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RESEARCH ARTICLE

Utility of gambling II: risk, paradoxes, and data

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Abstract We specialize our results on entropy-modified representations of event-based gambles to representations of probability-based gambles by assuming an implicit event structure underlying the probabilities, and adding assumptions linking the qualitative properties of the former and the latter. Under segregation and under duplex decomposition, we obtain numerical representations consisting of a linear weighted utility term plus a term corresponding to information-theoretical entropies. These representations accommodate the Allais paradox and most of the data due to Birnbaum and associates. A representation of mixed event-and probability-based gambles accommodates the Ellsberg paradox. We suggest possible extensions to handle the data not accommodated.

Keywords Duplex decomposition · Entropy · Functional equations · Linear weighted utility · Segregation · Expected utility · Utility of gambling · Utility paradoxes · Independence properties

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Luce et al. (2007) cited some history of the concern about the utility of gambling and the inherent value of some events, per se. There we emphasized uncertain[1] gambles and largely excluded the more widely studied, although perhaps less important in everyday life, case of risky[2] gambles.

The discussion of the utility of gambling found in the utility literature includes some ad hoc ideas mostly focused on the risky cases. Typically these involved special modifications of the expected utility representation. Summaries of references can be found in Conlisk (1993) from an economic perspective and in Luce and Marley (2000) from a more, though not entirely, psychological perspective. Two recent contributions are Diecidue et al. (2004) and Yang and Qiu (2005). In line with statements in Conlisk (1993), all these contributions tend not to be axiomatic, not to be mathematically very general, and not to apply to uncertain alternatives. To the best of our knowledge, Meginniss (1976) was the first author to arrive at a sensible theory incorporating what amounts to a concept of a utility of gambling for risky gambles. Until quite recently, his result appears to have been unknown, ignored, and/or forgotten by utility theorists, and the ability of his representations to account for anomalies has not been widely recognized (see Sect. 2).

Meginniss (1976) result is that the overall utility of a risky gamble is given by a linear weighted utility of consequences plus an (information-theoretic) entropy term dependent only on the probabilities. His proof of the result was clever but his assumptions seem quite special and they were not given any behavioral axiomatization. As already noted, he formulated it for risky rather than for uncertain gambles, and also invoked some apparently strong assumptions about the form of the utility representation of gambles. Without being aware of this work, Yang and Qiu (2005) proposed a model closely related to the special case where the entropy term corresponds to Shannon's entropy (Shannon 1948), explored some of its properties, and applied it to some of the well known anomalies. We develop similar explanations of several such anomalies in Sect. 2. Ng et al. (submitted) generalized Meginniss' approach in a variety of ways, but fundamentally they still followed his general ideas.

Here we base our analysis on the work in Luce et al. (2007) which dealt with uncertain alternatives in an axiomatic fashion, and with weights associated with events as in, e.g., Savage (1954) subjective expected utility theory. Our approach to risky gambles is to assume that there is an implicit event structure underlying the given probabilities. When segregation (6), which is a distribution property, holds, we arrive at an expected utility term plus a constant times the Shannon entropy (Shannon 1948). We call this representation entropy-modified expected utility (EM-EU). For segregation and non-finitely additive subjective weights, which gave rise to the rank-dependent utility (RDU) form in Luce et al. (2007), nothing further follows because an inconsistency arises in the risky condition. When duplex decomposition (9), which is a non-rational decomposition of binary gambles into two simpler ones, holds, we get, in addition, a linear weighted utility of consequences expression, with the weights a power of the probabilities, plus an entropy of degree equal to that power. We call this

[1] Those where the events have no readily agreed upon probabilities.

[2] Those for which events are simply replaced by probabilities, which is often described as there being a probability distribution associated with the possible consequences of the gamble.

representation entropy-modified linear weighted utility (EM-LWU). This is the representation discovered by Meginniss (1976), although our axiomatization and proof are quite different from his approach.

The "rational" expected utility, EU, is a special case with no utility of gambling term of the more descriptive representations of EM-EU of Theorem 3 and of the $\rho = 1$ case of EM-LWU of Theorem 4 (see Table 1 of Sect. 3). That is, the general models for risk (and uncertainty, for that matter), although descriptive in motivation, do not preclude classically rational actors.

The proofs draw heavily upon the mathematical theories of entropy. For segregation, the risky case uses a result of Aczél and Daróczy (1975) which is also summarized in Ebanks et al. (1998).

In Sect. 2 we show that EM-EU accommodates various descriptive anomalies including the Allais paradox as well as a number of violations of the independence properties studied more recently by Michael Birnbaum (for a summary of these properties, see Birnbaum 1997, Marley and Luce 2005). A special case of the event-based representations handles the Ellsberg paradox.

1 Entropy-modified utility representations under risk

The purpose of this section is to arrive at a more specific form for the elements of chance than we achieved in Luce et al. (2007). We do this for both the assumption of segregation, using Theorem 13, and for duplex decomposition, using Theorem 16, of Luce et al. (2007).

1.1 Risk and events

We make a distinction that decision theorists have long intuited, that between risky and uncertain (sometimes called ambiguous) situations. One of the first to make much of the distinction was Ellsberg (1961). Very recently, Hsu et al. (2005) provided both imaging and lesion evidence for a substantial brain distinction between the two.

The important special case of risky gambles entails an explicit set of probabilities, p_i, giving rise to consequences, x_i. Two reasons underlie the importance of this case: First, it is the class of gambles most often postulated by economists. Second, more often than not, it is what appears to be studied in laboratory experiments by both economists and psychologists in the sense that the events are completely implicit with no clear indication as to how exactly the probabilities are to be generated. This "appears to be" is amplified in Sect. 1.1.2.

1.1.1 Implicit events

Suppose that $\mathbf{p}_n = (p_1, p_2, \ldots, p_n)$ is any non-trivial, complete probability distribution, i.e., $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$. We assume, as is standard in the foundations of probability theory, that in a particular decision making context of gambles with explicitly given probabilities, the decision maker postulates a fixed, implicit, underlying algebra of events that is associated with a universal event Ω_0 such that

Assumption 1 There is a probability measure Pr on the algebra with universal event Ω_0 such that for every non-trivial, complete probability vector $\mathbf{p}_n = (p_1, p_2, \ldots, p_n)$, there is an ordered partition $(C_1, C_2, \ldots, C_n)$ of Ω_0, with $C_i \neq \varnothing$ in the algebra, and with[3] $\Pr(C_i \mid \Omega_0) = p_i$, $i = 1, \ldots, n$.

Let us be clear that this implicit algebra is assumed to be fixed for the decision making context, say a state lottery, independent of any particular lotteries that the decision maker may confront. Given the context of a fixed probability measure Pr, Ω_0 must be maximal. Of course, there may be another partition $(D_1, \ldots, D_n)$ of Ω_0, with $D_i \neq \varnothing$ in the algebra, such that $\Pr(D_i \mid \Omega_0) = p_i = \Pr(C_i \mid \Omega_0)$, $i = 1, \ldots, n$. The Observation following Proposition 1 shows that our assumptions are sufficient to overcome this ambiguity.

