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Ranked-Weighted Utilities and Qualitative Convolution

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Abstract

For gambles—non-numerical consequences attached to uncertain chance events—analogues are proposed for the sum of independent random variables and their convolution. Joint receipt of gambles is the analogue of the sum of random variables. Because it has no unique expansion as a first-order gamble analogous to convolution, a definition of qualitative convolution is proposed. Assuming ranked, weighted-utility representations (RWU) over gains (and, separately, over losses, but not mixtures of both), conditions are given for the equivalence of joint receipt, qualitative convolution, and a utility expression like expected value. As background, some properties of RWU are developed.

Keywords: gains decomposition, qualitative convolution, ranked weighted utility, rank-dependent utility, utility convolution

JEL Classification: C000, C440

Over the past decade, the second author has explored some consequences of adding to the usual gambling structure for uncertain alternatives the concept of the joint receipt of consequences and gambles (see Luce, 1991, 2000; Luce and Fishburn, 1991). In a way, joint receipt seems to play for gambles a role not unlike the sum of independent lotteries with money consequences and known probabilities. The latter is, of course, a special case of the sum of random variables. Yet, the analogy is by no means complete. For random variables we know in detail the distribution of their sum—the convolution expression. Recall that if the distribution of one random variable L is $F_L = (x_1, p_1; \ldots; x_i, p_i; \ldots; x_m, p_m)(x_i \in \mathbb{R}, p_i \geq 0, \sum_{i=1}^m p_i = 1)$ and that of a second random variable L' is $F_{L'} = (y_1, q_1; \ldots; y_k, q_k; \ldots; y_n, q_n)(y_k \in \mathbb{R}, q_k \geq 0, \sum_{k=1}^n q_k = 1)$, then if L and L' are independent random variables the distribution of

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the sum $L + L'$ is given by

$$(x_1 + y_1, p_1 q_1; \ldots; x_i + y_k; p_i q_k; \ldots; x_m + y_n; p_m q_n), \tag{1}$$

which is usually denoted $F_L * F_{L'}$. Further, computing the expected value of the convolution, we have

$$EV(L + L') = \sum_{i=1}^{m} \sum_{k=1}^{n} (x_i + y_k) p_i q_k. \tag{2}$$

This expression is correct whether or not L and L' are independent.

We explore generalizations of Eqs. (1) and (2) to general finite gambles in which the consequences need not be amounts of money and the underlying chance events need not be described in terms of probabilities. To do so, three main issues must be considered:

First, arithmetic addition, $+$, has to be replaced by some qualitative concept of "addition" such as having or receiving two (or more) things—joint receipt, which is denoted $\oplus$.

Second, one must replace the probabilities $p_i q_k$ in the convolution expression, Eq. (1), and in its expected value, Eq. (2), by something having to do with qualitative events.

Third, in calculating an analogue of EV one needs some numerical representation of non-numerical consequences, a utility function, as well as some numerical representation of chance events, a weighting function. This means that some form of utility expression must be postulated. We assume a quite general type, called ranked-weighted utility (RWU), which includes as special cases several familiar models. Some of the special cases have been axiomatized (see Section 2), but the general representation has not been.

Section 1 presents a detailed description of the underlying model of gambles, and we report, mostly known, results about the rank-dependent utility (RDU) of binary gambles and the utility of their joint receipt. RDU is one familiar model that is a special case of RWU; its structure is described in the next paragraph. A key concept that arises here is a kind of generalized additivity of U over joint receipts, $\oplus$, which is called p-additivity.

The second section concerns the broad class of RWU representations of finite gambles. That representation has the following general features. The consequence-event pairs (x_i, C_i), $i = 1, \ldots, m$, forming a gamble are ordered and indexed according to decreasing preference among the consequences. The utility U of a gamble with m consequence-event pairs is given by an ordered, weighted average of the utilities of the consequences, where the weights $S_{i,m}$ depend on the entire event partition. For some purposes, the weights are better rewritten in the form $S_{i,m} = W_{i,m} - W_{i-1,m}$, with $W_{0,m} = 0$ and $W_{m,m} = 1$. In the binary case, $m = 2$, the RWU model is, in fact, what is called in the literature the binary rank-dependent utility model. Some properties of some members of the RWU class are developed. The RDU model expresses $W_{i,m}$ as a function of $\bigcup_{j=1}^{i} C_j$. The gains decomposition utility (GDU) model expresses $W_{i,m}$ as a product of weights from lower order subgambles. Axiomatizations for RDU are known and we give a sample one for GDU.

The third section introduces our definitions of qualitative convolution and its "expected value." The major idea of qualititative convolution is that if the typical consequence-events pairs of two independent gambles are (x_i, C_i) with weight $S_{i,m}$ and (y_k, D_k) with weight $S'_{k,n}$ then a typical term of the convolution is $(x_i \oplus y_k, (C_i, D_k))$ and a typical term of the expected value (utility) is $U(x_i \oplus y_k)S_{i,m}S'_{k,n}$. The main result presents conditions on the form of the utility functions and weights of the RWU representations that cause $\oplus$ and the two other convolution concepts to be equivalent in the sense that their utility expressions agree. The exact conditions differ depending on whether or not U is additive over $\oplus$. When it is not additive, $\oplus$ and the operation defined as the natural utility analogue of expected value are equivalent if and only if U is p-additive (defined on page 140). Also qualitative convolution is equivalent to the expected utility operation if and only if the weights associated with the (i, k) term are products of the respective weights of the i and k terms of the individual gambles. What is tricky is to keep track of the ordering among the consequences of the convolved gamble. For additive U, the conditions for equivalence are somewhat weaker.

