldots, C_i, \ldots, C_n)$ denote a typical nontrivial, finite partition of Ω, i.e., $C_i \cap C_j = \varnothing$ if $i \neq j$, $C_i \neq \varnothing$, $\cup_{i=1}^{n} C_i = \Omega$. Unlike Savage (1954) and many subsequent treatments, we do not assume a single universal state space; rather we produce a more versatile model in which Ω is a variable, as is typical of both concrete examples of gambles, e.g., alternate modes of travel from A to B, and equally well of the experimental realizations of gambles in various experiments, e.g., spin of a wheel, withdrawal of a colored ball from a randomized urn, etc. One can, and airlines do it all the time, subtract and/or add alternatives to an existing set of flight alternatives. The versatility is essential to our approach using gamble decompositions.

An *uncertain alternative*, often called a *gamble* but with a far broader scope than ordinary usage, is defined inductively: A first-order one is a mapping $g_{[n]}$ from such a finite partition into X, a second-order one is a mapping to the union of X and first-order ones, which are not of first order, etc. We use only these two levels. The structure $\langle X, \oplus, \succsim \rangle$ can be extended to include all gambles and their joint receipts, $\mathcal{G}$, and we assume that the extended preference order continues to be a weak order, still called $\succsim$. And the additive representation over $\oplus$ also extends in the obvious way. With no loss of generality, we choose the indices so that the consequences of the gamble are ordered, i.e., $x_1 \succsim x_2 \succsim \ldots \succsim x_n$, and we assume that gambles are comonotonic in the sense of ordinary monotonicity so long as the ordering of

[4] Inductively, one constructs an algebraic version of commodity bundles of any size.

consequence is unchanged (Wakker, 1990). We may write a gamble explicitly in either of two equivalent ways:

$$g_{[n]} = \begin{pmatrix} C_1, \ C_2, \ \ldots, \ C_i, \ \ldots, \ C_n \\ x_1, \ x_2, \ \ldots, \ x_i, \ \ldots, \ x_n \end{pmatrix} \tag{3}$$

$$= (x_1, C_1; x_2, C_2; \ldots; x_i, C_i; \ldots; x_n, C_n). \tag{4}$$

We use which ever notation seems more useful at the occasion. Each consequence-event pair (x_i, C_i) is called a *branch* of the gamble. Thus, a gamble is a collection of n disjoint branches.

Although gambles are stated in ranked form, we note that such rankings are only a matter of convenience in stating both some axioms (e.g., comonotonicity) and some results (e.g., rank dependent representations). In fact, we assume that gambles differing only in a permutation of the branches are perceived as indifferent.

1.4 Assumptions about Kernel Equivalents and Elements of Chance

Following Luce and Marley (2000), any gamble for which every consequence is no change from the status quo, e, i.e., gambles of the form $(e, C_1; e, C_2; \ldots; e, C_n)$, is called an *element of chance.* This is simply the realization of a chance "experiment" with no assignment of consequences to the several events, meaning that the status quo is maintained, which we denote by e. A trivial example is watching a spin of a roulette wheel. For any gamble $g_{[n]} = (g_1, C_1; g_2, C_2; \ldots; g_n, C_n)$, where the g_i are first-order gambles, its *kernel equivalent,* denoted $KE(g_{[n]})$, is defined to be the pure consequence solution, which is assumed to exist, to the following indifference

$$g_{[n]} \sim KE(g_{[n]}) \oplus (e, C_1; e, C_2; \ldots; e, C_n). \tag{5}$$

Note that, because $KE(g_{[n]})$ is a pure consequence, the right hand expression involves only one realization of the experiment.

We see that (2) and (5) yield

$$U(g_{[n]}) = U(KE(g_{[n]})) + U(e, C_1; e, C_2; \ldots; e, C_n). \tag{6}$$

The utility of an element of chance is a possible measure of the UofG. Our goal is to discover something about its mathematical form. The first step in doing so is to weaken the classical assumptions about *idempotence*: The kernel equivalents are *idempotent* (KE-idempotent) if for any gamble, denoted $g_{[n]}(x)$, all of whose consequences are x,

$$KE(g_{[n]}(x)) \sim x. \tag{7}$$

The elements of chance are $e-$*idempotent* if

$$e \sim (e, C_1; e, C_2; \ldots; e, C_n). \tag{8}$$

Traditional theories of utility typically assume idempotence or prove it from other assumptions. We explicitly do *not* assume $e-$idempotence and it is not a consequence of our other assumptions.

Suppose that C_i, $i = 1,\ldots,n$, form a partition of a universal event Ω and that C_i' is the same partition but arising from an independent realization of the underlying experiment. We assume that

$$(e,C_1;e,C_2;\ldots;e,C_n) \sim (e,C_1';e,C_2';\ldots;e,C_n'). \tag{9}$$

This is obviously true if $e-$idempotence holds, but (9) does not imply $e-$idempotence.

Although Luce and Marley (2000) derived a number of properties about such a decomposition into KEs and elements of chance, they had no principled way of getting results about the utility of elements of chance, partly because they considered only binary gambles. We offer one remedy for that incompleteness.