Two observations about the assumption of the existence of an implicit algebra of events:

First, it is just that, an assumption. It is certainly conceivable that a decision maker may somehow deal with the probabilities without resorting at all to an underlying algebra of events, as for example in a binary gamble given as $(x, p; y, 1-p)$ where it is taken for granted that when carried out the decision maker gets exactly one of x and y.

Second, the assumption of an implicit algebra permits us to invoke the assumptions given in Luce et al. (2007) about events and the corresponding results, in particular Theorems 13 and 16. As we shall see, this means that there are several quite different types of decision makers. That fact has implications for the usual kind of data analysis that averages data over respondents instead of analyzing each respondent separately.

The risky gamble is presented as $g_{[n]} = (x_1, p_1; \ldots; x_n, p_n)$. We call each pair (x_i, p_i) a *branch* of the gamble. Let $\succsim$ denote the preference ordering over pure consequences and risky gambles. Further, we assume that $U(x_1, p_1; \ldots; x_n, p_n)$ is continuous in the p_i. This, of course, means that H_n, where

$$H_n(p_1, \ldots, p_n) := U(e, p_1; \ldots; e, p_n), \tag{1}$$

will also be continuous. Over $\mathcal{G}$, the event-based gambles, we assume a preference ordering $\succsim_{\mathcal{G}}$ exists that agrees with $\succsim$ over the structure of consequences and risky gambles and their joint receipt $\oplus$. For simplicity we drop the subscript $\mathcal{G}$.

1.1.2 A caution about pure risk

The concept of a purely risky gamble may be a fiction of the theorist and experimentalist in the sense that it need not really exist for a respondent. Indeed, when naïve respondents are being indoctrinated into the procedures of the laboratory or experiments on the web, the experimenter often "educates" them about specific ways—such as a spinner over a color-coded pie chart or random draws from an urn of colored or numbered balls—whereby the probabilities might be realized. In some cases, respondents are selectively rewarded by a random selection of some of their choices and

[3] Usually $\Pr(C_i \mid \Omega_0)$ is abbreviated to $\Pr(C_i)$, but we think it best in this article to keep the dependence explicit.

running the actual physical chance device described. In this way the respondent is invited to think in terms of events as well as probabilities. Moreover, for all we know, he or she may have superstitions about the colors or numbers used to identify events and that may well affect behavior.

1.2 Weights are powers of probabilities

The statement of the following proposition adds a qualitative assumption to those in Luce et al. (2007) that leads to a power function relating the weights in the representation of uncertain gambles to the probabilities in the risky gambles. Once we have this relationship, the representations for risky gambles follow quite directly from those for uncertain gambles in Luce et al. (2007).

Proposition 1 *Assume that* (i) *the gambles are risky and there is an implicit probability function* Pr *on* $\mathfrak{B}$ *satisfying Assumption 1,* (ii) *the background conditions stated in Theorems 13 and 16 of Luce et al. (2007), for event-based gambles on an algebra* $\mathfrak{B}$ *(namely, additivity over joint receipt, decomposability into kernel equivalents and elements of chance, kernel equivalents of unitary gambles have separable representations, and upper gamble decomposition and branching are satisfied),* (iii) *the preference ordering over event-based gambles is compatible with the preference ordering over the conditional-probability-based risky gambles:*

$$\begin{aligned}(x_1, C_1; \ldots; x_n, C_n) &\succsim (y_1, D_1; \ldots; y_m, D_m)\\ \Leftrightarrow \big(x_1, \Pr(C_1 | \cup_{i=1}^{n} C_i); \ldots; x_n, \Pr(C_n | \cup_{i=1}^{n} C_i)\big)&\\ \succsim \big(y_1, \Pr(D_1 | \cup_{i=1}^{m} D_i); \ldots; y_m, \Pr(D_m | \cup_{i=1}^{m} D_i)\big)&, \end{aligned} \tag{2}$$

and (iv) $p \mapsto U(e, p; e, 1-p)$ *is continuous. If* S_Ω *is the subjective weighting function in the separable representation* US_Ω *of Theorems 13 and 16 of Luce et al. (2007), then there is a constant* $\rho > 0$ *such that*

$$S_\Omega(C) = \Pr(C|\Omega)^\rho. \tag{3}$$

All proofs are in Sect. 4.1 of the Appendix.

Observation: The compatibility assumption (2) implies that the preference order over event-based gambles satisfies

$$\begin{aligned}(x_1, C_1; \ldots; x_n, C_n) &\succsim (y_1, D_1; \ldots; y_m, D_m)\\ \Leftrightarrow (x_1, C_1'; \ldots; x_n, C_n') &\succsim (y_1, D_1'; \ldots; y_m, D_m')\\ \text{whenever } \Pr(C_j | \cup_{i=1}^{n} C_i) &= \Pr(C_j' | \cup_{i=1}^{n} C_i') (\forall j = 1, \ldots, n)\\ \text{and } \Pr(D_j | \cup_{i=1}^{m} D_i) &= \Pr(D_j' | \cup_{i=1}^{m} D_i')\ (\forall j = 1, \ldots, m).\end{aligned} \tag{4}$$

The converse is also true: if (4) holds, then we may define the preference order over the conditional-probability-based risky gambles by (2).

To prove Theorems 3 and 4 below, we need the following results from Aczél and Daróczy (1975), namely, Corollary 3.3.5 of Theorem 6.3.9, and results from Diderrich (1978).

Proposition 2 *Suppose that the functions*

$$H_n : \left\{ (p_1, \ldots, p_n) \mid p_i > 0, \sum_{i=1}^{n} p_i = 1 \right\} \to \mathbb{R} \ (n = 2, 3, \ldots)$$

of (1) are symmetric and that the recursive property

$$H_n(p_1, p_2, p_3, \ldots, p_n) = (p_1 + p_2)^{\rho} H_2 \left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2} \right) + H_{n-1}(p_1 + p_2, p_3, \ldots, p_n) \quad (5)$$

holds. If $\rho \neq 1$*, then there exists a constant A such that*

$$H_n(p_1, \ldots, p_n) = \frac{A}{2^{\rho-1} - 1} \left(\sum_{i=1}^{n} p_i^{\rho} - 1 \right) \quad (n = 2, 3, \ldots).$$

If $\rho = 1$ *and the additional condition is satisfied that* $p \mapsto H_2(1 - p, p)$ *is locally bounded (bounded on an interval of arbitrarily small positive length), then there exists a constant A such that*

$$H_n(p_1, \ldots, p_n) = -A \sum_{i=1}^{n} p_i \log_2 p_i \quad (n = 2, 3, \ldots).$$

Observation: Note that for the case $\rho = 1$, continuity [(iv) of Proposition 1] has been replaced by the weaker property of local boundedness. Indeed, there exist symmetric solutions of (5) that are not locally bounded. No additional condition is needed when $\rho \neq 1$.