The upshot of these results is that the concepts introduced form a consistent extension to gambles of gains of the concepts of $+$, $*$, and $EV(*)$ for random variables. For lotteries and assuming either of the two special cases of RWU that have been axiomatized—RDU and GDU—the equivalence of ordinary convolution and qualitative convolution corresponds to utility representation actually being expected utility with a very specific form for the utility function. The case of mixed gains and losses is discussed briefly, establishing that nothing simple holds in that case.

The fourth section provides a summary, and the fifth gives proofs.

1. Framework and binary utility representation

1.1. Notation

The following primitives and structure of compound gambles and joint receipts are similar to those described in considerable detail in Luce (2000). The major new aspects are an enlarged domain of chance events and a different way of introducing joint receipt. The ultimate structure $\langle \mathscr{D}_2, e, \succsim \rangle$ is generated inductively in the following fashion. We begin with four primitives $\mathscr{C}$, e, $\mathscr{E}_{\mathbf{E}}$, and $\succsim$ and proceed inductively to the full structure being axiomatized.

- $\mathscr{C}$ is a set of pure (i.e., certain but in general non-numerical) consequences with $e \in \mathscr{C}$ a distinguished consequence which we interpret as no change from the status quo.
- $\mathscr{E}_{\mathbf{E}}$ is an algebra of chance events generated by an experiment (in the sense of statistics, namely, a source of chance outcomes, not in the sense of experimental science) or chance phenomenon $\mathbf{E}$ with universal event Ω.
- Define $\mathscr{D}_0 := (\mathscr{C} \times \mathscr{C}) \cup \mathscr{C}$. For $(x, y) \in \mathscr{C} \times \mathscr{C}$, we use the notation $x \oplus y := (x, y)$ and speak of *joint receipt*. The interpretation is having or receiving both x and y.

(Defined in this fashion, $\oplus$ is not a closed operation on $\mathscr{C}$; however, see the remarks at the beginning of Section 1.3.)

- $\succsim$ is a weak order over $\mathscr{D}_0$ which is interpreted as a preference order over pure consequences and their joint receipt. Because below we will assume that e is an identity under $\oplus$ in the sense that

$$x \oplus e \sim e \oplus x \sim x, \tag{3}$$

one may identify $\mathscr{C} \times \{e\}$ with $\mathscr{C}$.

- A general first-order, rank-ordered gamble generated from $\mathscr{D}_0$ and $\mathscr{E}_{\mathbf{E}}$ is denoted

$$g = (x_1, E_1; \ldots; x_i, E_i; \ldots; x_m, E_m), \tag{4}$$

where $x_i \in \mathscr{D}_0, x_1 \succsim x_2 \succsim \cdots \succsim x_m$, and $\{E_i\}$, where $E_i \in \mathscr{E}_{\mathbf{E}}$, is a partition of Ω. The interpretation is that x_i is the consequence if the outcome of running $\mathbf{E}$ lies in the event E_i. The notation for binary gambles is simplified to $(x, C; y)$ where $x = x_1, y = x_2, C = E_1$, and explicit mention of $E_2 = \Omega\backslash C$ is omitted. In particular, such gambles as $(x \oplus u, C; y \oplus v)$ are included.

- The concept of a first-order gamble can be generalized to include cases where the event is actually a pair of events, i.e., $E_i = (C_i, D_i) \in \mathscr{E}_{\mathbf{E}} \times \mathscr{E}_{\mathbf{E}}$, where $\{C_i\}, \{D_i\}$ are both partitions of E. We think of these pairs as arising from independently realized chance experiments. One may subsume the simple case within the more complex one by identifying (C_i, Ω) with C_i provided we assume that

$$(x_1, (C_1, \Omega); \ldots; x_m, (C_m, \Omega)) \sim (x_1, C_1; \ldots; x_m, C_m). \tag{5}$$

We also subsume $\mathscr{C}$ among the gambles by assuming the certainty property holds:

$$(x_1, \Omega; x_2, \varnothing; \ldots; x_m, \varnothing) \sim x_1. \tag{6}$$

Let $\mathscr{G}_1$ denote the set of all such gambles generated from $\mathscr{D}_0$ and $\mathscr{E}_{\mathbf{E}} \times \mathscr{E}_{\mathbf{E}}$.