1.5 Probabilities and Implicit Events

It is quite common to treat risky gambles in a fashion parallel to that for uncertain ones, but instead of providing the event structure, one simply replaces the state C_i by the probability p_i as, for example,

$$g_{[n]} = (x_1,p_1;\ldots;x_i,p_i;\ldots;x_n,p_n) \quad \left(\sum_{i=1}^{n} p_i = 1\right)$$

This is the form commonly invoked in most experiments and in most of the developments emanating from economics. Nevertheless, to provide a sound basis for such probabilities, there must be some implicit event structure – the risky gambles have to be realized in some fashion.

So we first summarize properties and results for event structures, and then specialize them to risky situations.

2 Key Properties

2.1 Separable Representations of Binary Gambles

For binary gambles, conjoint measurement assumptions are easily stated that lead to the following (multiplicative) *separable*, ordering preserving, (1), representation U^* over so-called *unitary binary gambles* in which one consequence is e:

$$U^*(KE(x,C;e,D)) = U^*(x)W_{C\cup D}(C), \tag{10}$$

where $W_{C\cup D}(C)$ is a subjective weighting of the event C, conditional on the event $C \cup D$. Because U of (1) and U^* each preserve the order $\succsim$, they are strictly monotonically related. Can one find a property linking the two underlying structures leading, respectively, to order preservation by U and additivity, (2), and to order preservation by U^* and separability, (10)? To that end we assume *simple joint-receipt decomposability*:

$$(f \oplus g, C; e, D) \oplus (e, C'; e, D') \sim (f, C; e, D) \oplus (g, C'; e, D'), \tag{11}$$

where the prime simply indicates an independent realization of the "experiment" underlying the partition (C, D). Then, the result is that there exists $\kappa > 0$ such that $U = (U^*)^{\kappa}$ and so we have the (multiplicative) separable form

$$U(KE(x, C; e, D)) = U(x) S_{C\cup D}(C) \tag{12}$$

where $S_{C\cup D} := W^{\kappa}_{C\cup D}$. The weights S_{Ω} for an event Ω are also involved in the representation of UofGs, explicitly for uncertain gambles, implicitly for risky gambles.

Next come two steps: the first extending (12) to unrestricted binary gambles, and the second extending the representation of binary gambles to general ones.

2.2 Two Alternative Binary Decompositions: Segregation and Duplex Decomposition

Luce (1997, 2000) has studied two closely related, but distinct, forms for extending unitary gambles $(x, C; e, D)$ to full binary gambles. The first is *segregation*:

$$(x \oplus y, C; y, D) \sim (x, C'; e, D') \oplus y. \tag{13}$$

Kahneman and Tversky (1979) invoked segregation during the preliminary editing phase of their prospect theory. Segregation with the earlier assumptions, where (10) holds for gains only, leads to: For $f \succsim g$

$$U[KE(f, C; g, D)] = U(f) S_{C\cup D}(C) + U(g)\left[1 - S_{C\cup D}(C)\right]. \tag{14}$$

The alternative decomposition, *duplex decomposition,* which first appeared in Slovic (1967) and in Slovic and Lichtenstein (1968), is:

$$(x, C; y, D) \oplus (e, C'; e, D') \sim (x, C; e, D) \oplus (e, C'; y, D'). \tag{15}$$

This with the earlier assumptions, where (10) is for both gains and losses, and results leads to:

$$U[KE(f, C; g, D)] = U(f) S_{C\cup D}(C) + U(g) S_{C\cup D}(D). \tag{16}$$

Note that segregation is significantly more restrictive than duplex decomposition in that it associates $1 - S_{C\cup D}(C)$ to the (g, D) branch whereas duplex decomposition associates $S_{C\cup D}(D)$ with no tie to $S_{C\cup D}(C)$.

One empirical study, Cho, Luce and Truong (2002), suggests that some people, perhaps 75% of them, satisfy one of these properties although that study was conducted under the assumption of that $(e, C'; e, D') \sim e$, which, of course, matters only for duplex decomposition.

2.3 Inductive Properties: Branching and Upper Gamble Decomposition

We invoke two inductive properties, neither of which has received experimental evaluation. They are both cases of the reduction of compound gambles in the context of events, not probabilities. Their mathematical role is to reduce the utility expressions for gambles of order $n > 2$ to the those for binary gambles, which were given in Sects. 2.1 and 2.2. In particular, they lead to equations characterizing the utility of gambling, UofG, terms. The first, called *upper gamble decomposition* (UGD), is:

$$g_{[n]} = \begin{pmatrix} C_1, & C_2, & \ldots, & C_i, & \ldots, & C_n \\ x_1, & x_2, & \ldots, & x_i, & \ldots, & x_n \end{pmatrix}$$
$$\sim \begin{pmatrix} C_1, & \Omega\backslash C_1 \\ x_1, & \begin{pmatrix} C_2, & \ldots, & C_i, & \ldots, & C_n \\ x_2, & \ldots, & x_i, & \ldots, & x_n \end{pmatrix} \end{pmatrix}. \quad (17)$$

One sees that if one is willing to consider compound gambles, it is highly rational in nature, the "bottom lines" being the same.

