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Functional equations arising in a theory of rank dependence and homogeneous joint receipts[☆]

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Dedicated to Lajos Tamássy on his 80th birthday

Abstract

This paper focuses on a class of utility representations of uncertain alternatives with two possible consequences (binary gambles) when they are linked via a distributivity property called segregation to an operation of joint receipt, which may be non-commutative. The assumption that the gambling structure and the joint receipt operation both have homogeneous representations that are order preserving leads to a functional equation that has too many solutions to be useful for characterizing a reasonably specific utility representation. A plausible restriction on the form of the utility of gambles leads to the functional equation

$$H[pw, qK(w)] = q\theta^{-1}\left[\psi(w)\theta\left(\frac{p}{q}\right)\right] \quad (w \in [0,1], q \in]0,k[, p \in [q,k[)$$

whose solution shows that it is equivalent to the widely studied rank-dependent representation. That representation is related to segregation via a particular homogeneous representation of joint receipt. In that case the above functional equation simplifies to

$$G[vF(z)] = A(v)G(vz) + G(v) \quad (v > 0, z \geqslant 0),$$

and it is solved under smoothness conditions. Two families of solutions arise, one commutative and associative and the other non-commutative and bisymmetric, but not associative. These are then interpreted in utility terms. The former is well studied. Parallel to earlier results for the commutative case, an axiomatic treatment of the non-commutative case is provided. An application to psychophysics favors non-commutativity. Several open problems are mentioned.

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The focus of our paper is a class of utility representations of uncertain alternatives (binary gambles) that are linked to a not necessarily commutative operation (joint receipt) by a distributivity property called segregation. The assumption of the gambling structure and the joint receipt operation both having order preserving homogeneous representations, together with further plausible restrictions, lead to functional equations. The two families of solutions that arise, one commutative and associative and the other non-commutative and bisymmetric, but not associative, are interpreted in utility terms. An application to psychophysics favors non-commutativity. At the end of the paper we state several open problems.

The basic utility background and motivation can be found in Luce (2000, pp. 1–61) and an alternative psychophysical interpretation, which motivated the study of the non-commutative case, is in Luce (2002).

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Here we repeat without much elaboration certain basic concepts, definitions, and assumptions that will be needed, using the utility interpretations and notations. We will, however, limit the assumptions made in order to accommodate the psychophysical interpretation as well, which is spelled out further in Section 4.4.

1. Background assumptions

1.1. The primitives

We have the following primitives:

- Let X denote a set with typical elements $x, y \in X$. In utility theory, these are taken to be valued entities that are free from uncertainty.
- Let $\mathscr{E}_{\mathbf{E}}$ denote an algebra of events arising from an "experiment" or "chance phenomenon" $\mathbf{E}$ with the null event $\emptyset$ and universal event Ω. A typical event is $C \in \mathscr{E}_{\mathbf{E}}$
- For each $x, y \in X$ and $C \in \mathscr{E}_{\mathbf{E}}$, let $(x, C; y)$ denote a binary gamble in which the holder of the gamble receives $x \in X$ if C occurs when $\mathbf{E}$ is executed and $y \in X$ if C fails to occur. If f, g denote (in some intuitive sense) independently realized binary gambles, then $(f, C; g)$ denotes a binary compound gamble, compound in the sense that its consequences are themselves gambles.
- For (independently realized) gambles f, g, let $f \oplus g$ denote having or receiving the pair (f, g). We speak of $\oplus$ as *joint receipt*. Over the domain $\mathscr{D}$ described next, we do not assume that $\oplus$ is a *closed* operation. We treat $f \oplus g$ as a gamble and, in particular (cf. below), $x \oplus y$ as a consequence in X, for $x, y \in X$.
- Define $\mathscr{D}_i$ inductively as follows: $\mathscr{D}_0 = X$, $\mathscr{D}_i = \mathscr{D}_{i-1} \cup \{(f, C; g), f \oplus g \mid f, g \in \mathscr{D}_{i-1}, C \in \mathscr{E}_{\mathbf{E}}\}$ $(i = 1, 2, \ldots)$. It will suffice to work with $\mathscr{D} = \mathscr{D}_3$ with, for the most part, $f, g \in \mathscr{D}_2$ and $(f, C; g),\ f \oplus g \in \mathscr{D}$.
- Let $\succsim$ be a *weak order*, i.e., transitive and connected, over $\mathscr{D}$ which is interpreted as a preference order. Define $\sim$ and $\succ$ by: $f \sim g \Leftrightarrow (f \succsim g$ and $g \succsim f)$; $f \succ g \Leftrightarrow (f \succsim g$ and not $g \succsim f)$, respectively.
- The set X is assumed to be so rich that for each gamble $f \in \mathscr{D}_3$, there exists an element $x(f) \in X$ such that $x(f) \sim f$; it is called the *certainty equivalent* of f.
- $e \in X$ is a distinguished element that is interpreted as no change from the status quo.
- Elements of $\mathscr{D}^+ := \{f \mid f \in \mathscr{D}$ and $f \succsim e\}$ are called *gains* and those of $\mathscr{D}^- := \{f \mid f \in \mathscr{D}$ and $f \precsim e\}$ are called *losses*. Note that $\mathscr{D}^+$ includes pure consequences (elements x of X with $x \succsim e$); gambles such as f and g, $f \succsim e$, $g \succsim e$; and compound gambles such as $(f, C; g) \succsim e$. Unless otherwise stated, consequences and gambles are assumed to be in $\mathscr{D}^+$.

1.2. Assumptions about gambles

The gambling structure is assumed throughout to satisfy for all $f, f', g, g' \in \mathscr{D}_2, C \in \mathscr{E}_{\mathbf{E}}$ the following.

- *Left-consequence monotonicity*:
$$f \succsim f' \Leftrightarrow (f, C; g) \succsim (f', C; g) \quad (C \neq \emptyset). \tag{1}$$
- *Right-consequence substitutability*:
$$g \sim g' \Rightarrow (f, C; g) \sim (f, C; g'). \tag{2}$$

[It is usual in utility theory to assume also the counterpart of (1), right-consequence monotonicity, which of course implies (2), but we do not assume this here in order to accommodate the psychophysical interpretation of Section 4.4.]

- *Idempotence*:
$$(f, C; f) \sim f.$$
- *Certainty*:
$$(f, \Omega; g) \sim f.$$
- *Nullity*:
$$(f, \emptyset, g) \sim g.$$

1.3. Assumptions about representations

Suppose that U maps $\mathscr{D}^+$ onto $[0, k[(k \in]0, \infty])$ and W maps $\mathscr{E}_{\mathbf{E}}$ onto $[0, 1]$.

- The pair (U, W) is said to form a *utility representation of* $\langle \mathscr{D}^+, \succsim, e \rangle$ if
  (i) U is order preserving,
  $$f \succsim g \Leftrightarrow U(f) \geqslant U(g) \quad (f, g \in \mathscr{D}^+);$$
  (ii) $U(e) = 0$;
  (iii) W is order preserving in the sense that
  $$(f, C; g) \succsim (f, D; g) \Leftrightarrow W(C) \geqslant W(D)$$
  $$(f \succ g, f, g \in \mathscr{D}_2^+);$$
  (iv) for $f \succsim g$, $U(f, C; g)$ depends upon f, g, and C only through $U(f)$, $U(g)$, and $W(C)$.

By the assumption of certainty equivalents, U maps the set X *onto* $[0, k[$. This fact is used below without further comment.

We have limited ourselves to the case $f \succsim g$ so that the results apply to the psychophysical context. For applications to utility theory, the case $f \prec g$ is often included by assuming the property of *complementarity*

$(f, C; g) \sim (g, \bar{C}; f) \quad (f \prec g),$

where $\bar{C} = \Omega \backslash C$.