1.3 Segregation and risk

In the following, $\oplus$ is the joint-receipt (addition) operation of Luce et al. (2007), with $\ominus$ the corresponding subtraction operation.

Theorem 3 *Suppose that the conditions of Proposition 1 hold (with (iv) modified as in the Observation following Proposition 2), and that the implicit event-based gamble structure satisfies segregation*

$$(x, C; y, D) \sim (x \ominus y, C'; e, D') \oplus y \quad (x \succsim y). \quad (6)$$

Then:

1. *the subjective weighting functions S_Ω are additive for all $\Omega \subseteq \Omega_0$, and*
2. *there exists a constant A with the unit of U, such that*

$$U(g_{[n]}) = U(x_1, p_1; \ldots; x_n, p_n) = \sum_{i=1}^{n} U(x_i)p_i + AI_n^{(1)}(p_1, \ldots, p_n), \tag{7}$$

where

$$I_n^{(1)} = -\sum_{i=1}^{n} p_i \log_2 p_i \tag{8}$$

is the Shannon entropy Shannon (1948) .

The representation (7) may be called *entropy-modified expected utility* (EM-EU). The EU term, $\sum U(x_i)p_i$, is standard expected utility and the entropy term is that of Shannon (1948) . The proof draws upon Theorem 13 of Luce et al. (2007) and Proposition 2.

1.4 Duplex decomposition and risk

Theorem 4 *Suppose that the conditions of Proposition 1 hold [(iv) reduced as in the Observation following Proposition 2] and that the implicit event-based gamble structure satisfies duplex decomposition*

$$(x, C; y, D) \oplus (e, C'; e, D') \sim (x, C; e, D) \oplus (e, C'; y, D'), \tag{9}$$

where (C', D') refers to an independent realization of the experiment underlying (C, D). Then:

1. *S_Ω satisfies (3), i.e.,*

$$S_\Omega(C) = \Pr(C|\Omega)^\rho.$$

2. *There exists a constant A with the unit of U, such that for any risky gamble $g_{[n]} = (x_1, p_1; \ldots; x_n, p_n)$,*

$$U(g_{[n]}) = \sum_{i=1}^{n} U(x_i)p_i^\rho + AI_n^{(\rho)}(p_1, \ldots, p_n), \tag{10}$$

where $I_n^{(\rho)}$, the entropy of degree ρ, is defined by

$$I_n^{(\rho)}(p_1, \ldots, p_n) := \begin{cases} -\sum_{i=1}^{n} p_i \log_2 p_i, & \rho = 1 \\ \frac{1}{2^{1-\rho}-1}\left[\sum_{i=1}^{n} p_i^\rho - 1\right], & 0 < \rho \neq 1 \end{cases}. \tag{11}$$

The entropy of degree 1 is the Shannon entropy.

We call the general class of cases given by (10) and (11) *entropy-modified linear weighted utility* (EM-LWU). In the $\rho = 1$ case, it coincides with EM-EU in the segregation case. The proof rests heavily on Theorem 16 of Luce et al. (2007).

1.5 Discussion of the representations under risk

These results raise an interesting concern about the almost exclusive focus of many utility theorists, when dealing with the risky case, on probabilities without any regard to the underlying event structure. Apparently, that focus can lead to overlooking cases with $\rho \neq 1$. Major exceptions are those utility theories, such as cumulative prospect theory, that include nonlinear probability weighting functions.

As noted in Luce et al. (2007), it is striking that we have not arrived at a utility of gambling version of RDU, like cumulative prospect theory, Tversky and Kahneman (1992), with an entropy term. This lack invites modifying the assumptions in some crucial way, in particular by replacing branching by some property, such as coalescing,[4] which is satisfied by the cumulative form.

Although purely rational considerations favor segregation and so EM-EU (7), over duplex decomposition, descriptively those considerations are not compelling and, as we shall see in the next Section, data reject EM-EU. Other data, Cho et al. (2002), strongly suggest that a substantial proportion of respondents are better described by duplex decomposition than segregation. In that case, individual differences abound, depending on the value of ρ. Therefore, it only makes sense to look at data on an individual basis without averaging them.

2 Representations, paradoxes, and data

This section discusses the relation of some of the representations we have arrived at to existing data sets. The vast majority of the empirical results are for risky gambles, and we show that EM-EU (7), can handle many, but by no means all, of them; the predictions of EM-EU that we present are for monetary consequences, where the utility function is linear with money. This material draws heavily on tests of properties developed by Birnbaum and his collaborators for risky gambles. The gambles used for one data set, namely that of the Ellsberg paradox, involve both risk and uncertainty—i.e., they involve events some of which are quite uncertain and others for which a probability is given. We use our results to motivate an entropy-modified form of subjective expected utility (SEU) as a—partial—explanation of those data.

A number of "paradoxes" have been raised over the years, each of which casts doubt on the descriptive adequacy of progressively more general theories. The most famous, the St. Petersburg paradox, which is very well known to utility theorists, questioned the descriptive adequacy of expected value (EV) (but see Sect. 2.3 of Luce et al. 2007); the Allais paradox questioned expected utility (EU); and the Ellsberg paradox questioned SEU. More recently Michael Birnbaum in collaboration with several others

[4] *Coalescing* simply means that, if a gamble has two branches (x, C_i) and (x, C_{i+1}) with the same consequence x, then they can be collapsed into a single branch $(x, C_i \cup C_{i+1})$. See Luce (2000).

has explored a series of "independence" properties (for a summary and references, see Marley and Luce 2005) that have cast considerable doubt on rank-dependent utility (RDU)—including, of course, cumulative prospect theory, SEU, and EU. Our aim in this section is to see how our entropy-modified representation fares against the empirical results.

We do not discuss the predictions of EM-EU vis-a-vis the St. Petersburg paradox because we have already discussed that paradox in Luce et al. (2007), and we do not see it as an issue. Here, we first discuss the Allais paradox, then the Ellsberg paradox, and finally various independence conditions.

Two properties are very important with respect to whether a model is able to handle the paradoxes or the data for an independence condition. One of these is *coalescing,* which was defined earlier in footnote 4. This is well known to be a characteristic of any rank-dependent utility representation (including EU and SEU as special cases); see Luce (2000). Further, Marley and Luce (2005) showed that the failure of coalescing may underlie the Allais and Ellsberg paradoxes. Because the EM-EU representation satisfies coalescing iff $A = 0$, we concentrate on the case $A \neq 0$.

Note that within the context of uncertain gambles, the definition of the concept of stochastic dominance depends upon using the property of coalescing. Thus, because the entropy terms differ between a gamble and its coalesced form, we do not predict stochastic dominance to hold in general.