- Let $\mathscr{D}_1 := \mathscr{G}_1 \times \mathscr{G}_1$ and extend the notation $\oplus$ so that if $f, g \in \mathscr{G}_1$ then $f \oplus g := (f, g)$, which is interpreted to mean that the decision maker receives or holds both of the gambles f and g.
- If only $m = 2$ is used in the above constructions, then the resulting set of binary, first-order gambles and their joint receipts is denoted $\mathscr{B}_1$.
- Let $\succsim$ be a weak preference order over $\mathscr{D}_1$ that when restricted to $\mathscr{D}_0$ agrees with the earlier order $\succsim$.
- $\mathscr{G}_2, \mathscr{D}_2$, and $\mathscr{B}_2$ are generated from $\mathscr{G}_1, \mathscr{D}_1$, and $\mathscr{E}_{\mathbf{E}}$ in an analogous fashion, resulting in compound gambles in the sense that the x_i are in $\mathscr{G}_1$. In the binary case, the notation is abbreviated as $(g, C; h)$ and the set of such gambles is denoted $\mathscr{B}_2$. By extending the certainty assumption, Eq. (5), and the identity assumption, Eq. (3), to gambles, we subsume the earlier structures into these. So, we see that $\mathscr{C} \subset \mathscr{D}_0 \subset \mathscr{D}_1 \subset \mathscr{D}_2$.
- Let $\succsim$ be a weak order over $\mathscr{D}_2$ that when restricted to $\mathscr{D}_1$ agrees with the earlier $\succsim$.
- Gains and losses are defined relative to e in terms of $\succsim$ as follows: $\mathscr{C}^+ := \{x\colon x \in \mathscr{C}$ and $x \succsim e\}$, and $\mathscr{B}_i^+, \mathscr{G}_i^+,\ i = 1, 2$, and $\mathscr{D}_i^+,\ i = 0, 1, 2$, are induced from the gains consequences, $\mathscr{C}^+$. Similar definitions hold for losses.

1.2. Basic behavioral properties

Certain basic behavioral properties are assumed. The first group involves just binary gambles and $\succsim$. Let $x, y, z \in \mathscr{G}_1$, $C \in \mathscr{E}_{\mathbf{E}}$,
Weak Order: $\succsim$ is transitive and complete.
Certainty: $(x, \Omega; y) \sim x$.
Idempotence: $(x, C; x) \sim x$.
Complementarity: $(x, C; y) \sim (y, \overline{C}; x)$.
Consequence Monotonicity: For $\varnothing \subset C \subset \Omega$, $x \succsim y \Leftrightarrow (x, C; z) \succsim (y, C; z)$.
Order Independence of Events: For $x, y \succ e$, $(x, C; e) \succsim (x, D; e) \Leftrightarrow (y, C; e) \succsim (y, D; e)$.

The second group includes the operation $\oplus$ of joint receipt as well as gambles and $\succsim$.
JR Commutativity: $x \oplus y \sim y \oplus x$.
JR Monotonicity: $x \succsim y \Leftrightarrow x \oplus z \succsim y \oplus z$.
JR Identity: $x \oplus e \sim x$.

1.3. Binary segregation, binary RDU, and p-additivity

In the following theorem strong continuity assumptions are incorporated. One of these is that for some $K > 0$ (including ∞ as a possibility) there is a ratio-scale utility function $U\colon \mathscr{C}, \mathscr{B}_2^+ \xrightarrow{onto} [0, K[$ that *preserves the order* $\succsim$. Observe that by the fact U is a ratio scale, there is no less of generality in assuming $K > 1$.

Also, the fact that U maps both $\mathscr{C}$ and $\mathscr{B}_2^+$ onto $[0, K[$ means that for any $g \in \mathscr{B}_2^+$ there exists a *certainty equivalent* $CE(g) \in \mathscr{C}$ such that

$$CE(g) \sim g. \tag{7}$$

Further, by the assumed monotonicity of $\oplus$,

$$CE(g \oplus h) \sim g \oplus h \sim CE(g) \oplus CE(h).$$

Also, by consequence monotonicity of gambles

$$(g, C; h) \sim (CE(g), C; CE(h)).$$

So any assertion that holds for $x, y \in \mathscr{C}$ extends to ones in $\mathscr{B}_1^+$ and $\mathscr{B}_2^+$. We will often state things for $x, y \in \mathscr{D}_0^+$, but in subsequent applications invoke the result for more complex gambles.

Given the existence of certainty equivalents, one may extend $\oplus$ from being a partial operation to a closed operation on $\mathscr{C}$ since $x \oplus y \sim CE(x \oplus y)$. In particular, this means that $(x \oplus y) \oplus z \sim CE(x \oplus y) \oplus z$ is defined. This plays a role in an observation below.

Theorem 1 (Luce and Fishburn, 1991, 1995)**.** *Suppose that $\langle \mathscr{B}_2^+, e, \succsim \rangle$ is a structure generated from $\mathscr{C}, e, \mathscr{E}_{\mathbf{E}}$ and $\oplus$ is defined as above. Assume that the structure satisfies weak order, consequence monotonicity, JR commutativity, JR monotonicity, and*

JR identity. In the following statements it is assumed that U: $\mathscr{C}, \mathscr{B}_2^+ \xrightarrow{onto} [0, K[$ *preserves the order* $\succsim$, $U(e) = 0$, *for experiment* $\mathbf{E}$, $W_{\mathbf{E}}$: $\mathscr{E}_{\mathbf{E}} \xrightarrow{onto} [0, 1]$, *and* $C \subseteq D$ *implies* $W_{\mathbf{E}}(C) \leq W_{\mathbf{E}}(D)$. *Then, any two of the following three statements imply the third: for* $x, y, x \oplus y \in \mathscr{D}_0^+$,