For a utility representation (U, W), satisfying (iv), we can define

$M_{W(C)}[U(f), U(g)] := U(f, C; g)$

$(f, g \in \mathscr{D}_2^+, f \succsim g, C \in \mathscr{E}_{\mathbf{E}}).$

Setting $w = W(C), p = U(f), q = U(g), \ (p, q \in [0, k[)$, this may be rewritten, by (i), as

$$M_w(p, q) = U(U^{-1}(p), W^{-1}(w); U^{-1}(q)) \quad (0 \leqslant q \leqslant p < k, w \in [0, 1]), \tag{3}$$

where $U^{-1}(p)$ denotes any element in the equivalence class of f under $\sim$, etc. By left-consequence monotonicity and right-consequence substitutability, respectively, the choice for $U^{-1}(p)$ and $U^{-1}(q)$ within their equivalence classes is immaterial. And by property (iii) of a utility representation, the choice for $W^{-1}(w)$ within its equivalence class is immaterial.

When dealing with $M_w(p, q)$ with all three variables free we speak of the function M and when w is fixed we speak of the function M_w. Note that the functions M_w are onto $[0, k[$ because $(f, C; f) \sim f$ is mapped onto $[0, k[$.

Assuming the existence of a utility representation (U, W), and parallel to the gambling structure, we will also call M a representation. This representation M forms:

- An *intern map* if

$$p = M_1(p, q) > M_w(p, q) > M_0(p, q) = q \quad (w \in]0, 1[, 0 \leqslant q < p < k), \tag{4}$$

and satisfies, in parallel to properties of $(f, C; g)$ of the same name,

- *Idempotence* if

$$M_w(p, p) = p \quad (0 \leqslant p < k).$$

- *Certainty* if

$$M_1(p, q) = p \quad (0 \leqslant q \leqslant p < k). \tag{5}$$

- *Nullity* if

$$M_0(p, q) = q \quad (0 \leqslant q \leqslant p < k). \tag{6}$$

Definition. We say that M is in Class *IIIC*, written $M \in IIIC$, if it is strictly increasing in p for $w \neq 0$, strictly increasing in w for $p > q$, intern (4), and continuous in p for each (w, q) and continuous in w for each (p, q) $(0 \leqslant w \leqslant 1, \ 0 \leqslant q \leqslant p < k)$.

This assumption is made throughout.

Note that the property of nullity shows that M is not strictly increasing in p when $w = 0$.

1.4. Assumptions about joint receipt

Recall that we also have an operation $\oplus$ of joint receipt. We assume the following throughout.

- *Strictly left-monotonic increasing*:

$$f \succsim f' \Leftrightarrow f \oplus g \succsim f' \oplus g \quad (f, f', g \in \mathscr{D}_2).$$

- The element e is a *left identity* of $\oplus$:

$$f \sim e \oplus f \quad (f \in \mathscr{D}_2).$$

The above has been stated entirely in terms of left-hand properties; a dual theory based on right-hand properties is obvious.

1.5. Other properties

A number of other properties will be explored in the results below.

For either all gains or all losses, the gambling structure is said to satisfy

- *Event commutativity* if

$$((f, C; g), D; g) \sim ((f, D; g), C; g)$$

$$(f, g \in \mathscr{D}_1^+ \text{ or } f, g \in \mathscr{D}_1^-, \ C, D \in \mathscr{E}_{\mathbf{E}}).$$

For the case of gains and restricted to $f \succsim g$, event commutativity is equivalent to

$$M_w[M_{w'}(p, q), q] = M_{w'}[M_w(p, q), q] \quad (0 \leqslant q \leqslant p < k, \ w, w' \in [0, 1]). \tag{7}$$

For either all gains or all losses, the gambling structure and the joint receipt operation are said to satisfy

- *Segregation* if

$$(f, C; e) \oplus g \sim (f \oplus g, C; g) \quad (f, g \in \mathscr{D}_2^+ \text{ or } f, g \in \mathscr{D}_2^-).$$

We do not assume segregation for mixed gambles of gains and losses.

A representation M is said to be

- *Separable* if

$$M_w(p, 0) = wp \quad (p \in [0, k[, w \in [0, 1]). \tag{8}$$

- *Homogeneous* if

$$M_w(zp, zq) = zM_w(p, q)$$

$$(z > 0, p, q, zp, zq \in [0, k[, p \geqslant q).$$

This means that the representation is of *ratio scale* type in the language of measurement theory (Krantz, Luce, Suppes, & Tversky, 1971, p. 10; Luce & Narens, 1985; Narens, 1981).

- A *binary rank-dependent utility* (RDU) representation is given by

$$M_w(p,q) = pw + q(1-w) \quad (0 \leqslant q \leqslant p < k). \tag{9}$$

Note that an RDU representation is in the class *IIIC* and is separable, homogeneous, and strictly increasing also in q for $w<1$. If we permit $w \in [0,\infty[$ as in the psychophysical model of Luce (2002) and Section 4.4, then it is continuous but not increasing in q for $w \geqslant 1$.

2. Homogeneity of joint receipt

This paper focuses to a great extent on what happens when we combine assumptions about the representation M with the assumption that $\oplus$ has a *unit representation* (V, F, d) in the sense that there exist $V: \mathscr{D}^+ \xrightarrow{onto} [0,\infty[$, $F:[0,\infty[\xrightarrow{onto} [1,\infty[$, $d>0$ such that V preserves the order of $\succsim$ and $V(f \oplus g)$ is a homogeneous function of $V(f)$ and $V(g)$, that is,

$$V(f \oplus g) = \begin{cases} V(g)F\left(\frac{V(f)}{V(g)}\right) & (V(g)>0), \\ dV(f) & (V(g)=0). \end{cases} \tag{10}$$

Note that $\oplus$ satisfies strictly left-monotonic increasing iff F is strictly increasing and strictly right-monotonic increasing iff $F(z)/z$ is strictly decreasing. We will suppose, however, unless otherwise stated, only that $F(z)/z$ *is non-constant* (rather than strictly decreasing) when speaking of unit representations.

Theorem 1. *Suppose a gains structure* $\langle \mathscr{D}^+, \succsim, \oplus, e \rangle$ *has a representation* (U, W) *such that* M, *defined by* (3), *is in class IIIC and is homogeneous and separable. Suppose further that* $\oplus$ *satisfies the following: strictly left-monotonic increasing, e is a left identity, segregation, and it has a unit representation* (V, F, d). *Define the strictly increasing function* $G:[0,\infty[\xrightarrow{onto} [0,k[$ *by* $U = G(V)$. *Then there exist functions* $\sigma:]0,k[\to]0,\infty[$ *and* $\lambda:[1,\infty[\to [0,\infty[$, *such that* λ *is strictly increasing and continuous with* $\lambda(1)=0$, *and the following functional equation is satisfied*:

$$\lambda\left(\frac{G[vF(z)]}{G(v)}\right) = \frac{G(vz)}{\sigma[G(v)]} \quad (v \in]0,\infty[, z \in [0,\infty[). \tag{11}$$

Furthermore, the relation

$$U(f \oplus e) = G(dG^{-1}[U(f)]) \tag{12}$$

holds.

Proof. Setting $u = V(f), v = V(g), z = u/v$, and using the upper part of (10), we invoke Theorem 4 of Ng, Luce, and Aczél (2002) which asserts that there are functions σ, λ, meeting the asserted conditions, such that

$$U(f \oplus g) = U(g)\lambda^{-1}\left(\frac{U(f)}{\sigma[U(g)]}\right).$$

Thus,

$$\begin{aligned} G[vF(z)] &= G\left[V(g)F\left(\frac{V(f)}{V(g)}\right)\right] \\ &= G[V(f \oplus g)] \\ &= U(f \oplus g) \\ &= U(g)\lambda^{-1}\left(\frac{U(f)}{\sigma[U(g)]}\right) \\ &= G(v)\lambda^{-1}\left(\frac{G(vz)}{\sigma[G(v)]}\right), \end{aligned}$$

and rewriting yields (11). Relation (12) is obtained from the lower part of (10):

$$U(f \oplus e) = G[V(f \oplus e)] = G[dV(f)] = G(dG^{-1}[U(f)]).$$

□

Eq. (11) is quite weak in the sense of having many solutions, probably too many to be of use in characterizing useful forms for utility representations. Although we do not know all solutions, the following family of them establishes that many exist. Consider a case of $k=\infty$ and $G(v) = \sigma(v) = v$. Then (12) reduces to $U(f \oplus e) = dU(f)$ (a generalization of the right identity property), while (11) reduces to $\lambda = F^{-1}$. Thus, if one of F or λ is selected to be an *arbitrary* strictly increasing function satisfying the domain and range assumptions, then the other is determined.