The second important property is called *common branch substitution (CBS)*: Suppose that $f_{[n]}, g_{[n]}$ are risky gambles that are such that $f_{[n]} \succsim g_{[n]}$ and that they have a common branch (z, p), $p > 0$. If $f'_{[n]}, g'_{[n]}$ are formed from $f_{[n]}, g_{[n]}$, respectively, by substituting (z', p) for the common branch (z, p), then $f'_{[n]} \succsim g'_{[n]}$. Observe that this holds for the class of EM-LWU models (10) and (11), which includes EM-EU. Note that in the risky case, the events are implicit and we assume that gambles under comparison have a common underlying Ω.

As already mentioned, in the remainder of this section we develop most of the arguments for EM-EU, i.e., with $\mathbf{p}_n = (p_1, \ldots, p_n)$,

$$U(g_{[n]}) = EU\left(g_{[n]}\right) + AI_n^{(1)}(\mathbf{p}_n), \tag{12}$$

where $I_n^{(1)}$ is the Shannon (1948) entropy. This case arises when either segregation or duplex decomposition is satisfied.

And, when gambles are based on uncertain events—i.e., they are presented in terms of events C_i rather than probabilities p_i—we consider the following entropy-modified form of SEU that is developed in Sect. 2.2.1 and embodied in (20) and (21):

$$U(g_n) = \sum_{i=1}^{n} U(x_i) S_\Omega(C_i) - A \sum_{i=1}^{n} S_\Omega(C_i) \log_2 S_\Omega(C_i), \tag{13}$$

where the utility of gambling term is the Shannon (1948) entropy of the subjective probabilities.

2.1 The Allais paradox

The following is a generic form for the gambles used in the study of the Allais paradox:

$$f = y, \quad g = (x, p; y, q; 0, s),$$
$$f' = (y,\ p + s; 0, q), \quad g' = (x, p; 0, q + s) \tag{14}$$

where $p + q + s = 1$, $p > 0, q > 0, s > 0$. The modal pattern of preference across people, also shown by some individuals, is a preference of f to g and g' to f', at least for the classic example where

$$x = \$\ 5M,\ \ y = \$\ 1M,\ \ 0 = \$\ 0,$$
$$p = 0.10,\ \ q = 0.89,\ \ s = 0.01.$$

This behavior violates coalescing together with monotonicity (see Luce 2000, p. 46, for a discussion of these concepts and how the example violates the condition). Thus, the typical behavior violates EU.

Now, we show that EM-EU (12) can account for the above choices. Note that the following argument applies equally to the joint choice of g over f and f' over g'.

The EM-EU representations of the gambles in (14) are

$$U(f) = U(y),$$
$$U(g) = U(x)p + U(y)q + AI_3^{(1)}(p, q, s),$$
$$U(f') = U(y)(p + s) + AI_2^{(1)}(p + s, q),$$
$$U(g') = U(x)p + AI_2^{(1)}(p, q + s).$$

Then, with

$$\mathbb{U} := U(y)(1 - q) - U(x)p, \tag{15}$$
$$K(p, q) := I_2^{(1)}(1 - q, q) - I_2^{(1)}(p, 1 - p), \tag{16}$$

the Allais paradox

$$[U(f) - U(g)]\left[U(f') - U(g')\right] < 0$$

is equivalent to the criterion

$$\left[\mathbb{U} - AI_3^{(1)}(p, q, s)\right][\mathbb{U} + AK(p, q)] < 0. \tag{17}$$

For the numerical example of the Allais paradox stated above, $I_3^{(1)}(p, q, s) > 0$ and $K(p, q) > 0$ follows from the fact that $I_2^{(1)}(u, v)$ increases as $u \leq 1/2$ increases and $\frac{1}{2} > 1 - q = p + s > p$. With these constraints, the argument leading to (17) then gives the classic pattern $f \succ g$ and $g' \succ f'$ iff

$$A < \min\left(\frac{\mathbb{U}}{I_3^{(1)}(p,q,s)}, \frac{-\mathbb{U}}{K(p,q)}\right).$$

Notice that this requires $A < 0$, which makes sense as it corresponds to an aversion for gambling.

2.2 The Ellsberg paradox

2.2.1 A special form for H_n in the uncertain case

The theory for uncertain alternatives in part (i) of Theorem 13 of Luce et al. (2007) leads to a representation of the form: with $\Omega = \bigcup_{i=1}^{n} C_i$,

$$U(g_n) = SEU\left(g_{[n]}\right) + K(\Omega) - \sum_{i=1}^{n} K(C_i) S_\Omega(C_i), \tag{18}$$

where

$$SEU\left(g_{[n]}\right) = \sum_{i=1}^{n} U(x_i) S_\Omega(C_i)$$

with S_Ω finitely additive.[5] Now we add conditions that allows us to derive a more specific form for the element of chance term.

Proposition 5 *Assume that* (i) *Assumption 1 holds* (ii) *the representation* (18) *holds* (iii) *the preference ordering over event-based gambles is compatible with the weights in the representation* (18) *in the sense that: with*

$$\Omega = \bigcup_{i=1}^{n} C_i,\ \ \Omega' = \bigcup_{i=1}^{n} C_i',\ \ \Omega_0 = \bigcup_{i=1}^{m} D_i,\ \ \Omega_0' = \bigcup_{i=1}^{m} D_i',$$

we have

$$\begin{aligned} &(x_1, C_1; \ldots; x_n, C_n) \succsim (y_1, D_1; \ldots; y_m, D_m) \\ \Leftrightarrow\ &(x_1, C_1'; \ldots; x_n, C_n') \succsim (y_1, D_1'; \ldots; y_m, D_m') \\ &\quad \textit{whenever}\ S_\Omega(C_j) = S_{\Omega'}(C_j')(\forall j = 1, \ldots, n) \\ &\qquad \textit{and}\ S_{\Omega_0}(D_j) = S_{\Omega_0'}(D_j')\ (\forall j = 1, \ldots, m). \end{aligned} \tag{19}$$

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[5] That Theorem has a part (ii) with S_Ω p-additive. Proposition 6 of Ng et al. (2007) includes a condition that leads to just the finitely additive case.

Then (18) becomes

$$U(g_n) = SEU\left(g_{[n]}\right) + AI_n^{(1)}(C_1, \ldots, C_n), \tag{20}$$

where

$$I_n^{(1)}(C_1, \ldots, C_n) = -\sum_{i=1}^{n} S_\Omega(C_i) \log_2 S_\Omega(C_i). \tag{21}$$

The $I_n^{(1)}$ term is the Shannon (1948) entropy in terms of the additive subjective probabilities.

2.2.2 Explanation of the paradox

We now provide an explanation of the Ellsberg paradox in terms of the entropy-modified form of SEU given in (20). Our explanation is incomplete as the paradox is usually studied with gambles that involve both events that are quite uncertain (often called ambiguous) and others for which a probability is given, i.e., they involve *mixed uncertainty and risk*, and it is an open problem to axiomatize such cases in our framework (see Ng et al. 2007 for a discussion of this problem and Maccheroni et al. 2006 for more general representations somewhat related to our own). Nonetheless, in our explanation, we take a tack common in the utility literature of restricting the finitely additive weights S_Ω to values that are compatible with the restrictions given by the specification of the paradox.