(i) Binary segregation *holds in the sense that*

$$(x \oplus y, C; y) \sim (x, C; e) \oplus y. \tag{8}$$

(ii) *(U, W) forms the following binary rank-dependent utility* (*RDU*) *representation,*

$$U(x, C; y) = \begin{cases} U(x)W_{\mathbf{E}}(C) + U(y)[1 - W_{\mathbf{E}}(C)], & x \succsim y \\ U(x)[1 - W_{\mathbf{E}}(\overline{C})] + U(y)W_{\mathbf{E}}(\overline{C}), & x \succ y. \end{cases} \tag{9}$$

(iii) *(U, W) forms the following representation: for a dimensional constant* δ *with unit* $=$ 1/*unit of U,*

$$U(x \oplus y) = U(x) + U(y) - \delta U(x)U(y), \tag{10}$$

and

$$U(x, C; e) = U(x)W_{\mathbf{E}}(C). \tag{11}$$

The form of Eq. (10) is called *p*(*olynomial*)*-additive* because it is the only polynomial form with $U(e) = 0$ that is transformable into a non-negative, additive representation V over gains. In particular, when Eq. (10) holds with $\delta = 0$, the additive representation is, of course, just U; when $\delta \neq 0$, it is $\kappa V(x) = -\text{sgn}(\delta)\ln[1 - \delta U(x)]$, $\kappa > 0$. Whenever we speak of U being p-additive it is implicit that this means relative to the operation $\oplus$ and not to any of the convolution operations later introduced. Thus, for case (iii), it follows that $\oplus$ is not only commutative but also associative for gains.

Similar results hold for losses. The case of mixed gains and losses, which is more complex, is discussed briefly in Section 3.3.

Observe that binary RDU implies all of the listed behavioral properties of gambles, in particular idempotence and certainty which are used in proofs.

The following observation, which proves a generalized form of segregation, rests on the assumption that the operation $\oplus$ has an order-preserving additive representation V with $V(e) = 0$. This, of course, holds under the conclusions of Theorem 1 and, because *CE*s exist, means that $\oplus$ is defined for all pairs and is associative. Next, observe that by the "onto" nature of U, for any $x, y \in D_0^+$ such that $x \succsim y$, there exists $w \in C^+$ such that $x \sim y \oplus w$. We now show that if segregation holds then it holds in the more general form $(x \oplus z, C; y \oplus z) \sim (x, C; y) \oplus z$. Consider the case $x \succsim y$. Then

$$\begin{aligned}(x \oplus z, C; y \oplus z) &\sim ((y \oplus w) \oplus z, C; y \oplus z) \sim (w \oplus (y \oplus z), C; y \oplus z) \\ &\sim (w, C; e) \oplus (y \oplus z) \sim [(w, C; e) \oplus y] \oplus z \\ &\sim (w \oplus y, C; y) \oplus z \sim (x, C; y) \oplus z.\end{aligned}$$

The case $x \prec y$ follows from complementarity.

2. Ranked-weighted utility representations

2.1. Ranked-weighted utility (RWU)

Let g be a gamble of order m based on the chance phenomenon $\mathbf{E}$ with universal event Ω. Let $\mathcal{P}_{\mathbf{E},m}$ denote the set of ordered partitions of Ω into m non-empty subsets.

We now consider a very broad class of representations, which others have mentioned but not studied very carefully. Two special cases are discussed in detail below.

Definition 1. Within the domain of second-order gambles of gains, $\mathcal{G}_2^+$, a utility representation with utility function U: $\mathcal{G}_2^+ \longrightarrow \mathbb{R}_+$ and weights $S_{i,m}$: $\mathcal{P}_{\mathbf{E},m} \to [0,1]$, $i = 1, 2, \ldots, m$, is said to be a member of the *class of ranked weighted utilities* (*RWU*) iff for each partition in $\mathcal{P}_{\mathbf{E},m}$,

$$0 < S_{i,m} < 1, i = 1, 2, \ldots, m, \sum_{i=1}^{m} S_{i,m} = 1 \tag{12}$$

and for $x_1 \succsim \cdots \succsim x_m \succsim e$,

$$U(x_1, E_1; \ldots; x_m, E_m) = \sum_{i=1}^{m} U(x_i) S_{i,m}. \tag{13}$$

Note that the argument of the $S_{i,m}$, namely the entire partition $\{E_1, E_2, \ldots, E_m\}$, is left implicit in Eq. (13). Note also that for $m = 2$, this reduces to the binary RDU form of Eq. (9).

We may write this expression in an alternative way. Define

$$W_{i,m} = \begin{cases} 0, & i = 0 \\ W_{i-1,m} + S_{i,m}, & 0 < i < m \\ 1, & i = m. \end{cases}$$

Clearly, $S_{i,m} = W_{i,m} - W_{i-1,m}$ which means that

$$0 = W_{0,m} < W_{i-1,m} < W_{i,m} < W_{m,m} = 1, \qquad i = 2, \ldots, m-1, \tag{14}$$

and that

$$U(x_1, E_1; x_2, E_2; \ldots; x_m, E_m) = \sum_{i=1}^{m} U(x_i)(W_{i,m} - W_{i-1,m}). \tag{15}$$

The reason for using the difference representation $W_{i,m} - W_{i-1,m}$ of weights, as in Eq. (15), rather than the apparently simpler $S_{i,m}$ one, will become apparent below.