A case somewhat more restrictive than (11), namely Eq. (36), is treated in the Corollary to Theorem 3 and solved under smoothness conditions in Theorem 4.

3. A restriction on the utility of gambles

Restrictions on the utility of a gamble, which seem quite general, are (I) that $W(\bar{C})$ depends only upon $W(C)$, i.e., $W(\bar{C}) = K[W(C)]$, where $K:[0,1] \xrightarrow{onto} [0,1]$ is strictly decreasing and thus continuous, and (II) that $U(f,C;g)$ depends upon $U(f)$, $U(g)$, and $W(C)$ only through the products $U(f)W(C)$ and $U(g)W(\bar{C}) = U(g)K[W(C)]$, i.e.,

$$U(f,C;g) = H(U(f)W(C), U(g)K[W(C)]) \quad (f \succsim g). \tag{13}$$

This form obviously includes RDU, (9), as the special case $H(s,t) = s+t$ and $K(w) = 1-w$.

Eq. (13) translates into

$$M_w(p,q) = H(pw, qK(w)) \quad (0 \leqslant q \leqslant p < k,\ w \in [0,1]), \tag{14}$$

where the domain of the function H is

$$D_H = \{(s,t) \mid s = pw, t = qK(w) \text{ for some } p,q \in [0,k[, p \geqslant q, w \in [0,1]\}$$

($p \geqslant q$ because $f \succsim g$). Clearly, $D_H \subseteq [0,k[\times[0,k[$. From K being strictly decreasing and onto $[0,1]$, it follows immediately that

$$K(0) = 1 \quad \text{and} \quad K(1) = 0. \tag{15}$$

In view of certainty (5) and nullity (6) we get

$$H(s,0) = s \quad \text{and} \quad H(0,t) = t. \tag{16}$$

This implies separability, $M_w(p,0) = wp$, in particular.

We begin by showing, without any added assumption, that H is homogeneous. As a preparation we define a strictly decreasing function $\Gamma :]0,1] \overset{onto}{\longrightarrow} [0,\infty[$ by

$$\Gamma(w) := \frac{K(w)}{w} \quad (w \in]0,1]), \tag{17}$$

and a strictly increasing $N : [0,\infty[\overset{onto}{\longrightarrow} [1,\infty[$ by

$$N\left(\frac{w}{K(w)}\right) = \frac{1}{K(w)} \quad (w \in [0,1[). \tag{18}$$

The two functions are related by

$$N(t) = \frac{t}{\Gamma^{-1}(1/t)} \quad (t>0). \tag{19}$$

Proposition 1. *Suppose that a gains structure $\langle \mathscr{D}^+, \succsim, e \rangle$ has a utility representation (U, W) such that $M \in IIIC$ satisfies* (14). *Then H is related to N in* (18) *by*

$$H(s,t) = tN\left(\frac{s}{t}\right) \quad ((s,t) \in D_H, t>0), \tag{20}$$

and H is homogeneous.

Proof. First, by idempotence, for all $r \in [0,k[$,

$$r = H[rw, rK(w)]. \tag{21}$$

Next, let $(s,t) \in D_H$ with $s>0$, $t>0$ be given. There exist $p,q \in]0,k[, p \geqslant q$, $w' \in]0,1[$, such that $s = pw'$ and $t = qK(w')$. We turn our attention to finding $r \in]0,k[, w \in]0,1[$ so that $s = rw$ and $t = rK(w)$. Recall that Γ of (17) is strictly decreasing, continuous, and maps $]0,1]$ onto $[0,\infty[$. Since $p \geqslant q$, $\Gamma(w') = K(w')/w' = (t/q)(p/s) = (t/s)(p/q) \geqslant t/s > 0$. Thus there exists a unique $w \in]0,1[$, with $w \geqslant w'$, such that $\Gamma(w) = t/s$. The next target is to find r so that $s = rw$, i.e. $pw' = rw$. Because $w' \leqslant w$, such r exists and $r \leqslant p$. Since $p \in]0,k[$, so is r. From the relations $s = rw$ and $t/s = \Gamma(w) = K(w)/w$ we get $t = rK(w)$ and this proves the existence of such r, w. Thus, by (21), $H(s,t) = H[rw, rK(w)] = r$. So, using (18),

$$H(s,t) = H[rw, rK(w)] = r = \frac{t}{K(w)} = tN\left(\frac{w}{K(w)}\right) = tN\left(\frac{s}{t}\right),$$

which is (20) for $(s,t) \in D_H, s>0$. It is extended to $s = 0$ by (16) and by $N(0) = 1$. The homogeneity follows from (20) and from $H(s,0) = s$. □

Theorem 2. *Suppose that a gains structure $\langle \mathscr{D}^+, \succsim, e \rangle$ has a utility representation (U, W) such that $M \in IIIC$. If event commutativity* (7) *holds and M is of the form* (14), *then there exists $\rho > 0$ such that*

$$H(s,t) = (s^\rho + t^\rho)^{1/\rho}, \quad K(w) = (1 - w^\rho)^{1/\rho} \quad (s,t \in D_H, w \in [0,1]). \tag{22}$$

Proof. By (14),

$$H[pw, qK(w)] = M_w(p,q).$$

By Proposition 1, H and thus M_w are homogeneous. Using the assumption of event commutativity, Theorem 3 of Ng et al. (2002) asserts that there exist a strictly increasing and continuous $\theta : [1,\infty[\to [0,\infty[$, with $\theta(1) = 0$, and a strictly increasing $\psi : [0,1] \overset{onto}{\longrightarrow} [0,1]$ such that

$$M_w(p,q) = q\theta^{-1}[\psi(w)\theta(p/q)]. \tag{23}$$

Thus

$$H[pw, qK(w)] = q\theta^{-1}\left[\psi(w)\theta\left(\frac{p}{q}\right)\right] \quad (0<q\leqslant p<k, \ w \in [0,1]). \tag{24}$$

On the other hand, from (20) we have

$$H[pw, qK(w)] = qK(w)N\left(\frac{pw}{qK(w)}\right) \quad \text{for } w \in [0,1[. \tag{25}$$

Putting these together, for $\xi = p/q \geqslant 1$ and $w \in [0,1[$,

$$K(w)N\left(\frac{\xi w}{K(w)}\right) = \theta^{-1}[\psi(w)\theta(\xi)]. \tag{26}$$

Expressing K and N in terms of Γ, using Eqs. (17) and (19), yields, for $\xi \geqslant 1$ and $w \in]0,1[$,

$$\frac{\xi w}{\Gamma^{-1}[\Gamma(w)/\xi]} = \theta^{-1}[\psi(w)\theta(\xi)].$$

Under the conversions $Z = 1/\xi, \Phi(Z) = \theta(1/Z), w = P$, this equation implies

$$\Phi((Z/P)\Gamma^{-1}[Z\Gamma(P)]) = \Phi(Z)\psi(P) \quad (Z, P \in]0,1[).$$

This equation was solved in Aczél, Maksa, and Páles (2000), Theorem 5, Eq. (5′), under the following conditions, which are satisfied here: ψ and Φ are strictly monotonic, positive, real-valued functions on $]0,1[$, and Γ is a strictly decreasing map of $]0,1[$ onto $]0,\infty[$. The general solution was given by

$$\Phi(Z) = A_1(Z^{-\rho} - 1)^c, \quad \Gamma(P) = B(P^{-\rho} - 1)^{1/\rho},$$

$$\psi(P) = P^{c\rho},$$

where ρ (k in Aczél et al., 2000) and A_1, B are arbitrary positive constants, c an arbitrary non-zero constant, in our case also positive because ψ is strictly increasing. Or, in the notation of the present paper, for $w\in]0,1[$ and $\xi\in]1,\infty[$,

$$\theta(\xi)=\Phi(1/\xi)=A_1(\xi^\rho-1)^c,\quad \Gamma(w)=B(w^{-\rho}-1)^{1/\rho},$$

$$\psi(w)=w^{c\rho}.$$

Thus, for $w\in]0,1[$ and $t\in]0,\infty[$,

$$K(w)=w\Gamma(w)=B(1-w^\rho)^{1/\rho},$$

$$N(t)=\frac{t}{\Gamma^{-1}(1/t)}=(B^{-\rho}+t^\rho)^{1/\rho}.$$

Since $K(0)=1, K(1)=0$ and K is continuous on $[0,1]$ we see that $B=1$ and K has for all $w\in[0,1]$ the form asserted in (22). Now we have, for $t>0$,

$$H(s,t)=tN\left(\frac{s}{t}\right)=(s^\rho+t^\rho)^{1/\rho}$$

and, by (16), $H(s,0)=s$. Thus we obtain the form of H in (22). [Substitution shows that these functions satisfy (26)]. □

If we set $U^*=U^\rho$ and $W^*=W^\rho$ then, by (13) and (22),

$$U^*(f,C;g)=U^*(f)W^*(C)+U^*(g)[1-W^*(C)],\qquad(27)$$

which means (U^*,W^*) is a RDU representation (9), and thus (U,W) *is equivalent to a RDU representation*.

