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Results in Mathematics

A FUNCTIONAL EQUATION ARISING FROM SIMULTANEOUS UTILITY REPRESENTATIONS

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Abstract

Suppose that two classes of utility representations of preferences, one additive and one increasing increments, hold simultaneously over uncertain binary alternatives (gambles). This assumption leads to the functional equation

$$f[h(x-y)+y]=f[h(x)]-f[h(y)]+f(y)\quad (\kappa > x\geq y\geq 0),$$

and to the inequality $h(z)\leq z$ ($z\in[0,\kappa[$), where the functions f and h are strictly increasing maps of the real interval $[0,\kappa[$ onto the real intervals $[0,\lambda[$ and $[0,\mu[$, respectively, $\kappa,\lambda,\mu\in]0,\infty]$. We present all solutions under the additional assumption of (first-order) differentiability.

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Introduction

Consider a preference order $\succsim$ over the class of binary gambles of the form $(\widetilde{x},C;\widetilde{y},\overline{C})$ where $\widetilde{x},\widetilde{y}$ are in X which is a set of pure consequences having no aspect of uncertainty and C is an event from a chance "experiment". The interpretation of the gamble is that, when the experiment is run, $\widetilde{x}$ is the consequence if the event C occurs and $\widetilde{y}$ is the consequence if the complementary event, $\overline{C}$, occurs. No probability distribution is assumed over events. There is a distinguished element $e\in X$ which represents the consequence of no change from the status quo. We limit ourselves to gains, i.e., for every $\widetilde{x}\in X$, $\widetilde{x}\succsim e$. We assume that X is so "rich" that there is an order-preserving map $U:X\overset{onto}{\longrightarrow}[0,\lambda[$ for some $\lambda\in]0,\infty]$. Note that it follows that $U(e)=0$.

The following family of utility representations, which is a classic one, is called *additive utility:*

$$U(\widetilde{x},C;\widetilde{y},\overline{C})=L_C[U(\widetilde{x})]+L_{\overline{C}}[U(\widetilde{y})],\tag{1}$$

where $L_C,L_{\overline{C}}:[0,\lambda[\rightarrow[0,\lambda[$ are strictly increasing functions with 0 a fixed point of each. If we assume that gambles are idempotent in the sense that for every $\widetilde{y}$ and every C, $(\widetilde{y},C;\widetilde{y},\overline{C})\sim\widetilde{y}$, then we see that $L_{\overline{C}}[U(\widetilde{y})]=U(\widetilde{y})-L_C[U(\widetilde{y})]$ and so (1) becomes

$$U(\widetilde{x},C;\widetilde{y},\overline{C})=L_C[U(\widetilde{x})]-L_C[U(\widetilde{y})]+U(\widetilde{y}).\tag{2}$$

The following more recent representation, which arises in the analysis of a psychophysical interpretation of the formalism, is called *increasing utility increments:*

$$U^*(\widetilde{x}, C; \widetilde{y}, \overline{C}) = M_C[U^*(\widetilde{x}) - U^*(\widetilde{y})] + U^*(\widetilde{y}) \quad (\widetilde{x} \succsim \widetilde{y}), \tag{3}$$

where $U^* : X \overset{onto}{\longrightarrow} [0, \kappa[$ is also order preserving and, of course, $U^*(e) = 0$ (Luce, 2003), and where $M_C : [0, \kappa[\to [0, \mu[\; (0 < \mu \leq \kappa)$.

Consider the possibility of both representations U and U^* holding. How do they relate? Because both are order preserving, there exists a strictly increasing function $f : [0, \kappa[\overset{onto}{\longrightarrow} [0, \lambda[$ (in particular, $f(0) = 0$) such that

$$U = f \circ U^*. \tag{4}$$

Set $\widetilde{y} = e$ in (2) and (3), recall that $L_C(0) = L_{\overline{C}}(0) = 0$, and use (4) to get

$$\begin{aligned} L_C(f[U^*(\widetilde{x})]) &= L_C[U(\widetilde{x})] = U(\widetilde{x}, C; e, \overline{C}) \\ &= f[U^*(\widetilde{x}, C; e, \overline{C})] = f(M_C[U^*(\widetilde{x})]) \end{aligned}$$

which shows that M_C is also strictly increasing, and which is equivalent to

$$f \circ M_C = L_C \circ f. \tag{5}$$

Thus, if we set

$$x := U^*(\widetilde{x}) = f^{-1}[U(\widetilde{x})], \quad y := U^*(\widetilde{y}) = f^{-1}[U(\widetilde{y})],$$

then we have

$$U(\widetilde{x}, C; \widetilde{y}, \overline{C}) = L_C[f(x)] - L_C[f(y)] + f(y) = f[M_C(x - y) + y].$$