This ranked form includes several well-known expressions. The two special cases discussed below in Sections 2.3 and 2.4 have been axiomatized, and the first has been carefully investigated. Both yield specific formulas for the weights $W_{i,m}$ in terms of the binary weights. It also includes the complex of "configural weight models" of M. H. Birnbaum for which no detailed axiomatic formulations have been provided (see, e.g., Birnbaum et al., (1992) and Luce (2000, §5.5.2.4) for additional references). We do not have any general classification of all RWU models.

The following is the explicit formula for the special case of RWU for four consequences which we use to illustrate general theorems. For $x_1 \succsim x_2 \succsim x_3 \succsim x_4 \succsim e$,

$$U(x_1, E_1; x_2, E_2; x_3, E_3; x_4, E_4) = U(x_1)W_{1,4} + U(x_2)(W_{2,4} - W_{1,4}) + U(x_3)(W_{3,4} - W_{2,4}) + U(x_4)(1 - W_{3,4}). \tag{16}$$

2.2. General segregation

Using the operation $\oplus$ of joint receipt, we may generalize the concept of binary segregation, Eq. (8), that arose in Theorem 1 to:

Definition 2. Within the domain of first-order gambles of gains, x_1, *segregation* (*of order m*) holds iff for $x_1 \succsim \cdots \succsim x_m \succsim e$,

$$(x_1 \oplus x_m, E_1; \ldots; x_{m-1} \oplus x_m, E_{m-1}; x_m, E_m) \sim (x_1, E_1; \ldots; x_{m-1}, E_{m-1}; e, E_m) \oplus x_m. \tag{17}$$

Segregation of order 2 agrees with binary segregation, Eq. (8).

Theorem 2. *Suppose the conditions of Theorem* 1 *are satisfied, binary segregation holds, and general gambles have an RWU representation. Then segregation of order m holds.*

The proof of this result is given, as are all proofs, in Section 5.

2.3. Two special cases

2.3.1. Rank-dependent utility (***RDU***)**.** In what follows, it is convenient to introduce the notation

$$E(j) = \bigcup_{i=1}^{j} E_i.$$

Note that

$$\Omega = E(m) = E(m-1) \cup E_m \qquad \text{and so} \qquad \Omega \backslash E_m = E(m-1).$$

One general rank-dependent representation that has been studied fairly extensively (Luce, 1998; Quiggin, 1993; Schmeidler, 1989) is:

Definition 3. Suppose that binary RDU, Eq. (9), holds on event E with weights $W_{\mathbf{E}}$: $\mathscr{E}_{\mathbf{E},m} \longrightarrow [0, 1]$. Within the domain of second-order gambles of gains, $\mathscr{G}_2^+$, a *rank-dependent* (*RDU*) *representation* holds iff the weights in RWU, Eq. (15), are given by

$$W_{i,m} = W_{\mathbf{E}}[E(i)]. \tag{18}$$

It is easy to verify that RDU implies that the behavior exhibits the following property of *coalescing* (*or event splitting*): whenever $x_{j+1} = x_j$, then

$$\begin{aligned}(x_1, E_1; \dots; x_j, E_j; x_j, E_{j+1}; \dots; x_m, E_m)\\ \sim (x_1, E_1; \dots; x_j, E_j \cup E_{j+1}; \dots; x_m, E_m).\end{aligned} \tag{19}$$

This property is clearly rational since the difference between the right and left sides is merely whether the two events with the common outcome are treated as separate or as their union. The fact of the matter, as discussed in Luce (2000, Ch. 5), is that RDU, and so coalescing, does not appear to be descriptively accurate.

Binary RDU, coalescing, and some added additivity conditions, e.g., RWU, imply RDU (Luce, 1998, 2000). Coalescing serves as an induction device.

2.3.2. Gains-decomposition utility (***GDU***). An alternative realization of the weights in the ranked weighted utility family is:

Definition 4. Within the domain of second-order gambles of gains, $\mathscr{G}_2^+$, a *gains-decomposition utility* (*GDU*) *representation* holds iff for a family of binary weights $W_{\mathbf{E}(\mathrm{i})}$: $\mathscr{E}_{\mathbf{E},i} \longrightarrow [0, 1]$, $i = 1, \dots, m$, the weights in Eq. (15) are given by

$$W_{i,m} = \begin{cases} 0, & i = 0 \\ \prod_{j=i}^{m-1} W_{\mathbf{E}(\mathrm{j}+1)}[E(j)], & 0 < i \le m-1 \\ 1, & i = m. \end{cases} \tag{20}$$

The definition of GDU representation is justified primarily by Part (i) of Theorem 3 in Section 2.4.2.