It should be noted that the above result is similar to one imbedded in the proof of the main theorem of Marley and Luce (2002). Both results draw on a result from Aczél et al. (2000). The major advantage of the present proof is that by proving directly the homogeneity of H the remainder of the proof is somewhat simpler to follow. On the other hand, the result in Marley and Luce (2002) is proved under density rather than continuity assumptions.

4. Rank-dependent utility and segregation

4.1. A characterization of RDU

As was noted earlier and as is motivated by Theorem 2 and (27), the rank-dependent one, (9), is an important representation. The following result shows a relation between it and joint receipts.

Theorem 3. *Suppose that a gains structure* $\langle\mathscr{D}^+,\succsim,\oplus,e\rangle$ *has a representation* (U,W) *such that* $M\in IIIC$, $\oplus$ *is strictly left-monotonic increasing, e is a left identity of* $\oplus$, *and for each* $f,g\in\mathscr{D}^+$ *with* $f\succ g$ *there exists* $h\in\mathscr{D}^+$ *such that* $f\sim h\oplus g$. *Then any two of the following three statements imply the third*:

(a) *The pair* (U,W) *forms a binary rank-dependent (RDU) representation of the binary gambles.*

(b) *Segregation is satisfied and* (U,W) *is separable.*

(c) *The function U forms a generalized left-weighted additive representation of* $\oplus$ *in the sense that there exists a positive valued function* Δ *on* $[0,k[$ *such that*

$$U(f\oplus g)=\Delta[U(g)]U(f)+U(g)\quad(f,g\in\mathscr{D}_2^+).\qquad(28)$$

Proof. Assume (a) and (b) and we prove (c). It was established in Theorem 2 of Ng et al. (2002) that separability and segregation yield the representations

$$M_w(p,q)=L[wl(p,q),q]\quad(w\in[0,1],0\leqslant q\leqslant p<k),\qquad(29)$$

where

$$L(p,q):=U\big(U^{-1}(p)\oplus U^{-1}(q)\big)\quad(p,q\in[0,k[).\qquad(30)$$

(Thus $U(f\oplus g)$ depends upon f and g only through $U(f)$ and $U(g)$.) The function L satisfies the condition that $L(\cdot,q):[0,k[\to[q,k[$ is strictly increasing and onto for all $q\in[0,k[$ (in particular, $L(0,q)=q$) and the boundary condition

$$L(p,0)=\alpha_0p\quad(p\in[0,k[)\qquad(31)$$

for some constant $\alpha_0>0$. The function l is the inverse of L in its first variable, defined by

$$l(p,q)=r\Leftrightarrow L(r,q)=p\quad(0\leqslant q\leqslant p<k).\qquad(32)$$

Comparing (29) with the RDU representation (9), we get

$$L[wl(p,q),q]=pw+q(1-w),$$

thus, by (32),

$$l(pw+q(1-w),q)=wl(p,q)\quad(w\in[0,1],0\leqslant q\leqslant p<k).$$

Holding $q>0$ temporarily fixed and writing $\xi:=p/q\in[1,k/q[$, $\lambda_q(\xi):=l(\xi q,q)$, $t:=\xi w+1-w$, (i.e. $w=(t-1)/(\xi-1)$), the above equation is of the form

$$\frac{\lambda_q(t)}{t-1}=\frac{\lambda_q(\xi)}{\xi-1}\quad\text{for all }1<t\leqslant\xi.$$

This implies the constancy of $\xi\mapsto\lambda_q(\xi)/(\xi-1)$ on the interval $]1,k/q[$, say $\lambda_q(\xi)/(\xi-1)=b(q)$. From $L(0,q)=q$ and (32), we have $\lambda_q(1)=l(q,q)=0$. The function λ_q being strictly increasing, we get $b(q)>0$. These yield

$$\lambda_q(\xi)=\frac{\xi-1}{b(q)}\quad(\xi\in[1,k/q[).\qquad(33)$$

Thus $l(p,q)=\lambda_q(p/q)=[(p/q)-1]/b(q)$. Using (32), we obtain

$$L\left(\frac{(p/q)-1}{b(q)},q\right)=p,\quad\text{that is,}\quad L(r,q)=q[rb(q)+1].$$

Putting this into (30) we have

$$\begin{aligned}U(U^{-1}(p)\oplus U^{-1}(q))&=L(p,q)\\&=q[pb(q)+1]=\Delta(q)p+q,\end{aligned}\qquad(34)$$

where

$$\Delta(q) = qb(q) \quad (q \in]0, k[). \tag{35}$$

This proves (28) for $U(g)>0$. For $U(g)=0$, Eq. (28) follows from (30) and (31) with the definition $\Delta(0)=\alpha_0$. Since b and α_0 are positive, so is Δ.

Assume (b) and (c) and we prove (a). Using generalized left-weighted additivity and separability on segregation and defining h by $f \sim h \oplus g$, we have for $f \succsim g$,

$$\begin{aligned} U(f, C; g) &= U(h \oplus g, C; g) \\ &= U[(h, C; e) \oplus g] \\ &= \Delta[U(g)]U(h, C; e) + U(g) \\ &= \Delta[U(g)]U(h)W(C) + U(g). \end{aligned}$$

Observe that, because $f \sim h \oplus g$, we have $U(f) = \Delta[U(g)]U(h) + U(g)$, so

$$U(f, C; g) = U(f)W(C) + U(g)[1 - W(C)],$$

which is binary RDU.

Assume (a) and (c) and we prove (b). Separability follows immediately from (a). To prove segregation, by generalized left-weighted additivity and RDU we have

$$\begin{aligned} U(f \oplus g, C; g) &= U(f \oplus g)W(C) + U(g)[1 - W(C)] \\ &= (\Delta[U(g)]U(f) + U(g))W(C) \\ &\quad + U(g)[1 - W(C)] \\ &= \Delta[U(g)]U(f)W(C) + U(g) \\ &= \Delta[U(g)]U(f, C; e) + U(g) \\ &= U[(f, C; e) \oplus g], \end{aligned}$$

and taking inverses yields segregation. □

Remark. If, in addition, $\oplus$ is also right monotonic ($f \oplus g$ strictly increases in g) and $k = \infty$, then Δ is increasing. In fact, if $\oplus$ is right monotonic, then $U(f \oplus g)$ is strictly increasing in g and so we have

$$(\Delta[U(g)] - \Delta[U(g')])U(f) + U(g) - U(g') > 0 \quad (g \succ g').$$

If $k = \infty$, then $U(f)$ can be arbitrarily large and so $\Delta[U(g)] - \Delta[U(g')]$ cannot be negative, proving that Δ is increasing in that case.

Note that (28) can be placed in the somewhat more symmetric form

$$U(f \oplus g) = U(f) + U(g) + U(f)\Lambda[U(g)]$$

by setting $\Lambda(q) = \Delta(q) - 1$. The commutative case is $\Lambda(q) = -\delta q$.