So, using (5) and renaming M_C as $h := M_C$, which maps $[0, \kappa[$ onto $[0, \mu[\subseteq [0, \kappa[$ (in particular $h(0) = 0$), we get the functional equation

$$f[h(x)] - f[h(y)] + f(y) = f[h(x - y) + y] \quad (\kappa > x \geq y \geq 0).$$

Equation (1) implies that *consequence monotonicity* holds for gambles, that is, $U^*(\tilde{x}, C; \tilde{y}, \bar{C})$ is strictly increasing in $\tilde{x}$ when $C \neq \emptyset$, and in $\tilde{y}$ when $\overline{C} \neq \emptyset$, i.e., when $C \neq \Omega$. Therefore, by (3), for $\tilde{x} \succ e$ and $C \neq \Omega$,

$$M_C[U^*(x)] = U^*(\tilde{x}, C; e, \bar{C}) < U^*(\tilde{x}, C; \tilde{x}, \bar{C}) = U^*(\tilde{x}).$$

Thus, $h(z) < z$ for $z > 0$, provided $C \neq \Omega$. Since $h(0) = 0$, we will assume the weaker inequality

$$h(z) \leq z \qquad \text{for all} \quad z \in [0, \kappa[.$$

We solve the above functional equation and inequality under the assumptions that f and h are differentiable as well as strictly increasing. An open problem is to find the solutions without assuming differentiability.

The Differentiable Solutions

Theorem. *The functional equation*

$$f[h(x)] - f[h(y)] + f(y) = f[h(x - y) + y] \quad (\kappa > x \geq y \geq 0). \tag{6}$$

and the inequality

$$h(z) \leq z \qquad (z \in [0, \kappa[) \tag{7}$$

hold under the assumptions that f and h are from $[0, \kappa[$ onto $[0, \lambda[$ or onto $[0, \mu[$ $(\kappa, \lambda, \mu \in [0, \infty]$ $\mu \leq \kappa)$, respectively, are strictly increasing, and are differentiable if, and only if, f and h belong to one of the following classes.

1.

$$h(z) = z, \quad f \text{ arbitrary (strictly increasing and differentiable).} \tag{8}$$

2. *There exist constants* $r > 0,\ s \in]0,1]$ *such that*

$$f(y) = ry, \tag{9}$$

$$h(z) = sz. \tag{10}$$

3. *There exist constants* $\alpha \neq 0,\ \gamma > 0,\ a \in]0,1[$ *such that*

$$f(y) = \frac{1}{\alpha\gamma}(e^{\alpha y} - 1), \tag{11}$$

$$h(z) = \frac{1}{\alpha}\ln\frac{ae^{\alpha z}+1}{a+1}. \tag{12}$$

In all three classes (in class 1 with the arbitrary f chosen accordingly) $\lambda = \infty$ iff $\kappa = \infty$ and $\mu = \infty$ also iff $\kappa = \infty$.

Proof: Notice that $\mu \le \kappa$ and (7) guarantee that $h(x), h(y)$, and $h(x-y)+y \le x$ are in the domain $[0,\kappa[$ of f.

Differentiating (6) with respect to x, we have

$$f'[h(x-y)+y]h'(x-y) = f'[h(x)]h'(x), \tag{13}$$

which with $y = x$ yields

$$f'(y)h'(0) = f'[h(y)]h'(y). \tag{14}$$

If $h'(0) = 0$, then from (14) $\frac{d}{dy}(f[h(y)]) = 0$, and so $f[h(y)] = c$, which contradicts strict monotonicity. So we assume $h'(0) \neq 0$. Now differentiate (6) with respect to y and take (14) into account to obtain

$$\begin{aligned} & f'[h(x-y)+y][1-h'(x-y)] \\ &= f'(y) - f'[h(y)]h'(y) \\ &= f'(y)[1-h'(0)]. \end{aligned} \tag{15}$$

Using (13) and (14), we also conclude

$$f'[h(x-y)+y]h'(x-y) = f'(x)h'(0), \tag{16}$$

so that, if f' or h' were 0 at a point, then they would be 0 on an interval of positive length, thus f or h, respectively, could not be strictly increasing. Therefore we can divide (15) by (16):

$$\frac{1-h'(x-y)}{h'(x-y)} = \frac{f'(y)}{f'(x)}\frac{1-h'(0)}{h'(0)}. \tag{17}$$

If $h'(0) = 1$, then $1 - h'(x-y) = 0$, i.e., $h'(z) = 1$ and so, because $h(0) = 0$, $h(z) = z$, which is (8).

Let

$$\overline{a} := \frac{h'(0)}{1-h'(0)} \neq 0,\ H(z) := \overline{a}\left(\frac{1}{h'(z)} - 1\right),\ F(y) = \frac{1}{f'(y)} \neq 0. \tag{18}$$

(Note that $\overline{a} \neq -1$ because otherwise $0 = -1$). Then (17) becomes

$$F(y+z) = F(y)H(z) \tag{19}$$

(where $z = x - y$). If F is constant, then so is H and $f'(y) = r$, a constant, and because $f(0) = 0$, therefore $f(y) = ry$, that is (9). By (18), also h' is constant and, using $h(0) = 0$, we have $h(z) = sz$, which is (10). By (7), $s \leq 1$.