Note two things: First, for $m = 2$ we have $W_{1,2} = W_{\mathbf{E(2)}}[E(1)]$, which of course is identical to the binary case of RDU. Second, since all of the weights are based on terms $W_{\mathbf{E}(\mathrm{j}+1)}[E(j)]$, they can all be estimated from data on binary gambles and the predictions for $m > 2$ are parameter-free. In particular, for the $m = 4$ case, we have

$$W_{1,4} = W_{\mathbf{E(2)}}[E(1)]W_{\mathbf{E(3)}}[E(2)]W_{\mathbf{E(4)}}[E(3)] \tag{21}$$

$$W_{2,4} = W_{\mathbf{E(3)}}[E(2)]W_{\mathbf{E(4)}}[E(3)] \tag{22}$$

$$W_{3,4} = W_{\mathbf{E(4)}}[E(3)]. \tag{23}$$

Note that by Eq. (14), $0 < W_{1,4} < W_{2,4} < W_{3,4}$, so we may rewrite the above equations as

$$W_{\mathbf{E(2)}}[E(1)] = \frac{W_{1,4}}{W_{2,4}} \qquad W_{\mathbf{E(3)}}[E(2)] = \frac{W_{2,4}}{W_{3,4}} \qquad W_{\mathbf{E(4)}}[E(3)] = W_{3,4}.$$

2.4. Axiomatization of GDU

2.4.1. Gains decomposition. Suppose $g^{(m)} \in \mathcal{G}_1^+$ is a gamble or order m. Define the following subgamble of $g^{(m)}$:

$$g^{(m-1)} = (x_1, E_1; \ldots; x_{m-1}, E_{m-1}). \tag{24}$$

Note that $g^{(m-1)} \in \mathcal{G}_1^+$ is based on the subexperiment $\mathbf{E(k-1)}$ with the universal event $E(k-1)$, i.e., the restriction of $\mathbf{E(k)}$ to outcomes in $E(k-1)$, and it is of order $m-1$.

The following definition modifies slightly the terminology used by Luce (2000, Ch. 5) which, in turn, generalized a property introduced by Liu (1995) for known probabilities to more general events:

Definition 5. Within the domain of second-order compound gambles of gains, $\mathcal{G}_2^+$, *gains decomposition* (*of order m*) holds if and only if for $g^{(m)} \in \mathcal{G}_1^+$ with $x_1 \succsim \cdots \succsim x_m \succsim e$,

$$g^{(m)} \sim \left(g^{(m-1)}, E(m-1); x_m, E_m\right), \tag{25}$$

where $(g^{(m-1)}, E(m-1); x_m, E_m) \in \mathcal{G}_2^+$. The right side is the compound gamble that first involves running the chance experiment $\mathbf{E(m)}$. If the outcome lies in the event $E_m = \Omega \backslash E(m-1)$, then the consequence is x_m. If, however, the outcome lies in $E(m-1)$, then the subgamble $g^{(m-1)}$ is the consequence attached to it, and so the experiment $\mathbf{E(m-1)}$ is next run to determine which consequence x_i, $i = 1, \ldots, m-1$, is received.

This is a rational property in the sense that the bottom lines associated with the two sides are identical, with the difference being whether one or two chance phenomena are carried out. In general, it is not consistent with coalescing, although each property seems rational in its own right. Gains decomposition has not, to our knowledge, received any empirical study.

2.4.2. Existence of GDU representation.

Theorem 3. *Within the domain of second-order compound gambles of gains, the following are true.*

(i) *GDU for gains* (*Def.* 4) *holds iff binary RDU and gains decomposition* (*Def.* 5) *hold.*
(ii) *RDU and GDU both hold iff the weights satisfy the following property: For all* $C \in \mathcal{E}_{\mathbf{D}}$ *and* $C, D \in \mathcal{E}_{\mathbf{E}}$, *i.e.,* $C \subseteq D \subseteq \Omega$,

$$W_{\mathbf{E}}(C) = W_{\mathbf{D}}(C) W_{\mathbf{E}}(D). \tag{26}$$

It should be noted that Eq. (26) is just the choice axiom investigated by Luce (1959) when the weights are finitely additive probabilities.

Corollary. *Let $g^{(m)}$ and $g^{(m-1)}$ be defined as in Eqs.* (4) *and* (24) *and $g^{(1)} = x_1$. Then if GDU holds*

$$W_{\mathbf{E(m)}}[E(m-1)] = \frac{U(g^{(m)}) - U(x_m)}{U(g^{(m-1)}) - U(x_m)}.$$

2.4.3. Double gains decomposition. Another form of gains decomposition involves separating off x_1 and $g_{\overline{E}_1}^{(m-1)} = (x_2, E_2; \ldots; x_m, E_m)$ and assuming

$$g^{(m)} \sim (x_1, E_1; g_{\overline{E}_1}^{(m-1)}, \overline{E}_1). \tag{27}$$

Theorem 4. *Suppose binary RDU holds. Then both forms of gains decomposition, Eq.* (25) *and* (27)*, hold for $m = 3$ iff the binary weights based on* E(3) *and its subsets satisfy the choice axiom, Eq.* (26)*, and for some constant $\rho \neq 0$*

$$W_{\mathbf{E(3)}}(E_1 \cup E_2) = W_{\mathbf{E(3)}}(E_1) + W_{\mathbf{E(3)}}(E_2) - \rho W_{\mathbf{E(3)}}(E_1) W_{\mathbf{E(3)}}(E_2), \tag{28}$$

when $E_1 \cap E_2 = \varnothing$.