Parts (b) and (c) provide an argument for RDU, but this is mainly of interest when we can develop an axiomatic basis in terms of the primitives. Doing this rests on three things: an axiomatization of representation (28), an axiomatization of separability, and a qualitative condition that shows that both representations can be achieved using the same utility function U. Axiomatizing separability is well understood since it amounts to additive conjoint measurement put in multiplicative form (Krantz et al., 1971, pp. 248–266). This is unique up to positive powers. However, we do not know how to axiomatize (28) in general. As we shall see, if joint receipt is homogeneous, then under certain smoothness conditions the situation devolves to two special cases, and we are able to carry out this program for each of them (Luce (1996, 2000, pp. 153–154) and Corollary to Theorem 4 in Section 4.3; cf. also Section 5).

Corollary to Theorem 3. *Under the conditions of the theorem, suppose that* (28) *holds and that* $\oplus$ *also has a unit representation* (V, F, d)

$$V(f \oplus g) = \begin{cases} V(g)F\left(\frac{V(f)}{V(g)}\right) & (V(g)>0), \\ dV(f) & (V(g)=0), \end{cases}$$

where $d>0$ *is a constant,* F *is strictly increasing, continuous, and* $F(z)/z$ *is non-constant. Let* G *be defined by*

$$U(f) = G[V(f)].$$

Then, with $u = V(f), v = V(g)$, *and, for* $v>0$, $z = u/v$, $A(v) := \Delta[G(v)]$, *the following functional equations are satisfied*:

$$A(v)G(vz) + G(v) = G[vF(z)] \quad (v>0, z \geqslant 0), \tag{36}$$

$$G(du) = \alpha_0 G(u) \quad (u \geqslant 0, \alpha_0 = \Delta(0)). \tag{37}$$

Proof. For $v = V(g) > 0$, we have

$$\begin{aligned} G[vF(z)] &= G[V(g)F(V(f)/V(g))] \\ &= G(V(f \oplus g)) = U(f \oplus g) \\ &= \Delta[U(g)]U(f) + U(g) \\ &= \Delta[G(v)]G(u) + G(v) = A(v)G(u) + G(v). \end{aligned}$$

This proves (36). For $v = V(e) = 0$, we have

$$\begin{aligned} G(du) &= G(dV(f)) = G(V(f \oplus e)) = U(f \oplus e) \\ &= \Delta[U(e)]U(f) + U(e) = \Delta(0)G(u), \end{aligned}$$

proving (37). □

Note that by setting $z = 1$ in (36) we have

$$A(v) = \frac{G[vF(1)]}{G(v)} - 1 \quad (v \in]0, \infty[). \tag{38}$$

We will determine the solutions F, G of Eq. (36) in the next section under smoothness assumptions. They also satisfy (37). Under those assumptions either $\oplus$ is commutative or G is a power function.

4.2. Continuously differentiable solutions of (36)

We have not been able to solve (36) under just the monotonicity conditions of the problem, but we have solved it with added differentiability assumptions.

Theorem 4. *The following two statements are equivalent*:

(a) *The functions* $F:[0,\infty[\to[1,\infty[, G:[0,\infty[\to[0,k[$ $(k\in]0,\infty])$ *are strictly increasing, surjective* (“*onto*”), *F is once and G twice differentiable on* $]0,\infty[$, F' *and* G'' *are continuous, and* $F(z)/z$ *is non-constant, and the functional equation* (36) *is satisfied where A is given by* (38).

(b) *There exist constants* $\alpha,\beta,\gamma,\kappa,\mu$ *with* $\alpha>0$, $\beta>0,\gamma>0,\mu\kappa>0$ *such that either for all* $z\in[0,\infty[, v\in[0,\infty[$,

$$F(z)=(\alpha z^{\beta}+1)^{1/\beta}, \tag{39}$$

$$G(v)=\gamma v^{\beta} \tag{40}$$

or for all $z\in[0,\infty[, v\in[0,\infty[$,

$$F(z)=(z^{\beta}+1)^{1/\beta}, \tag{41}$$

$$G(v)=\mu[\exp(\kappa v^{\beta})-1]. \tag{42}$$

Proof. It is easy to check that (b) implies (a).

To prove that (a) implies (b), we first note that, since G is twice differentiable, so too is A, by (38). We differentiate (36) with respect to z, yielding

$$G'[vF(z)]F'(z)=A(v)G'(vz)\quad (v\in]0,\infty[, z\in]0,\infty[) \tag{43}$$

and then differentiate that with respect to v to get

$$G''[vF(z)]F(z)F'(z)=A'(v)G'(vz)+A(v)zG''(vz). \tag{44}$$

We claim that none of the functions A, F', G' can be 0 anywhere on $]0,\infty[$. Indeed, if we had $A(v_0)=0$ for some $v_0\neq 0$ then, from (36), $G[v_0F(z)]=G(v_0)$ for all $z\in]0,\infty[$. The surjectivity of F implies that G would be constant on an interval and that would contradict the strict monotonicity of G.

If $G'(v_1)=0$ were true for a $v_1\neq 0$ then for all $v=v_1/F(z)$ $(z\in]0,\infty[)$ we would have $G'[vF(z)]=G'(v_1)=0$. Thus, by (43) and since A is nowhere 0, we would have $G'[(z/F(z))v_1]=G'(vz)=0$. But $z/F(z)$ is continuous and non-constant so, again, G' would be 0 on an interval, which is a contradiction. The fact that A and G' vanish nowhere on $]0,\infty[$ implies that F' does not vanish anywhere either, as asserted.

Furthermore, F, G being strictly increasing, all G', F', A terms in (43) are positive.

We divide (44) by (43) to yield

$$\frac{G''[vF(z)]}{G'[vF(z)]}F(z)=\frac{A'(v)}{A(v)}+z\frac{G''(vz)}{G'(vz)}.$$

Letting

$$\eta(v):=v\frac{G''(v)}{G'(v)},\quad \zeta(v):=v\frac{A'(v)}{A(v)}, \tag{45}$$

we have

$$\eta[vF(z)]=\zeta(v)+\eta(vz)\quad (v\in]0,\infty[, z\in]0,\infty[). \tag{46}$$

We solve this auxiliary equation by proving (in the appendix) the following.

Proposition 4. *All continuous solutions of* (46) $\eta:]0,\infty[\to\mathbb{R}$, $\zeta:]0,\infty[\to\mathbb{R}$, $F:]0,\infty[\to]0,\infty[$ *where* $z\mapsto F(z)/z$ *is non-constant on* $]0,\infty[$, *are given by*

$$\eta=\text{constant},\quad \zeta=0,\quad F \text{ arbitrary} \tag{47}$$

(*with* $z\mapsto F(z)/z$ *continuous and non-constant on* $]0,\infty[$), *and by*

$$\eta(v)=c_1v^{\beta}+c_2,\ \zeta(v)=b_1v^{\beta},\ F(z)=\left(z^{\beta}+\frac{b_1}{c_1}\right)^{1/\beta}, \tag{48}$$

where $\beta\neq 0, b_1\neq 0, c_1\neq 0$ *and* c_2 *are constants with* $b_1/c_1>0$.

We now determine G, A, F from (45) and from (47) or (48). As we see, the conditions in Proposition 4 are weaker than what we have established about η,ζ and F, thus the proposition is somewhat stronger than what we need. In particular, instead of requiring F to map $]0,\infty[$ *onto* $]1,\infty[$, we assume only that it maps *into* $]0,\infty[$.

Take first solution (47) of Eq. (46). By (45),

$$A'(v)=0,\quad \text{so } A(v)=\alpha \text{ (constant)}\ (v\in]0,\infty[) \tag{49}$$

and

$$\frac{G''(v)}{G'(v)}=\frac{\tilde\beta}{v}\quad (\tilde\beta \text{ constant})\ (v\in]0,\infty[).$$

Integrating the latter equation repeatedly we get, since G' is positive,

$$G'(v)=\tilde\gamma v^{\tilde\beta}\quad\text{and}\quad G(v)=\gamma v^{\beta}+c\quad (v\in]0,\infty[) \tag{50}$$

$[\beta=\tilde\beta+1,\ \gamma=\tilde\gamma/(\tilde\beta+1)]$ for the case $\tilde\beta\neq-1$. For the case $\tilde\beta=-1$ we get $G(v)=\tilde\gamma\ln v+c$, which is impossible because G is supposed to be strictly increasing and map $[0,\infty[$ onto $[0,k[$. As to (50), for the same reason $\gamma>0,\beta>0$ and $c=0$, which yields $G(v)=\gamma v^{\beta}$, and, with $G(0)=0$ following from the surjectivity of the strictly increasing function G, we have (40).