The second property, *branching*, is

$$\begin{pmatrix} C_1, & C_2, & \ldots, & C_i, & \ldots, & C_n \\ x_1, & x_2, & \ldots, & x_i, & \ldots, & x_n \end{pmatrix}$$
$$\sim \begin{pmatrix} C_1 \cup C_2, & C_3, \ldots, C_i, \ldots, C_n \\ \begin{pmatrix} C_1, & C_2 \\ x_1, & x_2 \end{pmatrix}, & x_3, \ldots, x_i, \ldots, x_n, \end{pmatrix}. \quad (18)$$

This, too, is highly rational.

Note that each property involves a binary gamble, the first with the partition $(C_1, \Omega\backslash C_1)$ and the second with (C_1, C_2). Thus, we are able to invoke either (14) or (16).

Under these two properties for $n = 3$, one is able to prove (Luce et al., in press, a) the *choice property*[5]: for events $C \subseteq D \subseteq E$,

$$S_E(C) = S_D(C)S_E(D). \tag{19}$$

One can construct a function μ from events to the real numbers such that for all $C \subseteq E$, $E \neq \varnothing$,

$$S_E(C) = \mu(C)/\mu(E). \tag{20}$$

2.4 Main General Result

Under these assumptions one is able to arrive at a number of representations depending on which decomposition is assumed and on whether or not S_Ω is finitely additive (FA).

A first, important, result is that, under segregation, the representation has to be p-additive in the sense that for an appropriate choice for the unit of μ, there exists a constant Δ such that

$$S_\Omega(C \cup D) = S_\Omega(C) + S_\Omega(D) + \Delta\mu(\Omega)S_\Omega(C)S_\Omega(D). \tag{21}$$

The weights are finitely additive iff $\Delta = 0$.

Then the resulting representations are summarized in Table 1, which is to be read as follows: It is the cell wise sum of two 2×2 matrices corresponding, respectively, to the utility of kernel equivalents and to the utility of gambling terms. The matrix rows are whether or not S_Ω is finitely additive. The columns are by whether segregation or duplex decomposition is assumed. The cell entries are the representations listed below the table itself.

Table 1 Summary of representations for uncertain gambles*

Codes: DD = Duplex Decomposition, FA = Finitely Additive, KE = Kernel Equivalent, Seg = Segregation, UofG = Utility of Gambling

		U(KE)			UofG	
		Seg (13)	DD (15)		Seg	DD
S_Ω	FA	SEU	SEU	+	H	H
	Not FA	RDU	LWU		$0, A$	H

*Adapted from Table 1 of Luce, Ng, Marley, and Aczél (2008a), with kind permission of Springer Science+Business Media.

[5] With finitely additive weights, which we do not yet have, Luce (1959) called it the *choice axiom.* Here we use a more neutral term.

where

$$LWU\left(g_{[n]}\right) := \sum_{i=1}^{n} U(x_i) S_\Omega(C_i).$$

$$SEU(g_{[n]}) := LWU(g_{[n]}) \text{ with } \sum_{i=1}^{n} S_\Omega(C_i) = 1.$$

$$RDU(g_{[n]}) := \sum_{i=1}^{n} U(x_i)\left[S_\Omega(C(i)) - S_\Omega(C(i-1))\right]$$

$$= \sum_{i=1}^{n} U(x_i) S_\Omega(C_i)\left[1 + \Delta\mu(\Omega) S_\Omega(C(i-1))\right] \left(C(i) := \bigcup_{j=1}^{i} C_j\right),$$

and when Ω is maximal $H = A,$ a constant; otherwise it is 0.

$$H(C_1,\ldots,C_n) := U(e,C_1;\ldots;e,C_n) = K(\Omega) - \sum_{i=1}^{n} K(C_i) S_\Omega(C_i).$$

These results are based on theorems reported in Davidson and Ng (1981), Ebanks (1982), and Ebanks, Kannappan and Ng (1988). The representation $SEU + H$ is known as generalized subjective expected utility (G-SEU) with H the utility of gambling. The function K that arises in the form of H, the last form listed, is not otherwise specified. The nonfinitely additive representation under segregation has RDU as its kernel equivalent and H is a constant A assigned to Ω that is 0 when Ω is not maximal. As we will see, the results under risk, given in Table 2, are far more specific.

3 Risky Elements of Chance and An Application

3.1 Risk and Implicit Events

Next, we discuss the more specific forms for the UofG that we have derived in the case of risky gambles (Luce et al., 2008 b), and later (Sect. 5) summarize the evaluation of one of those forms vis-a-vis available data. As already mentioned, the case of risk entails an explicit set of probabilities, p_i, and a risky gamble is a function assigning a consequence x_i to $p_i, i = 1,\ldots,n$. These cases are important because, first, they are the class of gambles most often postulated by economists, and second, more often than not, these cases are studied in laboratory experiments by both economists and psychologists. Usually in experiments, the events are implicit with no clear indication as to exactly how the probabilities are to be generated except to the extent that participants in the experiment are "educated" about how the probabilities might be realized through mechanisms such as spins of a color-coded pie chart or random draws from an urn of colored balls. In this sense, we might suggest that there is an "implicit" event space underlying the probability distributions. In fact, we now assume that, even when the probabilities are presented explicitly, the participant postulates an underlying implicit event space. Then, we add assumptions concerning the linkage between uncertain gambles over the event space and risky gambles over the probability space that allow us to use our previous results about the representation of uncertain gambles to obtain representations of risky gambles with specific entropy-based representations of the elements of chance.