Corollary. *Under the conditions of the theorem,*

$$\begin{aligned} 1 = {} & W_{\mathbf{E(3)}}(E_1) + W_{\mathbf{E(3)}}(E_2) + W_{\mathbf{E(3)}}(E_3) \\ & - \rho[W_{\mathbf{E(3)}}(E_1)W_{\mathbf{E(3)}}(E_2) + W_{\mathbf{E(3)}}(E_1)W_{\mathbf{E(3)}}(E_3) + W_{\mathbf{E(3)}}(E_2)W_{\mathbf{E(3)}}(E_3)] \\ & + \rho^2 W_{\mathbf{E(3)}}(E_1)W_{\mathbf{E(3)}}(E_2)W_{\mathbf{E(3)}}(E_3). \end{aligned} \tag{29}$$

The p-additive form of Eq. (28) relative to disjoint unions means that there is a finitely additive function P such that $\rho W_{\mathbf{E(3)}}$ is exponentially or negatively exponentially related to P. The normalization that $W_{\mathbf{E(3)}}(\varnothing) = P(\varnothing) = 0$ can be imposed without any constraint on the parameters. The normalization $W_{\mathbf{E(3)}}[E(3)] = P[E(3)] = 1$ together with $\rho W_{\mathbf{E(3)}}(C) = 1 - \exp[-\kappa P(C)]$, $\rho\kappa > 0$ imposes the constraint $\rho = 1 - e^{-\kappa}$, $\rho\kappa > 0$. Presumably in the case of known probabilities, we may assume $P(E_i) = p(E_i)$, in which case the weights do not exhibit the inverse S-function for which there is considerable evidence at both individual and group levels. The group data are, however, consistent with an average of weighting functions with some above and some below the identity line. (For more detail, see Luce, 2000, Ch. 3.) But the fact that individuals seem to have inverse S-shaped weights makes the use of gains decomposition in both directions a dubious assumption.

3. Generalizations of convolution

In this section we explore conditions under which various pairs of three different binary operations on gambles are always judged indifferent. One operation is joint receipt, which is a qualitative analogue of the addition of independent random variables. The second is a qualitative analogue of convolution, the first-order distribution of the sum of random variables. And the third is a utility analogue of the expected value of a convolution.

3.1. Two definitions

To introduce qualitative analogues of convolution and the expected value of a convolution using $\oplus$ rather than $+$, we must suppose that the experiment **E** is run twice and independently. Let g and h be gambles based, respectively, on the partitions $\{C_1, \ldots, C_i, \ldots, C_m\}$ with consequences $x_i \succ x_j$ iff $i > j$ and $\{D_1, \ldots, D_k, \ldots, D_n\}$ with consequences $y_k \succ y_l$ iff $k > l$. Let (C_i, D_k) denote that event C_i occurs in the first realization of **E** and that event D_k occurs in the second realization of **E**.

Definition 6. *Qualitative convolution* $\circledast$ *on* $\mathcal{G}_1^+$ is defined by

$$g \circledast h := (x_1 \oplus y_1, (C_1, D_1); \ldots; x_i \oplus y_k, (C_i, D_k); \ldots; x_m \oplus y_n, (C_m, D_n)). \tag{30}$$

When we assume that RWU representations obtain, we denote the weight associated with the ith component of $U(g)$ by $S_{i,m}$, that of the kth component of $U(h)$ by $S'_{k,n}$, and that of the rth component of $U(g \circledast h)$ by $S''_{r,mn}$.

A minor observation is that if, for gambles $g = (x_1, C_1; \ldots; x_i, C_i; \ldots; x_m, C_m)$ and $h = (y_1, D_1; \ldots; y_i, D_i; \ldots; y_n, D_n)$, we assume that reduction of compound gambles holds in the form

$$(\ldots; (\ldots; x_i \oplus y_k, D_k; \ldots), C_i; \ldots) \sim (\ldots; x_i \oplus y_k, (C_i, D_k); \ldots),$$

then defining

$$g \otimes h := (\ldots; (\ldots; x_i \oplus y_k, D_k; \ldots), C_i; \ldots),$$

leads immediately to $g \otimes h \sim g \circledast h$.

Next we consider a generalization of the right side of the expected value of the sum of two independent random variables, Eq. (2). In the proposed generalization we let $U(x_i \oplus y_k)$ play the role of $x_i + y_k$, $S_{i,m}$ play the role of $p_i = \Pr(C_i)$, and $S'_{k,n}$ the role of $q_k = \Pr(D_k)$.

Definition 7. Suppose that RWU holds. *Utility convolution* $\boxplus$ *on* $\mathcal{G}_1^+$ is defined by

$$U(g \boxplus h) := \sum_{i=1}^{m} \sum_{k=1}^{n} U(x_i \oplus y_k) S_{i,m} S'_{k,n}. \tag{31}$$

3.2. The equivalence of convolution concepts for gains

Our focus will be on conditions that cause pairs of $\oplus, \boxplus, \circledast$ to agree in the following sense:

Definition 8. Define the *equivalence*, $\approx$, of any two binary operations $\circledcirc_1, \circledcirc_2$ on $\mathcal{G}_1^+$ by:

$$\circledcirc_1 \approx \circledcirc_2 \Leftrightarrow \forall g, h \in \mathcal{G}_1^+ (g \circledcirc_1 h \sim g \circledcirc_2 h). \tag{32}$$

One of the assumed conditions will be the existence of continuous RWU. The other concerns $\oplus$ and it is stated in terms of its utility representation. For $\zeta, \eta \in [0, K[$, the image of U, define

$$F(\zeta, \eta) := U[U^{-1}(\zeta) \oplus U^{-1}(\eta)]. \tag{33}$$

For $\oplus$ satisfying the background assumptions, note four things about F:

1. Because of the monotonicity of $\oplus$, F is well defined and strictly increasing in each variable.
2. By the commutativity of $\oplus$, F is symmetric.
3. By the fact e is an identity of $\oplus$ and $U(e) = 0$, $F(\zeta, 0) = F(0, \zeta) = \zeta$.
4. By choice $K > 1$, so $F(1, 1)$ is defined.