To determine F, substitute (49) and (50) into (36) yielding $\gamma v^{\beta}F(z)^{\beta}=\alpha\gamma v^{\beta}z^{\beta}+\gamma v^{\beta}$, that is, for $z\in]0,\infty[$,

$$F(z)=(\alpha z^{\beta}+1)^{1/\beta},$$

which is (39) for $z>0$. As to the constants, we know already that $\beta>0$ and $\gamma>0$. Since A is positive on $]0,\infty[$, we have also $\alpha>0$. Finally, because in Theorem 4(a), the strictly increasing F was supposed to map $[0,\infty[$ onto $[1,\infty[$, we have $F(0)=1$. Thus (39) holds on $[0,\infty[$.

Second, we consider solutions (48) of Eq. (46). The F there becomes

$$F(z)=(z^{\beta}+1)^{1/\beta}\quad (\beta>0),$$

because, by supposition, $F(0) = 1$ and F is continuous. We thus have obtained (41) and $b_1 = c_1 \neq 0$ in (48). Further, from (45) and (48), on $]0, \infty[$

$$\frac{A'(v)}{A(v)} = \frac{\zeta(v)}{v} = c_1 v^{\beta-1}, \quad \frac{G''(v)}{G'(v)} = \frac{\eta(v)}{v} = c_1 v^{\beta-1} + \frac{c_2}{v}.$$

Integrating and noting that A, G' are positive we get, with $\kappa = c_1/\beta \neq 0$,

$$A(v) = \exp\left(c_1 \frac{v^\beta}{\beta} + c_3\right) = a_1 \exp(\kappa v^\beta) \quad (a_1 = \exp c_3 > 0) \tag{51}$$

and

$$G'(v) = \exp\left(c_1 \frac{v^\beta}{\beta} + c_2 \ln v + c_4\right) = a_2 v^{c_2} \exp(\kappa v^\beta) \ (a_2 = \exp c_4 > 0). \tag{52}$$

Putting these A, G' into (43), we obtain

$$a_2 v^{c_2} F(z)^{c_2} \exp[\kappa v^\beta (z^\beta + 1)] F(z)^{1-\beta} z^{\beta-1} = a_1 \exp(\kappa v^\beta) a_2 v^{c_2} z^{c_2} \exp(\kappa v^\beta z^\beta)$$

because

$$F'(z) = (1/\beta)(z^\beta + 1)^{(1/\beta)-1} \beta z^{\beta-1} = F(z)^{1-\beta} z^{\beta-1}$$

or, after cancellations,

$$[F(z)/z]^{c_2+1-\beta} = a_1.$$

Since, by supposition, $F(z)/z$ is non-constant, this implies

$$c_2 = \beta - 1 \quad \text{and} \quad a_1 = 1.$$

Taking these into consideration, (51) and (52) become

$$A(v) = \exp(\kappa v^\beta) \quad (v \in]0, \infty[) \tag{53}$$

and

$$G'(v) = a_2 v^{\beta-1} \exp(\kappa v^\beta),$$

respectively. Integrating the latter again, we get, since by supposition $G(0) = 0$ and G is continuous,

$$G(v) = \mu \exp(\kappa v^\beta) - \mu, \quad (v \in [0, \infty[),$$

where $\mu := a_2/(\kappa\beta)$, and $\mu\kappa = a_2/\beta > 0$. Thus we have obtained (42). We got (41) earlier. This concludes the proof of Theorem 4. □

Three comments about the result:

First, if $\kappa < 0$ and $\mu < 0$ in the second equation for G then $k = -\mu$. For all other solutions $k = \infty$.

Second, as we will see in the Corollary to Theorem 4, the first solution with $\alpha_0 = \alpha = 1$ and the second solution are the cases where $\oplus$ is commutative (cf. observation 5 at the end of Section 4.3).

Third, if G is of the form (40) then, by (37), $\alpha_0 = d^\beta$, while if G is of the form (42) then, also by (37), $\alpha_0 = d = 1$. In the latter case, the left identity e is also a right identity. This, obviously, does not imply that $\oplus$ is commutative; however the family of commutative $\oplus$, which case was dealt with in Luce (2000, pp. 131–172), falls within the class of structures with a two sided identity. Of course, there may be other, non-smooth solutions with or without a two-sided identity.

4.3. The resulting utility forms

We now obtain the general permissible forms of $U(f \oplus g)$.

Corollary to Theorem 4. *Suppose that the assumptions and conditions* (a)–(c) *of Theorem* 3 *hold with* (U, W) *an RDU representation, that* $\oplus$ *has a unit representation* (V, F, d), *and* $U = G(V)$. *If* F *is once and* G *is twice continuously differentiable, then either there exist constants* $\alpha > 0$ *and* $\alpha_0 > 0$ *such that*

$$U(f \oplus g) = \begin{cases} \alpha U(f) + U(g) & (g \succ e), \\ \alpha_0 U(f) & (g \sim e) \end{cases} \tag{54}$$

or a constant $\delta \in \mathbb{R}$ *such that*

$$U(f \oplus g) = U(f) + U(g) - \delta U(f) U(g). \tag{55}$$

In case (54) *we have* $k = \infty$.

Proof. Given that $\oplus$ has a unit representation, (10), we know by the Corollary to Theorem 3 that the functional eqs. (36) and (37) hold. The solutions, under differentiability assumptions, to Eq. (36) are given in Theorem 4. Substituting them in (10) with $U = G(V)$ and taking (37) into consideration we get representations equivalent to (54) and (55). In (54), letting both $U(f)$ and $U(g)$ tend to k while observing that $U(f \oplus g) < k$, we conclude that $k = \infty$. □

Several observations:

1. The first solution for $U(f \oplus g)$, (54), with $\alpha \neq 1$ or $\alpha_0 \neq 1$, is new and non-commutative yielding a form called *left-weighted additive*, abbreviated *lw-additive*, over $\oplus$.
2. If we were also to assume right positivity in the sense that $f \oplus g \succsim f$, then $\alpha \geqslant 1$ and $\alpha_0 \geqslant 1$.
3. Had we assumed that $\oplus$ has a right identity, then right positivity of $\oplus$ would hold and we would have been led to a right-weighted additive form

   $$U(f \oplus g) = U(f) + \alpha' U(g) \ (f \succ e), \quad U(e \oplus g) = \alpha'_0 U(g). \tag{56}$$

   Adding to that the assumption that $\oplus$ is left positive, $f \oplus g \succsim g$, leads to $\alpha' \geqslant 1$ and $\alpha'_0 \geqslant 1$.
4. Axiomatic conditions on $\langle \mathcal{D}^+, \succsim, \oplus \rangle$ that give rise to the form (54) for $g \succ e$ or to (56) for $f \succ e$ are well known (Aczél, 1966, pp. 278–292; Krantz et al., 1971, pp. 293–301). The key necessary condition is

bisymmetry,

$$(f \oplus f') \oplus (g' \oplus g) \sim (f \oplus g') \oplus (f' \oplus g). \quad (57)$$

This, in the presence of other assumptions that are satisfied in the present situation, leads for $g \succ e$ to a representation of the general form

$$U(f \oplus g) = \alpha U(f) + \alpha' U(g) + \iota.$$

Using the left identity property and $U(e) = 0$, then, for $g \succ e$,

$$U(g) = U(e \oplus g) = \alpha U(e) + \alpha' U(g) + \iota = \alpha' U(g) + \iota,$$

and so $\alpha' = 1$ and $\iota = 0$.

5. The second solution of the Corollary to Theorem 4, (55), is the commutative one previously developed in Luce (2000, pp. 151–152) drawing on Luce and Fishburn (1991).