3.2 Probabilities Realized by Implicit Events

Let $\mathbf{p}_n = (p_1, p_2, \ldots, p_n)$ be any nontrivial, complete probability distribution, i.e., $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$. We assume, as is standard in the foundations of probability theory, that in a particular decision making context of gambles with explicitly given probabilities, the decision maker postulates a fixed, implicit, underlying algebra of events that is associated with a maximal universal event Ω_0 and a probability measure Pr on that algebra such that there is an ordered partition $\mathbf{C}_n := (C_1, C_2, \ldots, C_n)$ of Ω_0, with $C_i \neq \varnothing$, in the algebra, and with[6] $\Pr(C_i|\,\Omega_0) = p_i$, $i = 1, \ldots, n$. This implicit algebra is assumed to be *fixed* for the decision making context, e.g., a state lottery, independent of any particular lotteries that the decision maker may confront. Of course, there may be another partition $\mathbf{D}_n = (D_1, \ldots, D_n)$ of Ω_0, with D_i in the algebra, such that $\Pr(D_i|\,\Omega_0) = p_i = \Pr(C_i|\,\Omega_0)$, $i = 1, \ldots, n$. Our assumptions, below, overcome this ambiguity.

The risky gamble is presented as $g_{[n]} = (x_1, p_1; \ldots; x_n, p_n)$. Let $\succsim$ denote the preference ordering over pure consequences and risky gambles, and assume that a preference ordering $\succsim_{\mathcal{G}}$ exists over the event-based gambles $\mathcal{G}$. We assume that $\succsim_{\mathcal{G}}$ agrees with $\succsim$ over the structure of pure consequences, risky gambles, and their joint receipt, so for simplicity we drop the subscript $\mathcal{G}$.

We make two observations about the assumption of the existence of an implicit algebra of events:

First, it is just that, an assumption. It is certainly conceivable that a decision maker may somehow deal with the probabilities without resorting at all to an underlying algebra of events, as for example in a binary gamble given as $(x, p; y, 1-p)$ where it is taken for granted that when carried out the decision maker gets exactly one of x and y.

Second, the assumption of an implicit algebra permits us to invoke the earlier assumptions about events and the corresponding results. As we shall see, this means that there are several quite different types of decision makers, which has important implications for the usual kind of data analysis that averages data over respondents instead of analyzing each respondent separately.

Now we need the preference ordering over event-based gambles to be compatible with the preference ordering over the conditional-probability-based risky gambles in the following sense where we write $C(n) = \cup_{i=1}^{n} C_i$, $D(m) = \cup_{i=1}^{m} D_i$:

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[6] Usually $\Pr(C_i|\,\Omega_0)$ is abbreviated to $\Pr(C_i)$, but we think it best in this article to keep it explicit.

$$\begin{aligned}&(x_1,C_1;\ldots;x_n,C_n)\succsim(y_1,D_1;\ldots;y_m,D_m)\\&\Leftrightarrow(x_1,\Pr(C_1|C(n));\ldots;x_n,\Pr(C_n|C(n)))\\&\quad\succsim(y_1,\Pr(D_1|D(m));\ldots;y_m,\Pr(D_m|D(m)))\,.\end{aligned}\tag{22}$$

Under the background conditions (but not including segregation or duplex decomposition) and with (22) and $p\mapsto U(e,p;e,1-p)$ continuous, Luce et al. (2008 b) show that there is a constant $\rho>0$ such that

$$S_\Omega(C)=\Pr(C|\Omega)^\rho,\tag{23}$$

where S_Ω is the subjective weighting function in the representation of the uncertain gambles.

Under the above conditions and those leading to the results summarized in Table 1 for uncertain gambles, we obtain the representations for risky gambles that are summarized in Table 2, which is read in a fashion similar to Table 1.
where

$$EU(g_{[n]})=\sum_{i=1}^{n}U(x_i)p_i,$$

and

$$I^{(\rho)}(p_1,\ldots,p_n):=\begin{cases}-\sum_{i=1}^{n}p_i\log_2 p_i, & \rho=1\\ \frac{1}{2^{1-\rho}-1}[\sum_{i=1}^{n}p_i^\rho-1], & 0<\rho\neq 1\end{cases}.$$

The UofG term when $\rho=1$ is a constant A times the well-known Shannon (1948) entropy. The sign of A determines whether UofG is positive or negative and the magnitude of A determines the importance of UofG relative to the expected utility term. The proof of these results rest upon the mathematical theory of information (entropy) discussed by Aczél and Daróczy (1975). The sum in Table 2 corresponding to $\rho=1$, $EU+AI^{(1)}$, we call *entropy-modified expected utility* (EM-EU), and the sum corresponding to $\rho\neq 1$ under duplex decomposition, $\sum_{i=1}^{n}U(x_i)p_i^\rho+AI^{(\rho)}$, we call *linear power weighted utility* (LPWU), which, clearly, coincides with EM-EU when $\rho=1$. As indicated in the table, the case where segregation holds and $\rho\neq 1$ cannot occur under our assumptions.