Now suppose that F is not constant, then the general integrable solution of (19) is given by (see Aczél, 1966, pp. 142-143)

$$F(y) = \gamma e^{-\alpha y}, \; H(z) = e^{-\alpha z} \quad (\alpha\gamma \neq 0). \tag{20}$$

Thus, by (18) $f'(y) = \frac{1}{\gamma} e^{\alpha y}$ and, taking into account $f(0) = 0$,

$$f(y) = \frac{1}{\alpha\gamma}(e^{\alpha y} - 1).$$

This f is positive and strictly increasing iff $\gamma > 0, \alpha \neq 0$ (both if $\alpha > 0$ and if $\alpha < 0$), so we have (11). Also, by (18) and (20), $e^{-\alpha z} = H(z) = \overline{a}\left(\frac{1}{h'(z)} - 1\right)$. Thus $h'(z) = \frac{\overline{a}e^{\alpha z}}{1+\overline{a}e^{\alpha z}}$ and so, taking $h(0) = 0$ into account, we get

$$h(z) = \frac{1}{\alpha} \ln \left| \frac{\overline{a}e^{\alpha z} + 1}{\overline{a} + 1} \right|. \tag{21}$$

We distinguish several cases. If $\overline{a} > 0$, then the absolute value symbol is redundant and

$$h(z) = \frac{1}{\alpha} \ln \frac{\overline{a}e^{\alpha z} + 1}{\overline{a} + 1}$$

is strictly increasing for both $\alpha > 0$ and $\alpha < 0$, and also positive in both cases (for $\alpha > 0$, because $\overline{a}e^{\alpha z} + 1 > \overline{a} + 1$, and for $\alpha < 0$ because $\overline{a}e^{\alpha z} + 1 < \overline{a} + 1$). With $a := \overline{a}$ this gives (12), which always satisfies (7).

If $-1 < \overline{a} < 0$ or, with $a := -\overline{a}$, $0 < a < 1$, and $\alpha < 0$, then (21) becomes

$$h(z) = \frac{1}{\alpha} \ln \frac{1 - ae^{\alpha z}}{1 - a}. \tag{22}$$

Because this is decreasing, it is excluded. Also, for $\alpha > 0$ (and still $0 < a < 1$) $\frac{1-ae^{\alpha z}}{1-a} > 0$ at least for $0 < z < \frac{1}{\alpha} \ln \frac{1}{a}$ and h, as given by (22), still decreases for these z, and is thus again excluded.

Because $\overline{a} = 0$ and $\overline{a} = -1$ were excluded, the remaining case is $\overline{a} < -1$, i.e., $a := -\overline{a} > 1$. Then (21) becomes

$$h(z) = \frac{1}{\alpha} \ln \left| \frac{ae^{\alpha z} - 1}{a - 1} \right|. \tag{23}$$

If $\alpha > 0$, then $\frac{ae^{\alpha z}-1}{a-1} > 0$, and so the absolute value symbol is not needed. But (7) is not satisfied, because it would mean

$$h(z) = \frac{1}{\alpha} \ln \frac{ae^{\alpha z} - 1}{a - 1} \leq z,$$

that is, $ae^{\alpha z} - 1 \leq (a-1)e^{\alpha z}$, $e^{\alpha z} \leq 1$, $\alpha \leq 0$, in contradiction to $\alpha > 0$. Therefore this solution is excluded.

Finally, if $\alpha < 0$ (and still $a > 1$), then the absolute value symbol might be needed because $ae^{\alpha z} - 1 \gtreqless 0$ if $z \lesseqgtr \frac{1}{\alpha} \ln \frac{1}{a}$ $(a > 1)$. We show that h in (23) does not satisfy the inequality (7), $h(z) \leq z$, for $\alpha < 0$ either. Every z in $[0, \frac{1}{\alpha} \ln \frac{1}{a}[$ is a counter example. On that interval the fraction in (23) is positive (remember $\alpha < 0$), so no absolute value symbol is needed, and

$$h(z) = \frac{1}{\alpha} \ln \frac{ae^{\alpha z} - 1}{a - 1} \leq z$$

would imply that $ae^{\alpha z} - 1 \geq (a-1)e^{\alpha z}$, i.e. $e^{\alpha z} \geq 1$, $\alpha \geq 0$, in contradiction to $\alpha < 0$. Therefore also this solution is excluded.

Substitution shows that the functions given in 1, 2, and 3 indeed satisfy (6) and (7), and looking at their form we see that they (in class 1 with appropriate f) are bounded iff $\kappa < \infty$. □

Remark If the inequality (7) is replaced by $h(z) < z$ for $0 < z < \kappa$ (the inequality presented in the Introduction), then in solution (10) $0 < s < 1$ has to hold, while (8) clearly does not satisfy this sharper inequality.

Conclusions

Solution 2 is what Luce and Marley (2003) found to be the only solution when the same ideas are applied to gambles of order $n > 2$, and they had expected that it might also be true in the binary context. As solution 3 shows, that is untrue, so that solution is something that has to be born in mind in the theory of binary gambles in isolation. Luce and Marley (2003) work out implications of (11) and (12). The above Remark establishes that for gambles the solution (8) is irrelevant and solution (10) is limited to $s < 1$.

As noted earlier, it would be of interest to find the solutions without assuming differentiability. The general solution will, of course, include those of the present theorem but, perhaps, others.

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