4.4. Results for $W(p) \in [0, \infty[$

Luce (2002) proposed one psychophysical interpretation of the formalism with $\oplus$ constructed from presentations of different intensities of pure tones to the two ears. Preliminary data established conclusively that this defined operation is not usually commutative, which fact motivated the current work. In that reinterpretation, $\succ, \succsim$ become $>, \geqslant$, respectively, and the analogue of a binary gamble is interpreted as the intensity $(x, p; y)$ that a respondent says makes the "interval" $[y, (x, p; y)]$ stand in the proportion $p > 0$ to the "interval" $[y, x]$. Were respondents literally to follow the instructions, we should have

$$\frac{(x, p; y) - y}{x - y} = p.$$

Empirically, this is not what they do. A better approximation to the empirical results is the assumption in Luce (2002) that there exists a psychophysical measure U of intensities and a measure W of proportions such that

$$\frac{U(x, p; y) - U(y)}{U(x) - U(y)} = W(p) \quad (x > y \geqslant 0),$$

where $W : [0, \infty[\xrightarrow{onto} [0, \infty[$, $W(1) = 1$, holds. Of course, this is just a rewritten version of RDU. As we know from earlier work with commutative $\oplus$, the only impact of extending the domain and range of W from $[0, 1]$ to $[0, \infty[$ is to exclude the form (55) with $\delta > 0$, and so U is necessarily superadditive. Here we ask whether such an extension in any way restricts the weighted additive forms (54) and (56) that are already either superadditive or subadditive as well as non-commutative.

If one examines the proof of Theorem 3, we did not impose on $M_w(p, q)$ monotonicity in q which is the only difference for W onto $[0, 1]$ versus onto $[0, \infty[$. Everything else is unaffected.

5. Linking left-weighted additivity and separability

Recall that in Theorem 3 condition (b) asserts that (U, W) forms a separable representation and condition (c) asserts that U is a generalized left-weighted additive representation of $\oplus$, i.e.,

$$U(f \oplus g) = \Delta[U(g)]U(f) + U(g).$$

As noted earlier, axiomatic conditions are known that give rise to the separable form, but we do not know how to axiomatize the generalized left-weighted additive form except for the two special cases of the Corollary to Theorem 4. For the case of commutative $\oplus$, the axioms are well known for an additive representation $V(f \oplus g) = V(f) + V(g)$ and so for U to satisfy (55). They go under the title of extensive measurement in the measurement literature (Krantz et al., 1971, pp. 71–135). We turn now to the case where Δ is a constant in (28). In that case, as we have seen, $\oplus$ is bisymmetric. As noted under observation 4 of Section 4.3, axiomatizations of the bisymmetric case are well known (Aczél, 1966, pp. 278–292; Krantz et al., 1971, pp. 153–154).

The only remaining question is the conditions under which the two independent axiomatizations lead to the same utility function. In Luce (1996, 2000, pp. 153–154) this question was answered for the commutative case. The relevant necessary and sufficient condition, called *joint-receipt decomposition*, is that, for all $f \in \mathscr{D}^+$ and $C \in \mathscr{E}_{\mathbf{E}}$, there exists a $D \in \mathscr{E}_{\mathbf{E}}$: such that for an independent realization of C, say C', and for all $g \in \mathscr{D}^+$

$$(f \oplus g, C; e) \sim (f, C'; e) \oplus (g, D; e).$$

Note that D is not a function of g. In this setting the independence of C and C' is taken to imply that $W(C) = W(C')$.

The special case of $D = C''$, where C'' is another independent realization of C, is called *simple joint-receipt decomposition*, i.e.,

$$(f \oplus g, C; e) \sim (f, C'; e) \oplus (g, C''; e). \quad (58)$$

The next result shows that this is the relevant condition in the lw-additive case. As shown in the Corollary to Theorem 4, in this case $k = \infty$.

Theorem 5. *Suppose that the conditions of Theorem* 3 *are met and that U is an lw-additive representation* (54) *of $\oplus$ onto $[0, \infty[$ and that (U_*, W_*) is a separable one* [*cf.* (8)], *with U_* onto $[0, k_*[$. Then the following two statements are equivalent*:

(a) *There exist $\varepsilon > 0, \rho > 0$ such that $U = \varepsilon U_*^\rho$ (thus $k_* = \infty$) and, with $W := W_*^\rho$, the pair (U, W) forms a separable representation.*
(b) *For all $f, g \in \mathscr{D}^+$ and $C \in \mathscr{E}_{\mathbf{E}}$ simple joint-receipt decomposition*, (58), *is satisfied.*

Proof. Assume (a), where U is lw-additive over $\oplus$ and (U, W) forms a separable representation. Then, for $C \neq \emptyset$ and $g \succ e$, implying also $(g, C''; e) \succ e$ by left-consequence monotonicity and idempotence, we have

$$\begin{aligned} U(f \oplus g, C; e) &= U(f \oplus g) W(C) \\ &= \alpha U(f) W(C) + U(g) W(C) \\ &= \alpha U(f) W(C') + U(g) W(C'') \\ &= \alpha U(f, C'; e) + U(g, C''; e) \\ &= U[(f, C'; e) \oplus (g, C''; e)], \end{aligned}$$

yielding (58) for the case $C \neq \emptyset$ and $g \succ e$. In the case $C \neq \emptyset$ and $g \sim e$, (58) can be confirmed in a similar manner using α_0 instead of α. In the case $C = \emptyset$, (58) holds by nullity and by e being a left identity. This proves simple joint-receipt decomposition.

Assume (b), i.e., simple joint-receipt decomposition. By assumption, we have a representation (U_*, W_*) that is separable and a function U that is lw-additive over $\oplus$. Since U and U_* are order preserving, there exists a strictly increasing function $Q : [0, k_*[\xrightarrow{onto} [0, \infty[$ such that $U = Q(U_*)$. Since Q is strictly increasing and maps an interval onto an interval, it is continuous. Consider, again for $C \neq \emptyset$ and $g \succ e$,

$$\begin{aligned} U(f \oplus g, C; e) &= Q[U_*(f \oplus g, C; e)] \\ &= Q[U_*(f \oplus g) W_*(C)] \\ &= Q[Q^{-1}(U(f \oplus g)) W_*(C)] \\ &= Q[Q^{-1}[\alpha U(f) + U(g)] W_*(C)] \end{aligned}$$

and

$$\begin{aligned} &U[(f, C'; e) \oplus (g, C''; e)] \\ &= \alpha U(f, C'; e) + U(g, C''; e) \\ &= \alpha Q[U_*(f, C'; e)] + Q[U_*(g, C''; e)] \\ &= \alpha Q[U_*(f) W_*(C')] + Q[U_*(g) W_*(C'')] \\ &= \alpha Q[Q^{-1}(U(f)) W_*(C')] \\ &\quad + Q[Q^{-1}(U(g)) W_*(C'')]. \end{aligned}$$

Assuming simple joint-receipt decomposition, we equate these and set $p = U(f) \in [0, \infty[$, $q = U(g) \in]0, \infty[$, $w = W_*(C) = W_*(C') = W_*(C'') \in]0, 1]$, thus yielding the functional equation

$$Q[Q^{-1}(\alpha p + q) w] = \alpha Q[Q^{-1}(p) w] + Q[Q^{-1}(q) w]$$

$$(p \geqslant 0, q > 0, w \in]0, 1]).$$

For each fixed w we apply the Uniqueness Theorem in the appendix to this equation (setting $I = [0, \infty[, J =]0, \infty[$, $T(p, q) = \alpha p + q$, $\varphi_0(p) = \alpha p$, $\pi_0(q) = \chi_0(q) = q$, $\varphi(p) = \alpha Q[Q^{-1}(p) w]$, $\pi(q) = \chi(q) = Q[Q^{-1}(q) w]$) and obtain

$$\alpha Q[Q^{-1}(p) w] = \mu(w) \alpha p + v_1(w),$$

$$Q[Q^{-1}(q) w] = \mu(w) q + v_2(w),$$

$$Q[Q^{-1}(q) w] = \mu(w) q + v_1(w) + v_2(w).$$

Consistency in these three relations gives $v_1(w) = v_2(w) = 0$ and

$$Q[Q^{-1}(p) w] = \mu(w) p \quad (p \geqslant 0, w \in]0, 1]).$$

Letting $p_* = Q^{-1}(p)$ it takes the form of a Pexider equation:

$$Q(p_* w) = \mu(w) Q(p_*) \quad (p_* \in [0, k_*[, w \in]0, 1]).$$

Because $Q : [0, k_*[\xrightarrow{onto} [0, \infty[$ is strictly increasing, this implies that $Q(p_*) = \varepsilon p_*^\rho$ on $[0, k_*[$ for some constants $\rho > 0, \varepsilon > 0$ (see e.g. Aczél, 1987, p. 81). Thus, $U = \varepsilon U_*^\rho$. It is easy to check that (U, W), with $W = W_*^\rho$, form a separable representation. □