These results raise an interesting concern about the almost exclusive focus of many utility theorists on probabilities without any regard to the underlying event structure. Apparently, that focus can lead to overlooking cases with $\rho\neq 1$.

It is striking that we have not arrived at a risky version of RDU, such as cumulative prospect theory, plus a UofG term. This lack invites modifying the assumptions in some crucial way, in particular by replacing branching by some property, such as coalescing, satisfied by the kernel equivalent of such a form.

Although purely rational considerations favor segregation and so EM-EU over duplex decomposition, descriptively those considerations are not compelling and, as we shall see in Sect. 5, some data reject EM-EU. Other data (Cho, Luce, & Truong, 2002) strongly suggest that a substantial proportion of respondents are better described by duplex decomposition than segregation. In that case, individual

Table 2 Summary of representations for risky gambles*

Codes: DD = Duplex Decomposition, FA = Finitely Additive, KE = Kernel Equivalent, Seg = Segregation, UofG = Utility of Gambling

		U(KE)				UofG	
		Seg (13)	DD (15)			Seg	DD
$S_\Omega^{1/\rho}$	$\rho = 1$	EU	EU			$I^{(1)}$	$I^{(1)}$
is				$+$	$A\times$		
FA	$\rho \neq 1$	—	$\sum U(x_i)p_i^\rho$			—	$I^{(\rho)}$

*Adapted from Table 1 of Luce, Ng, Marley, and Aczél (2008 b), with kind permission of Springer Science+Business Media.

differences abound, depending on the value of ρ. Therefore, it only makes sense to look at data on an individual basis without averaging them. Despite that admonition, most of the available data are for sets of people, not individuals.

3.3 An Application: Short-Term Gambling

Let $b = b(g_{[n]})$ denote the maximum buying price of the gamble $g_{[n]} = (x_1, C_1; \ldots; x_n, C_n)$, where we have in mind a quick resolution of the uncertainty. Thus, we are not treating such long-term "gambles" as life insurance, long-term health disability, and long-term financial investments. Our theory is timeless and so no financial discounting is involved. The following definition of b is natural (Luce, 2000, and earlier references), where the subjective weights may or may not be finitely additive:

$$e \sim (x_1 \ominus b, C_1; \ldots; x_n \ominus b, C_n).$$

It is obvious that when one buys a gamble one acquires the gamble with each consequence reduced by the buying price.

In the following, to make clear that the utility and weighting functions belong to the buyer, who is the gambler, we use the subscript b.

In those cases where $S_{\Omega,b}$ is assumed to be finitely additive, as in this subsection, we know that $\sum S_{\Omega,b}(C_i) = 1$, and so this definition is equivalent to

$$\begin{aligned} U_b(b) &= \sum_{i=1}^{n} U_b(x_i) S_{\Omega,b}(C_i) + H_b(C_1, \ldots, C_n) \\ &= U_b(KE(g_{[n]})) + H_b(C_1, \ldots, C_n), \end{aligned} \tag{24}$$

which is equivalent to $b \sim g_{[n]}$.

The case of selling prices is a good deal more subtle and we do not take it up here.

Let us apply this to the issue of commercial gambling. Suppose that the seller is either a state (lottery) or a casino and the buyer, i.e., a gambler, is an individual.

Assume, as seems to be the case, that pricing money lotteries by both states and casinos is based on some factor times the expected rate of return, i.e., $s = (1+\alpha)EV$, $\alpha > 0$. Assuming the special case where the buyer's utility for money is the identity function, then (24) yields

$$\begin{aligned} b \geq s \Leftrightarrow U_b(b) &\geq U_b\left((1+\alpha)EV(g_{[n]})\right) = (1+\alpha)EV(g_{[n]}) \\ \Leftrightarrow U_b(KE(g_{[n]})) &+ H_b(C_1,\ldots,C_n) \geq (1+\alpha)EV(g_{[n]}) \\ \Leftrightarrow KE(g_{[n]}) &+ H_b(C_1,\ldots,C_n) \geq (1+\alpha)EV(g_{[n]}). \end{aligned}$$

Let us suppose that, except for enjoying gambling, the gambler is fully rational and identifies the kernel equivalent of the gamble with its expected value:

$$KE(g_{[n]}) = EV(g_{[n]}).$$

Then, s/he will gamble if and only if

$$H_b(C_1,\ldots,C_n) \geq \alpha EV(g_{[n]}),$$

namely, whenever the gambler's utility of gambling exceeds the profit to the seller. This suggests that the utility of gambling is a strong determinant of behavior, as, of course, has been widely recognized if not previously modeled so formally.