The Ellsberg paradox of Ellsberg (1961) in coalesced form[6] involves choices between f vs. g and f' vs. g' where, with money consequences ($x := \$x, x \in \mathbb{R}$),[7]

$$\begin{aligned} f &\sim (x, R; 0, G \cup Y) \equiv (x, p; 0, 1-p) \\ g &\sim (x, G; 0, R \cup Y) \\ f' &\sim (x, R \cup Y; 0, G) \\ g' &\sim (x, G \cup Y; 0, R) \equiv (x, 1-p; 0, p) \end{aligned}$$

with $x \succ 0$. The probabilities of G and of Y are not specified, whereas $\Pr(R) = 1 - \Pr(G \cup Y) = p$. In the classic example, where $\Pr(R) = p = 1/3$ and $\Pr(G \cup Y) = 1 - p = 2/3$, and $x = \$100$, people typically pick f over g and g' over f'. The usual argument suggesting that this result is paradoxical is to note that for any p one can (rationally) ignore event Y because the consequences in each pair are identical given Y. Once the event Y is ignored, the remaining structure in each pair is identical.

[6] If the gambles are presented in uncoalesced form, then the following explanations of the paradox require the additional assumption that the participants convert the gambles to their coalesced forms.

[7] The event notation R, G, Y arose from the interpretation of the chance experiment being a draw from an urn with red, green, and yellow balls.

The choice of f over g and g' over f' is clearly incompatible with SEU since, using the finite additivity of $S_\Omega = S_{R\cup G\cup Y}$, which we write as S, we have

$$U(f) > U(g) \Leftrightarrow S(R) > S(G),$$
$$U(f') < U(g') \Leftrightarrow S(R\cup Y) < S(G\cup Y) \Leftrightarrow S(R) < S(G),$$

which is impossible.

Note that the following argument applies equally to the joint choice of g over f and f' over g'. Assume that the decision maker selects finitely additive $S(R)$, $S(G)$, $S(Y)$ that are consistent with the given probability constraints. In particular, the decision maker sets $S(R) = p = 1/3$, $S(G) + S(Y) = S(G \cup Y) = 1 - p = 2/3$, $S(G) = q$, $0 \le q \le 1 - p$, and so implies $S(Y) = 1 - p - q$. Then, the entropy-modified form of SEU given by (20) becomes

$$\begin{aligned}
U(f) &= U(x)p + AI_2^{(1)}(p, 1-p),\\
U(g) &= U(x)q + AI_2^{(1)}(q, 1-q),\\
U(f') &= U(x)(1-q) + AI_2^{(1)}(q, 1-q),\\
U(g') &= U(x)(1-p) + AI_2^{(1)}(p, 1-p).
\end{aligned} \tag{22}$$

Then with K defined by (16), the Ellsberg paradox

$$[U(f) - U(g)]\left[U(f') - U(g')\right] < 0$$

is equivalent to the criterion

$$A^2K(p,q)^2 > U(x)^2(p-q)^2.$$

Thus, the paradox does not occur if $A = 0$ or $K(p,q) = 0$. It does occur if $K(p,q) \neq 0$ and A is sufficiently large, either positively or negatively. Similarly, for fixed A and $p \neq q$, the paradox is predicted to disappear for sufficiently large (positive or negative) $U(x)$. And the paradox occurs in the classic form of $f \succ g$ and $g' \succ f'$ iff

$$AK(p,q) < U(x)\min(p-q, q-p).$$

2.3 Data and independence properties

Two basic principles are useful in deriving the properties of EM-EU and in comparing them with those of EU. First, the properties of EM-EU agree with those of EU when either $A = 0$ or when the Shannon entropy terms $I_n^{(1)}$ in the various gambles under consideration are related in specific ways (some of which we illustrate below). Second, the properties of EM-EU are likely to differ from those of EU when $A \neq 0$ and the Shannon entropy terms $I_n^{(1)}$ in the various gambles under consideration are not equal and do not "cancel" in appropriate ways. Note that such $I_n^{(1)}$ values are numerically equal either if the probability distributions underlying the gambles are identical or "by chance," i.e., due to the particular values of the probabilities involved in the gambles, even if the relevant probability distributions are not identical. We do not need to concern ourselves with the latter possibility as all the independence conditions are stated for general probability distributions. Thus, we may assume that if the two gambles being compared involve different probability distributions, then the $I_n^{(1)}$ value associated with each gamble is different. We then need to check whether or not we can countermand the dictates of EU by the choice of the magnitude and sign of $A \neq 0$. Of course, one needs to study specific data, preferably for single participants, to see whether the pattern of failures of EU can be fit by the EM-EU model with the value of A possibly different for different participants.

We now apply the above basic principles to the main theoretical results, and some of the data, for a set of "independence" conditions that have been studied extensively by Birnbaum in collaboration with others (see, e.g., Birnbaum 2007, and see Marley and Luce 2005 for an integrative summary of these conditions and of their relations to current rank-dependent utility theories). We present the arguments for EM-EU in detail for a class of conditions that includes *branch independence* (*BI*), below, as a special case, and then state the results for the other conditions as they follow from similar arguments. Definitions of all these conditions are given in the Sect. 4.2 of the Appendix.

Each condition involves two pairs of gambles $(f_{[n]}, g_{[n]})$ and $(f'_{[n]}, g'_{[n]})$ with $f_{[n]}$ and $f'_{[n]}$ based on one non-trivial, complete probability distribution $\mathbf{p}_n$, i.e., $p_i > 0$ and $\sum_{i=1}^{n} p_i = 1$, and with $g_{[n]}$ and $g'_{[n]}$ based on a second, possibly different, non-trivial, complete probability distribution $\mathbf{q}_n$. Also, in particular cases, such as branch independence, certain specified relations hold between the pure consequences in the four gambles. We consider when the following condition holds or fails:

$$f_{[n]} \succsim g_{[n]} \Leftrightarrow f'_{[n]} \succsim g'_{[n]}. \tag{23}$$

A violation of (23) for EM-EU (12), occurs if:[8]

$$0 > [U(f) - U(g)]\left[U(f') - U(g')\right]$$

which is equivalent to

$$\begin{aligned} 0 > &\left[EU(f) - EU(g) + AI(\mathbf{p}) - AI(\mathbf{q})\right] \\ &\times \left[EU(f') - EU(g') + AI(\mathbf{p}) - AI(\mathbf{q})\right]. \end{aligned}$$

Then, with

$$\mathbb{U} := EU(f) - EU(g), \mathbb{U}' := EU(f') - EU(g'), \overline{I} := I(\mathbf{p}) - I(\mathbf{q}),$$