6. Conclusions and open problems

On the assumption that a separable, homogeneous representation (U, W) exists for uncertain alternatives with two consequences (binary gambles) and that (V, F, d) is a unit representation of a (in general) non-commutative joint receipt operation, we derived a functional equation relating the two structures (Theorem 1). This has not been solved in general and we viewed it as probably too weak to be of great inherent interest. So we sought to impose additional structure. We began with a representation (Eq. (13) in Section 3) weaker than the RDU representation. We showed, first, that it is in fact homogeneous and then, using results in the literature, that it is equivalent to RDU (Theorem 2, Eq. (27); see also Marley & Luce, 2002). Next, generalizing a result of Luce and Fishburn (1991), we characterized RDU in terms of segregation and a particular representation ("left generalized additive", Eq. (28) in Section 4.1) of joint receipt (Theorem 3). Imposing on that the assumption that joint receipt is homogeneous leads to a more restrictive functional equation than that of Theorem 1 (Corollary to Theorem 3). The solutions to this equation under certain continuous differentiability assumptions were given (Theorem 4). This led to the well-studied case of commutative (and associative) $\oplus$ and to a new non-commutative, non-associative, bisymmetric solution that is of interest in psychophysics (Corollary to Theorem 4). For the new solution, we showed how U over $\oplus$ can be axiomatized in such a way that (U, W) is separable as well (Theorem 5).

The following problems are open:

- Find all solutions to the functional equation (11) that arose in Theorem 1.
- Develop an axiomatization in terms of the primitives of the generalized left-weighted additive representation, (28).

- Discover all strictly monotonic solutions to the functional equation (36) arising in the rank-dependent utility case. Clearly, those we do not know cannot be very smooth, but some of these may nonetheless be of interest.
- A less precise problem is to formulate homogeneous representations that meet the following three conditions: They are more general than RDU, they are less general than the form given in (23), and they are of interest in either utility theory or psychophysics. Such a property has to be less restrictive than assuming that $U(f, C; g)$ is some function of $U(f)W(C)$ and $U(g)K[W(C)]$, where K is strictly decreasing from $[0,1]$ onto $[0,1]$, because these conditions are equivalent to RDU (Section 3 and Marley & Luce, 2002). Luce (2003) has arrived at such an example in which, with V an additive representation of $\oplus$, a generalization of segregation, right distributivity, yields in addition to RDU

  $$V(x, C; y) = V(y) + \Psi_C[V(x) - V(y)],$$

  where $\Psi_C(0) = 0$ and Ψ_C is strictly increasing.

Appendix

We quote here a uniqueness theorem (see Ng, 1973), which was applied in the proof of Theorem 5 and will be used in proving the Proposition 4 that was invoked in the proof of Theorem 4.

Uniqueness Theorem. *Let I and J be real intervals, $T: I \times J \to \mathbb{R}$ be continuous, and consider the functional equation*

$$\varphi(x) + \pi(y) = \chi(T(x, y)) \quad (x \in I, y \in J), \tag{A.1}$$

where $\varphi: I \to \mathbb{R}$, $\pi: J \to \mathbb{R}$ and $\chi: T(I \times J) \to \mathbb{R}$. If (A.3) has a solution $(\varphi_0, \pi_0, \chi_0)$ with continuous, non-constant φ_0 and π_0, then

$$\varphi = \mu\varphi_0 + \nu_1, \quad \pi = \mu\pi_0 + \nu_2, \quad \chi = \mu\chi_0 + \nu_1 + \nu_2,$$

with constants μ, ν_1, ν_2, give the general solutions (φ, π, χ) with continuous φ and π.

Proposition A.1. *All continuous solutions $\eta:]0,\infty[\to \mathbb{R}$, $\zeta:]0,\infty[\to \mathbb{R}$, $F:]0,\infty[\to]0,\infty[$ of the equation*

$$\eta[vF(z)] = \zeta(v) + \eta(vz) \quad (v \in]0,\infty[, z \in]0,\infty[), \tag{46}$$

where $z \mapsto F(z)/z$ is non-constant on $]0,\infty[$, are given by

$$\eta = \text{constant}, \quad \zeta = 0, \quad F \text{ arbitrary} \tag{47}$$

(with $z \mapsto F(z)/z$ continuous and non-constant on $]0,\infty[$), and by

$$\eta(v) = c_1 v^\beta + c_2, \quad \zeta(v) = b_1 v^\beta, \quad F(z) = \left(z^\beta + \frac{b_1}{c_1}\right)^{1/\beta}, \tag{48}$$

where $\beta \neq 0, b_1 \neq 0, c_1 \neq 0$ and c_2 are constants with $b_1/c_1 > 0$.

Proof. Clearly, (46) is of the form (A.1) with $\varphi = \chi = \eta$, $\pi = \zeta$, and with $y = vz$, $T(v, y) = vF(y/v)$.

Case 1: If η is constant then, by (46), $\zeta(v) \equiv 0$ and F can be arbitrary (with $F(z)/z$ non-constant). This gives the solution (47).

Case 2: The function η is non-constant.

Case 2.1: The function ζ is constant, say $\zeta(v) = \zeta_1$, and so

$$\eta[vF(z)] = \zeta_1 + \eta(vz), \tag{A.2}$$

which is a particular case of the equation

$$\eta\left[vz\left(\frac{F(z)}{z}\right)\right] = \tau(z) + \eta(vz). \tag{A.3}$$

We compare the latter to

$$\ln\left[vz\left(\frac{F(z)}{z}\right)\right] = \ln\left(\frac{F(z)}{z}\right) + \ln(vz). \tag{A.4}$$

Since $F(z)/z$ is non-constant, all terms in (A.4) are non-constant. By the uniqueness theorem quoted above, the functions τ and η in (A.3) must be both constant or both non-constant. By comparing (A.2) and (A.3), we see that τ is a constant and, therefore, η would have to be a constant, contrary to the Case 2 assumption. Thus Case 2.1 is not possible.

Case 2.2: Suppose ζ is non-constant. Comparing (46), that is,

$$\eta[vF(z)] = \zeta(v) + \eta(vz) \tag{A.5}$$

to the equation

$$\eta[tvF(z)] = \zeta(tv) + \eta(tvz),$$

obtained by replacing v by tv in (A.5), we see that the functions η_t, ζ_t, defined by $\eta_t(v) := \eta(tv)$, $\zeta_t(v) := \zeta(tv)$ are also solutions of (A.5). So, fixing t temporarily, there exist, again by the uniqueness theorem, "constants" μ_t, ν_t such that $\eta_t(v) = \mu_t\eta(v) + \nu_t$, $\zeta_t(v) = \mu_t\zeta(v)$. Letting t vary again, we have for all $t \in]0,\infty[$ and for all $v \in]0,\infty[$

$$\eta(tv) = \mu(t)\eta(v) + \nu(t) \tag{A.6}$$

and

$$\zeta(tv) = \mu(t)\zeta(v). \tag{A.7}$$

For the solution of such equations see, e.g., Aczél (1987, pp. 81, 130–131). The general continuous non-constant solutions are given by $\mu(t) = t^\beta$, $\nu(t) = c_2(1 - t^\beta)$, and

$$\eta(v) = c_1 v^\beta + c_2, \quad \zeta(v) = b_1 v^\beta,$$

$(\beta \neq 0, b_1 \neq 0, c_1 \neq 0)$, as in (48). Putting these into (46) yields the remaining equation of (48):

$$F(z) = \left(z^\beta + \frac{b_1}{c_1}\right)^{1/\beta}.$$

In order that F map $]0, \infty[$ into $]0, \infty[$ and that $z \mapsto F(z)/z$ be non-constant, $b_1/c_1 > 0$ must hold.

Since the functions given by (47) and (48) satisfy Eq. (46) and are continuous, with $F(z)/z$ non-constant, the Proposition is proved. □

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