4 Utility of Gambling with Valued Uncertain Events

The problem to be addressed in this section is motivated by the obvious, but widely ignored, fact that in many important real-world situations not only do event partitions have consequences attached to the events, but some events themselves are inherently valued by the decision maker. An example is airplane travel in which some of the chance events, such as the trip being terminated in a crash, are themselves of (negative) value. Such a value is independent of any bet, – e.g., insurance on the flight – that is placed on the trip. Moreover, we know of no principled way that allows for the separate measurement of the inherent value of events. Nonetheless, by a novel conceptual device we are able to use the results of Table 1 to arrive at the more specific forms given below.

The conceptual device is an ordering $\succsim_X$, which has an additive representation over joint receipts, and a family $\mathcal{O}$ of order extensions of $\succsim_X$ to include gambles. Also, the model presumes, for each and every $\succsim \in \mathcal{O}$, the formulation of Sects. 1 and 2 and the results summarized in Table 1. A difference arises because the weights now depend on $\succsim$, i.e., we have $S_{\succsim,\Omega}$ rather than S_Ω. We make assumptions that are sufficient for there to be some $\sigma > 0$ such that, for each pair $(\mathbf{p}_n, \mathbf{C}_n)$, there is some $\succsim \in \mathcal{O}$ for which $S^{\sigma}_{\succsim,\Omega}(C_i) = p_i$. With these, and other assumptions, Ng, Luce, & Marley (2008b) show that, for each order $\succsim \in \mathcal{O}$ with additive $S^{\sigma}_{\succsim}$, we can define

$$H\begin{pmatrix} C_1, C_2, \ldots, C_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} := H_{\succsim}(C_1,\ldots,C_n) := U_{\succsim}(e,C_1;\ldots;e,C_n).$$

Then, under the assumptions about the family of orders, this family of functions satisfies the conditions of what is known as inset entropy, introduced by Aczél and Daróczy and Aczél in 1975 and Kannappen in 1978. In particular, (18) with $x_i = e$, $i = 1,\ldots,n$, becomes

$$H\begin{pmatrix} C_1, C_2, \ldots, C_n \\ p_1, p_2, \ldots, p_n \end{pmatrix} = H\begin{pmatrix} C_1 \cup C_2, C_3, \ldots, C_n \\ p_1 + p_2, p_3, \ldots, p_n \end{pmatrix} + H\begin{pmatrix} C_1, & C_2 \\ \frac{p_1}{p_1+p_2} & \frac{p_2}{p_1+p_2} \end{pmatrix}, (p_1+p_2)^{1/\sigma}. \tag{25}$$

Using this, the utility of gambling becomes more specialized than in Table 1.

For both segregation and duplex decomposition with finitely additive S_Ω, the utility of gambling term becomes

$$\sum_{i=1}^{n} [V(\Omega) - V(C_i)]\, S_{\succsim,\Omega}(C_i) - A\sum_{i=1}^{n} S_{\succsim,\Omega}(C_i)\log_2 S_{\succsim,\Omega}(C_i), \tag{26}$$

where V maps events to numbers and A is a constant, both in the same units as U. We call the left term subjective expected value and, of course, the right term is the subjective Shannon entropy. As with risk, the sign of A determines whether the subjective entropy is seen as having utility or dis-utility, whereas the magnitude of A controls its importance relative to the two expectations.

For additive S_Ω^σ where $\sigma \neq 1$, the segregation case is impossible and the duplex decomposition one yields

$$V(\Omega) - \sum_{i=1}^{n} V(C_i) S_{\succsim,\Omega}(C_i) - A\left[1 - \sum_{i=1}^{n} S_{\succsim,\Omega}(C_i)\right]. \tag{27}$$

In this case, the term following A is called subjective entropy of degree $1/\sigma$ (Havrda and Charvát, 1967). The role of A is as before.

5 Data: Accommodated and Not Accommodated

In this section we focus mostly on the case of risk and illustrate the relation of EM-EU to existing data sets, although we do consider one case involving uncertainty (Sect. 5.2). Details for both the risky and the uncertain cases are presented in Luce et al. (2008 b). We focus on risk because, in the vast majority of experiments, the gambles are formulated as risky. Nonetheless, Luce et al. (2008 b) note that the concept of a purely risky gamble may be a fiction of the theorist and experimentalist in the sense that it need not really exist for a respondent. For instance, as discussed in Sect. 3.1, the experimenter often "educates" respondents about specific event spaces whereby the probabilities stated in the risky gambles might be realized. A respondent may have superstitions about the qualities, such as colors or numbers, used to identify such events and that may well affect behavior. Also, in naturalistic settings, people are confronted with valued events, such as a standard blood test, where the unpleasantness of the test is independent of the probability of the possible test results, and the above comments suggest that they may also impute values to events that an experimenter considers valueless. We are not aware of experimental studies of gambles, with human respondents, that explicitly involve valued events, nor have we thought through what impact imputing values to valueless events has for data analysis.