[8] Where we have dropped $_{[n]}$.

a violation occurs if

$$0 > \left[\mathbb{U} + A\overline{I}\right][\mathbb{U}' + A\overline{I}].$$

Note that if $\mathbb{U} = \mathbb{U}'$, then the right hand expression is a square, hence cannot be less than zero. If $\mathbb{U} \neq \mathbb{U}'$, then a violation occurs if

$$\text{either} \quad \mathbb{U} > -A\overline{I} > \mathbb{U}' \quad \text{or} \quad \mathbb{U}' > -A\overline{I} > \mathbb{U}.$$

As an example where (23) holds, consider $n = 3$, $\mathbf{p}_3 = (p_1, p_2, r)$ and $\mathbf{q}_3 = (q_1, q_2, r)$ arbitrary non-trivial complete probability distributions, and consequences $x_1, y_1, x_2, y_2, z, z'$ with $y_1 \succ x_1 \succ x_2 \succ y_2 \succ e$ and $y_2 \succ z \succ e$, $y_2 \succ z' \succ e$. Then *branch independence of type* $(3, 3)^2$ states that:

$$f_{[3]} \sim (x_1, p_1; x_2, p_2; z, r) \succsim (y_1, q_1; y_2, q_2; z, r) \sim g_{[3]} \tag{24}$$

iff

$$f'_{[3]} \sim (x_1, p_1; x_2, p_2; z', r) \succsim (y_1, q_1; y_2, q_2; z', r) \sim g'_{[3]}. \tag{25}$$

Note that under EM-EU, the common term associated with the branches (z, r) and (z', r) both cancel, and so the above gambles are such that

$$\begin{aligned} EU(f_{[3]}) - EU(g_{[3]}) &= U(x_1)p_1 + U(x_2)p_2 - U(y_1)q_1 - U(y_2)q_2 \\ &= EU(f'_{[3]}) - EU(g'_{[3]}), \end{aligned}$$

which was shown above to be sufficient for the condition to hold. In fact, all cases of branch independence when $n = 3$ reduce to such a condition, and hence EM-EU predicts that they all hold. EM-EU also satisfies all forms of the natural generalization of the condition to gambles for each $n \geq 3$.

Thus, the following are EM-EU's predictions for branch independence as compared to the data: the 5 types of unrestricted BI hold—there are no data for these conditions; the 2 types of restricted BI hold, contrary to the data; the 3 types of co-ranked BI hold, which agrees with the data.

Using similar arguments for other properties that have been studied empirically, we find that EM-EU predicts the following (Definitions are given in Sect. 4.2 of the Appendix): that 3-distribution independence (lower, upper and lower/upper) and 4-distribution independence hold, contrary to some of the data. The property called k-interval independence holds, which is contrary to the data. Lower and upper tail independence holds. This property has not been tested adequately—see Marley and Luce (2005).

When the Shannon entropy terms $I_n^{(1)}$ in EM-EU impact the hypothesis and conclusion of a condition in different ways, we need to check whether there is at least one value A for which the condition fails. In this way, we obtain the following predictions for EM-EU for the other independence conditions in Marley and Luce (2005): None of common consequence independence, common ratio independence or cumulative independence hold, in agreement with the data.

In summary, EM-EU accommodates various of the data obtained in tests of independence conditions, but, contrary to the available data, it predicts that the following hold: the 2 types of restricted BI; 3-distribution independence (lower, upper and lower/ upper) and 4-distribution independence; and k-interval independence.

2.4 Additional data not currently accommodated

In this subsection we discuss a new set of data presented in Birnbaum (2007) that appear not to be accounted for by any of the standard models including our new ones. These data are consistent with Birnbaum's ad hoc rank-dependent TAX representation. We summarize the results, consider whether the appropriate data have been collected, and, assuming that the results are valid, speculate on a way that the final representations might be modified so as to explain the data better.

Consider $x_1 > x_2 > x_3$, the gamble $g = (x_1, p; x_2, 1 - 2p; x_3, p)$ and the two event splittings of that gamble that yield

$$\overline{g} = (x_1, \frac{p}{2}; x_1, \frac{p}{2}; x_2, 1 - 2p; x_3, p)$$

and

$$\underline{g} = (x_1, p; x_2, 1 - 2p; x_3, \frac{p}{2}; x_3, \frac{p}{2}).$$

Birnbaum (2007) reports data on binary choices between gambles (averaged over participants) for which the estimates show $\overline{g} \succ g \succ \underline{g}$. Note that coalescing implies that $g \sim \overline{g} \sim \underline{g}$**,** and thus any of the usual rank dependent models including as special cases EU, which satisfies coalescing, cannot account for these data. Also, the three gambles all have the same EU and $\overline{g}$ and $\underline{g}$ have the same entropy of degree ρ, $I_n^{(\rho)}$, because, by the symmetry of $I_n^{(\rho)}$, $I_4^{(\rho)}(\frac{p}{2}, \frac{p}{2}, 1 - 2p, p) = I_4^{(\rho)}(p, 1 - 2p, \frac{p}{2}, \frac{p}{2})$. Therefore, either $\overline{g} \sim \underline{g} \succ g$ or $\prec g$, and so modifying any utility representation that fails by adding entropy of degree ρ does not account for the data. Finally, using the symmetry of the entropy of degree $\rho \neq 1$, it is easily checked that, when the pure consequences are all gains (respectively, all losses), EM-LWU cannot account for the data.

Two issues that must be raised are the nature of the data, namely binary choices, and the data analysis, namely, averages over participants. We would be more convinced if the results were reproduced (preferably within participants) in an experiment where monetary $CE(\overline{g})$ and $CE(\underline{g})$ are determined using a standard procedure for constructing a psychometric function by collecting choices between money amounts and $\overline{g}$ and $\underline{g}$ are estimated as the 50% point of the resulting psychometric function. The statistical test is whether or not $CE(\overline{g}) - CE(\underline{g}) = 0$.

Another possibility within the present theoretical context may be that this finding, if sustained, cannot be accommodated because we do not in any way distinguish between gains and losses relative to a reference point, which may, or may not, be the null consequence e. These data may mean that for a more descriptive theory, the

Table 1 Summary of representations for risky gambles

	U(KE)			UofG	
$S_{\Omega}^{1/\rho}$ is FA	Seg	DD		Seg	DD
$\rho = 1$	EU	EU	$+ \ A\times$	$I^{(1)}$	$I^{(1)}$
$\rho \neq 1$	—	LWU		—	$I^{(\rho)}$

,

where the entropy terms are defined by

$$I_n^{(\rho)}(p_1, \ldots, p_n) := \begin{cases} -\sum_{i=1}^{n} p_i \log_2 p_i, & \rho = 1 \\ \frac{1}{2^{1-\rho}-1}[\sum_{i=1}^{n} p_i^{\rho} - 1], & 0 < \rho \neq 1 \end{cases}.$$

Codes: DD = Duplex Decomposition, EU = Expected Utility, FA = Finitely Additive, KE = Kernel Equivalent, LWU = Linear Weighted Utility, Seg = Segregation, UofG = Utility of Gambling

differential role of gains and losses have to be taken into account. Such a partition is, at best, fairly complicated (see Luce 2000) but vastly more so when there is a utility of gambling term, which itself can be positive or negative. This simply has not been thought through at this time.

3 Conclusions

The purpose of the article was to show the form of the utility of gambling in the context of risk, rather than uncertainty. In Sect. 1 we specialized the representation for uncertain gambles obtained in Luce et al. (2007) to risky (known probabilities) gambles on the assumption of an implicit event structure. This led to the representations shown in Table 1 involving standard forms of entropy.

In Sect. 2, we saw that EM-EU (12), and the entropy-modified form of SEU (13), together, can accommodate the classic empirical paradoxes plus some of the other more recent experiments described in Sect. 2.3. These representations seem to have a substantial advantage over most of the earlier proposals.

4 Appendix

4.1 Proofs

Proof of Proposition 1 Let U be the additive representation over consequences, event-based gambles, and their joint receipt. We may extend U to risky gambles on $\Omega \subseteq \Omega_0$ as follows:

$$U^*(x_1, \Pr(C_1|\Omega); \ldots; x_n, \Pr(C_n|\Omega)) := U(x_1, C_1; \ldots; x_n, C_n). \qquad (26)$$

Because of (4), U^* is well-defined and it clearly preserves the ordering on risky gambles. As U agrees with U^* over pure consequences, they also agree over certainty equivalents, and so we are justified in simply writing U rather than U^* for risky gambles. In particular, with $p_i = \Pr(C_i|\Omega)$, $i = 1, \ldots, m$, we can define

$$H_n(p_1, \ldots, p_n) := U(e, C_1; \ldots; e, C_n) := H_n(C_1, \ldots, C_n). \tag{27}$$

The event-based gamble representation

$$U(x, C; e, \Omega \backslash C) = U(x)S_\Omega(C) + U(e, C; e, \Omega \backslash C)$$

is converted by (26) to

$$\begin{aligned} &U(x, \Pr(C|\Omega); e, 1 - \Pr(C|\Omega)) \\ &\quad = U(x)S_\Omega(C) + U(e, \Pr(C|\Omega); e, 1 - \Pr(C|\Omega)). \end{aligned}$$

Fixing x at $x_0 \neq e$, and defining

$$f(t) := \frac{1}{U(x_0)}[U(x_0, t; e, 1 - t) - U(e, t; e, 1 - t)],$$

we get

$$S_\Omega(C) = f(Pr(C|\Omega)). \tag{28}$$

Using the choice property of S_Ω (proved to hold in Theorems 13 and 16 of Luce et al. 2007), and of Pr as a conditional probability, we see that f preserves division:

$$\begin{aligned} f\left(\frac{\Pr(C|\,\Omega_0)}{\Pr(\Omega|\,\Omega_0)}\right) &= f(\Pr(C|\Omega)) = S_\Omega(C) \\ &= \frac{S_{\Omega_0}(C)}{S_{\Omega_0}(\Omega)} = \frac{f(\Pr(C|\,\Omega_0))}{f(\Pr(\Omega|\,\Omega_0))}. \end{aligned}$$

The solution f of the above functional equation, being a bijection on [0, 1], is given by $f(t) = t^\rho$ for some constant $\rho > 0$, (3) holds. □

Proof of Theorem 3 1. Proposition 1 proves that (3) holds. However, by Theorem 13 of Luce et al. (2007), we know that S_Ω is either finitely additive, in which case $\rho = 1$, or p-additive. The latter is ruled out as it is not compatible with (3). Thus, S_Ω is finitely additive.

2. Next, consider the branching relation (47), of Proposition 11 of Luce et al. (2007), which is satisfied under the assumptions of Theorem 13 of Luce et al. (2007):

$$H_n(C_1, \ldots, C_n) = H_2(C_1, C_2)S_\Omega(C_1 \cup C_2) + H_{n-1}(C_1 \cup C_2, C_3, \ldots, C_n),$$

in which $\cup_{i=1}^{n} C_i = \Omega$. Using (3) with $\rho = 1$, i.e., $S_\Omega(C_i) = \Pr(C_i|\Omega) = p_i$, and (27) this becomes the classic (probability) recursive formula ((5) with $\rho = 1$)

$$H_n(p_1, \dots, p_n) = (p_1 + p_2) H_2\left(\frac{p_1}{p_1 + p_2}, \frac{p_2}{p_1 + p_2}\right) + H_{n-1}(p_1 + p_2, p_3, \dots, p_n),$$

which with condition (iv) of Proposition 1 (or just local boundedness) leads, by Proposition 2, to a constant times the Shannon entropy (8). The representation (7) now follows from the event-based representation of Case (i) of Theorem 13 of Luce et al. (2007). □

Proof of Theorem 4 1. The proof of (3) is given by Proposition 1.
2. The representation

$$U(g_{[n]}) = \sum_{i=1}^{n} U(x_i) S_\Omega(C_i) + K(\Omega) - \sum_{i=1}^{n} K(C_i) S_\Omega(C_i), \tag{29}$$

with $\Omega = \cup_{i=1}^{n} C_i$, obtained in Theorem 16 of Luce et al. (2007) gives, in particular, the representation

$$H_n(C_1, \dots, C_n) = K(\Omega) - \sum_{i=1}^{n} K(C_i) S_\Omega(C_i). \tag{30}$$

We can easily confirm that (30) implies the branching relation

$$H_n(C_1, \dots, C_n) = S_\Omega(C_1 \cup C_2) H_2(C_1, C_2) + H_{n-1}(C_1 \cup C_2, C_3, \dots, C_n).$$

Using (27) and (3), this becomes the classic (probability) recursive formula (5) leading, by Proposition 2, to

$$H_n = A I_n^{(\rho)} \tag{31}$$

for some constant A. The representation (10) now follows from (29), with (3) and (31). □

Proof of Proposition 5 We can confirm easily that

$$H_n(C_1, \dots, C_n) = K(\Omega) - \sum_{i=1}^{n} K(C_i) S_\Omega(C_i)$$

satisfies the recursion

$$H_n(C_1, C_2, C_3, \dots, C_n) = S_\Omega(C_1 \cup C_2) H_2(C_1, C_2) + H_{n-1}(C_1 \cup C_2, C_3, \dots, C_n). \tag{32}$$

Also, because of (19), we may define $\tilde{H}_n(S_\Omega(C_1), \ldots, S_\Omega(C_n)) := H_n(C_1, \ldots, C_n)$ and, since S_Ω is finitely additive, we may also define $p_j := S_\Omega(C_j)$. Then, using (32), invoke Proposition 2 for the $\rho = 1$ case, giving (20). □

4.2 Definitions of independence conditions

The following definitions are all found in Marley and Luce (2005), which gave general formulations of Birnbaum's specific examples.