A number of "paradoxes" have been raised over the years, each of which casts doubt on the descriptive adequacy of progressively more general theories. The oldest and most famous, the St. Petersburg paradox, questioned the descriptive adequacy of expected value (EV); the Allais paradox questioned expected utility (EU); and the Ellsberg paradox questioned SEU. More recently Michael Birnbaum in collaboration with several others has explored a series of "independence" properties (for a summary and references, see Marley & Luce, 2005) that have cast considerable doubt on rank-dependent utility (RDU) — including, of course, cumulative prospect theory, SEU, and EU. The vast majority of these data are for the risky case, and Luce et al. (2008 b) show that EM-EU can handle many, but by no means all, of the empirical results. Here we summarize the results implied by EM-EU for the Allais paradox and the independence conditions, all situations of risk. For the Ellsberg paradox, which is based in part on uncertain events, we turn to the special case of G-SEU, given below as (29), where H is the subjective Shannon entropy. One can view this as a specialization of the finite additive cases of either Table 1 or of the representation (26) for which the value function V is a constant.

Two basic principles are useful in deriving the properties of EM-EU and in comparing them with those of EU and various data.[7] First, the properties of EM-EU agree with those of EU when either $A = 0$ or when the Shannon entropy terms $I^{(1)}$ in the various gambles under consideration are related in specific ways (some of which we illustrate below). Second, the properties of EM-EU are likely to differ from those of EU when $A \neq 0$ and the Shannon entropy terms $I^{(1)}$ in the various gambles under consideration are not equal and do not "cancel" in appropriate ways. We illustrate these principles with the Allais paradox, the Ellsberg paradox and one of Birnbaum's "independence" conditions.

As already mentioned, in the remainder of this section we develop most of the arguments for EM-EU, i.e., for

$$U(g_{[n]}) = EU\left(g_{[n]}\right) + AI^{(1)}(p_1,\ldots,p_n), \tag{28}$$

where $I^{(1)}$ is the Shannon (1948) entropy. This case arises under both segregation and duplex decomposition.

[7] Parallel principles apply to G-SEU, especially the special case that we apply to the Ellsberg paradox.

And, when gambles are based on uncertain events, – i.e., they are presented in terms of events C_i rather than probabilities p_i – we consider the following very special, but important, case of G-SEU:

$$U(g_{[n]}) = SEU\left(g_{[n]}\right) - A\sum_{i=1}^{n} S_\Omega(C_i)\log_2 S_\Omega(C_i), \tag{29}$$

where the UofG term is the Shannon (1948) entropy of the subjective probabilities.

5.1 The Allais Paradox

The classic example of the Allais paradox arises when an individual has the following pair of preferences (where M means million):

$$\begin{aligned}&\$1M \succ (\$5M, 0.10; \$1M, 0.89; \$0, 0.01),\\&(\$5M, 0.10; \$0, 0.90) \succ (\$1M, 0.11; \$0, 0.89),\end{aligned}$$

a pattern of choices that is shown easily to violate EU. However, note that each gamble is based on a different probability distribution, which means that the entropy terms do not in general "cancel" when $A \neq 0$. In fact, Luce et al. (2008 b) show that the above preference pattern is compatible with EM-EU for a sufficiently large negative A value. Such use of a negative A value makes sense as it corresponds to an aversion to gambling.

5.2 The Ellsberg Paradox

We now provide an explanation of the Ellsberg paradox in terms of the entropy-modified form of SEU given in (29).

The Ellsberg (1961) paradox in coalesced[8] form is of the following form with the choices between f vs. g and f' vs. g' where[9]

$$\begin{aligned}f &= (x, R; 0, G\cup Y) \equiv (x, p; 0, 1-p)\\g &= (x, G; 0, R\cup Y)\\f' &= (x, R\cup Y; 0, G)\\g' &= (x, G\cup Y; 0, R) \equiv (x, 1-p; 0, p)\end{aligned}$$

[8] If there are two (or more) branches (x,C), (x,D) in a gamble, with the common consequence x, then their coalesced form replaces the two by the single branch $(x, C\cup D)$. If the gambles are presented in uncoalesced form, then the following explanation of the paradox requires the additional assumption that the participants convert the gambles to their coalesced forms.

[9] The event notation R, G, Y arose from the interpretation of the chance experiment being a draw from an urn with red, green, and yellow balls.

with $x \succ e$. Note that the probability of G, and so of Y, is not specified beyond being bounded to the interval $(0, 1-p)$. In the classic example, where $\Pr(R) = p = 1/3$ and $\Pr(G \cup Y) = 1 - p = 2/3$, people typically pick f over g and g' over f'. It is checked easily that this pattern of choices is incompatible with SEU.

Paralleling the reasoning for the Allais paradox, note that the gambles f and g are based on different partitions of the events, as are f' and g'. This suggests that the entropy terms given by the entropy-modified form of SEU, (29), do not in general "cancel" when $A \neq 0$. In fact, Luce et al. (2008 b) show that the above preference pattern is compatible with (29) provided that, in (29), the Shannon entropy $I^{(1)}(S_\Omega(R), 1 - S_\Omega(R)) \neq I^{(1)}(S_\Omega(G), 1 - S_\Omega(G))$ and A is sufficiently large, either positively or negatively.