First, we need to define various terms that help in stating the definition of branch independence conditions compactly. We say that a branch (z, E) of a 3-component gamble is in *position* i, $i = 1, 2, 3$, if z is in rank position i. A branch (z, E) that occurs in each of two gambles based on the same "universal" event is of *type* (i, j) if z is in position i in the first gamble and in position j in the second gamble. For convenience, we refer occasionally to the given pair of gambles as of type (i, j). The definitions of branch cancellation and independence given below considers pairs of such gambles with the restriction that $x' \succ x \succ y \succ y' \succ e$, in which case there are 5 distinct possibilities for the type of z, namely (1, 1), (1, 2), (2, 2), (3, 2), (3, 3). The three symmetric cases, (1, 1), (2, 2), and (3, 3), are called *co-ranked* with, respectively, upper (U), intermediate (I), and lower (L) positions.

Finally, for two pairs of pairs of such gambles, say

$$(x, C; y, D; z, E), (x', C'; y', D'; z, E),$$

and

$$(x, C; y, D; z', E), (x', C'; y', D'; z', E),$$

we say that the branches (z, E) and (z', E) have a *common location* (i, j) iff each is of type (i, j). In the theoretical development below, we restrict attention mainly to the 5 common locations, which we denote by $(1, 1)^2$, $(1, 2)^2$, $(2, 2)^2$, $(3, 2)^2$, $(3, 3)^2$. Consistent with our prior use of the term, the cases $(1, 1)^2$, $(2, 2)^2$, $(3, 3)^2$ are called *co-ranked*.

Definition 6 Branch independence (BI) of type $(i, j)^2$ is defined by: Given consequences x, x', y, y', z, z' with $x' \succ x \succ y \succ y' \succ e$, z, z' with common location (i, j) in the gambles below but otherwise arbitrary, and all events C, C', D, D', and non-null E where (C, D, E) and (C', D', E) are both partitions of the same event,

$$(x, C; y, D; z, E) \succsim (x', C'; y', D'; z, E)$$

iff

$$(x, C; y, D; z', E) \succsim (x', C'; y', D'; z', E).$$

Restricted BI of type $(i, j)^2$ holds if the above definition holds with the restriction that $C' = C$, $D' = D$.

Co-ranked BI of type $(i, i)^2$ holds if BI of type $(i, i)^2$ holds. These cases are called upper, intermediate, and lower BI, respectively, for $i = 1, 2, 3$. If this condition is satisfied for all 3 types, then we simply say that **co-ranked BI** holds.

Definition 7 **3-distribution independence (3-DI)** is defined by: For $x' \succ x \succ y \succ y' \succ e, z \succ e, z' \succ e, p, p' \in (0, 1/2]$,

$$(x, p; y, p; z, 1-2p) \succ (x', p; y', p; z, 1-2p),$$

iff

$$(x, p'; y, p'; z', 1-2p') \succ (x', p'; y', p'; z', 1-2p').$$

Lower 3-DI (L3-DI) holds if $y' \succ z = z'$.
Upper 3-DI (U3-DI) holds if $z = z' \succ x'$.
Lower/Upper 3-DI (L/U3-DI) holds if $z' \succ x', y' \succ z$.

Definition 8 **4-distribution independence (4-DI)** is defined by: For $z' \succ x' \succ x \succ y \succ y' \succ z \succ e$ and $p, r, r', r+2p, r'+2p \in [0, 1]$, then

$$(z', 1-r-2p; x, p; y, p; z, r) \succ (z', 1-r-2p; x', p; y', p; z, r),$$

iff

$$(z', 1-r'-2p; x, p; y, p; z, r') \succ (z', 1-r'-2p; x', p; y', p; z, r').$$

Definition 9 **Common consequence independence** (CCI) is satisfied if, *for all* $p, q, r, p+r, q+r \in [0, 1]$,

$$(x, p; e, 1-p) \succsim (y, q; e, 1-q) \tag{33}$$

is equivalent to

$$(x, p; z, r; e, 1-p-r) \succsim (y, q; z, r; e, 1-q-r).$$

Definition 10 **Common ratio independence** is satisfied if, for all $p, q \in [0, 1]$, (33) is equivalent to

$$(x, ap; e, 1-ap) \succsim (y, aq; e, 1-aq) \qquad (a \leq 1/\max(p, q)).$$

Definition 11 The following forms of **cumulative independence (CI)** are defined where $z' \succ x' \succ x \succ y \succ y' \succ z \succ e$:

Lower CI

$$(x, C; y, D; z, E) \succ (x', C; y', D; z, E)$$

implies

$$(x, C \cup D; y', E) \succ (x', C; y', D \cup E).$$

Upper CI

$$(z', E; x, C; y, D) \prec (z', E; x', C; y', D)$$

implies

$$(x', E; y, C \cup D) \prec (x', C \cup E; y', D).$$

Definition 12 Let

$$\begin{aligned}
A_k &= (x_1, C_1; x_2, C_2; \cdots; x, C_k; ; \ldots; x_n, C_n),\\
B_k &= (x_1, C_1; x_2, C_2; \cdots; y, C_k; \ldots; x_n, C_n),\\
A'_k &= (y_1, D_1; y_2, D_2; \cdots; x, D_k; \ldots; x_n, D_n),\\
B'_k &= (y_1, D_1; y_2, D_2; \cdots; y, D_k; \ldots; x_n, D_n),
\end{aligned}$$

where

$$\bigcup_{j=1}^{n} C_j = \bigcup_{j=1}^{n} D_j, \text{ and } C_k = D_k.$$

Then **interval independence at position** k **(kII)** is satisfied provided that

$$U(A_k) - U(B_k) = U(A'_k) - U(B'_k).$$

Lower II holds when $k = n$ and **upper II** holds when $k = 1$.

To formulate the following definition, consider a gamble

$$A = (x_1, C_1; \ldots; x_k, C_k; x_{k+1}, C_{k+1}; \ldots; x_n, C_n)$$

with $x_1 \succsim x_2 \succsim \ldots \succsim x_n \succsim e$. Let k be an index with $2 \le k \le n - 1$. The upper tail of A is the portion $A_u(k) := (x_1, C_1; \ldots; x_k, C_k)$ and the lower tail is the portion $A_l(k) := (x_{k+1}, C_{k+1}; \ldots; x_n, C_n)$.

Definition 13 Let A, B, A', B' be gambles of size n with a common universal set C and the same ordered consequences. Let $k \in \{2, \ldots, n - 1\}$.

Lower tail independence (LTI) holds if $A_l(k) = B_l(k)$, $A'_l(k) = B'_l(k)$, $A_u(k) = A'_u(k)$, and $B_u(k) = B'_u(k)$ implies that $A \succsim B$ iff $A' \succsim B'$.

Upper tail independence (UTI) holds if the same condition is true with the u and l interchanged.

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