5.3 Independence Properties

Consider $n = 3$, (p_1, p_2, r) and (q_1, q_2, r) arbitrary nontrivial complete probability distributions, and consequences $x_1, y_1, x_2, y_2, z, z'$ with $y_1 \succ x_1 \succ x_2 \succ y_2 \succ e$ and $y_2 \succ z \succ e$, $y_2 \succ z' \succ e$. Then *branch independence of type*[10] $(3,3)^2$ states that:

$$f_{[3]} \sim (x_1, p_1; x_2, p_2; z, r) \succsim (y_1, q_1; y_2, q_2; z, r) \sim g_{[3]} \tag{30}$$

iff

$$f'_{[3]} \sim (x_1, p_1; x_2, p_2; z', r) \succsim (y_1, q_1; y_2, q_2; z', r) \sim g'_{[3]}. \tag{31}$$

Note that, under EM-EU, the above gambles are such that

$$\begin{aligned} EU(f_{[3]}) - EU(g_{[3]}) &= U(x_1)p_1 + U(x_2)p_2 - U(y_1)q_1 - U(y_2)q_2 \\ &= EU(f'_{[3]}) - EU(g'_{[3]}). \end{aligned} \tag{32}$$

Now, it is routine to show that, under EM-EU, (32) is sufficient for branch independence of type $(3,3)^2$ to hold, i.e., (30) iff (31). In fact, all cases of branch independence when $n = 3$ reduce to such a condition, and hence EM-EU predicts that they all hold, contrary to some data.

Applying similar arguments to other independence conditions, Luce et al. (2008b) show that EM-EU accommodates various, but by no means all, of the data obtained in tests of independence conditions not leading to simple cancellation of the UofG terms.

[10] The notation $(3,3)^2$ indicates that the consequence z (respectively, z') is the third consequence of the ranked gamble.

6 Conclusions

The major results, which are formally stated as theorems with proofs in our cited papers, are the four representations found in Table 1 for uncertain gambles and the three in Table 2 for risky ones. Those of Table 2 are, essentially, the same ones that Meginniss (1976) first discovered in his long ignored paper. The difference is that we have found an axiomatic basis for the results whereas he began by assuming a representation of the form $U(g_{[n]}) = \sum_{i=1}^{n} f(U(x_i), p_i)$, and that the common function f is differentiable. In the proof he invoked, with little comment, what amounts to GDU. His proof is far simpler and briefer than ours, but we feel it is less illuminating.

By using a conceptual construct of an (infinite) family $\mathcal{O}$ of order extensions of $\succsim_X$, plus other assumptions, we were able to develop, for each order extension, a representation of the UofG term as a subjectively weighted value of events plus a subjective entropy term involving the same weights.

Four major problems are worth mentioning that are unresolved here. First, the case where utility is p-additive rather than additive, i.e., $U(x \oplus y) = U(x) + U(y) + \delta U(x)U(y)$, is of considerable interest because the impact of the elements of chance is amplified by the utility of the kernel equivalents:

$$U(g_{[n]}) = U(KE(g_{[n]})) + U(e, C_1; \ldots; e, C_n)\left[1 + \delta U(KE(g_{[n]}))\right].$$

Ng, Luce, and Marley (2008c, submitted) obtains a very nice representation in the uncertain case for segregation but obtains essentially nothing interesting under duplex decomposition. Second, we need a fuller understanding of why RDU (including, of course, cumulative prospect theory), which has been so popular, admits only a very restricted UofG for uncertain gambles and simply does not arise for risky ones. To have a richer utility of gambling environment that permits rank dependent utility with utility of gambling must require some changes in the axioms.

Third, the conceptual device invoked in Sect. 4 cannot be empirically realized and tested because it applies to infinitely many orderings satisfying the same axioms and agreeing over $\langle X, \oplus \rangle$, whereas an individual generates just one. To make the conceptual device seem a bit more concrete, some people are comfortable in imagining an infinite family of individuals whose preference orders differ only due to differences in their assignment of probability distributions to event partitions. Others find it easier to think of a single individual whose extension is simply unknown to a theorist who must be prepared to model whatever extension happens to be true. The open problem is find some testable way to arrive at those results where the utility of a gamble was partitioned into the sum of three subjective terms: a linear weighted utility of consequences plus a linear weighted value of events per se plus an entropy term.

Fourth, although we have invoked the rank ordering induced by the consequences of a gamble, we have also assumed invariance under permutations and so that constraint actually imposed no real limitation. It was done merely as a convenience in stating certain assumptions and theorems. However, some of Birnbaum's data strongly suggest that whether an event underlies the best or the worst consequence actually matters greatly in how it is evaluated. Thus, a major open problem is to work out a theory for the inherently ordered case. One possibility is to try to arrive at weighted entropy,

$$\sum_{i=1}^{n} a_i S_\Omega(C_i) \log S_\Omega(C_i),$$

which has been mentioned in the literature. But this is certainly not the only possibility.

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