Definition 9. Suppose U is a continuous RWU function, then the operation $\oplus$ is said to be *additive* iff for all $\zeta, \eta \in]0, k[$

$$\begin{aligned} &F(\zeta, \eta) = \zeta + \eta \\ \Leftrightarrow\ &U(x \oplus y) = U(x) + U(y), \end{aligned}$$

and *nonadditive* otherwise.

3.2.1. Bookkeeping. Given two gambles, we need to have a systematic way of keeping track of the order among consequences $x_i \oplus y_k$. To that end, we define Φ to be a mapping of $\{1, \dots, m\} \times \{1, \dots, n\}$ to $\{1, \dots, mn\}$ such that for $i, j \in \{1, \dots, m\}$, $k, l \in \{1, \dots, n\}$,

$$\Phi(i, k) < \Phi(j, l) \Leftrightarrow x_i \oplus y_k \succ x_j \oplus y_l, \tag{34}$$

and for any indifference the order is resolved randomly. Note that if f, g are the two gambles under consideration, then (obviously) Φ depends on those gambles. Later, when we deal with several pairs of related gambles, we briefly need to use the notation $\Phi_{f,g}$ to clearly indicate this fact. However, in general we can use the simpler notation of Eq. (34).

3.2.2. The Main Result.

Theorem 5. *Suppose that $\langle \mathscr{G}_2^+, e, \succsim \rangle$ is a structure satisfying the background hypotheses listed in Theorem* 1 *and that there is a continuous RWU representation with utility function U.*

(i) *Suppose that U is not additive.*

(a) $\oplus \approx \boxplus$ *iff U over $\oplus$ is p-additive with $\delta \neq 0$.*

(b) *Suppose, in addition, that the first partial derivatives of F exist everywhere on $[0, K[\times[0, K[$. Then, $\boxplus \approx \circledast$ iff for all $i \in \{1, \ldots, m\}, k \in \{1, \ldots, n\}$,*

$$S''_{\Phi(i,k),mn} = S_{i,m} S'_{k,n}. \tag{35}$$

(ii) *Suppose that U is additive. Then*

(a) $\oplus \approx \boxplus$.
(b) $\boxplus \approx \circledast$ *iff for all $i \in \{1, \ldots, m\}, k \in \{1, \ldots, n\}$,*

$$S_{i,m} = \sum_{k=1}^{n} S''_{\Phi(i,k),mn} \text{ and } S'_{k,n} = \sum_{i=1}^{m} S''_{\Phi(i,k),mn}. \tag{36}$$

One suspects that part (i)(b) is true without the differentiability assumption, but at present we do not know how to prove it except by assuming differentiability.

3.2.3. Binary RDU and GDU expressions. Next, consider the form of weights derived in Theorem 5 on the assumption that RWU is specialized to RDU and U is nonadditive. In this case we may simplify the notation on the right side of Eq. (35) just to W. For $m = n = 2$ and assuming that $x_1 \oplus y_2 \succ x_2 \oplus y_1$, the RDU form yields for the left side:

$$W_{1,4} = W''(C, D) = W(C)W(D), \tag{37}$$

$$W_{2,4} = W''[(C, D) \cup (C, \overline{D})] = W(C), \tag{38}$$

$$W_{3,4} = W''[(C, D) \cup (C, \overline{D}) \cup (\overline{C}, D)] = W(C) + W(D) - W(C)W(D). \tag{39}$$

In the other case $x_2 \oplus y_1 \succ x_1 \oplus y_2$, the only change is that $W_{2,4} = W(D)$. Note that if W'' is finitely additive, i.e., the subjective expected utility model, then the last two equations follow from the first one.

In the additive case, i.e., when Eq. (36) holds, all one obtains is Eq. (38) and the sum of Eqs. (37) and (39)

$$W_{1,4} + W_{3,4} = W(C) + W(D) \tag{40}$$

which is unaffected by the order of the second and third terms.

On the assumption of the GDU representation, we know from Section 2.3.2 that in the nonadditive case, for $m = n = 2$ and $x_1 \oplus y_2 \succ x_2 \oplus y_1$,

$$\frac{W_{1,4}}{W_{2,4}} = W_{\mathbf{E(2)}}[E(1)] = W(D)$$

$$\frac{W_{2,4}}{W_{3,4}} = W_{\mathbf{E(3)}}[E(2)] = \frac{W(C)}{W(C) + W(D) - W(C)W(D)}$$

$$W_{3,4} = W_{\mathbf{E(4)}}[E(3)] = W(C) + W(D) - W(C)W(D),$$

from which Eqs. (37), (38), and (39) follow immediately. For $x_2 \oplus y_1 \succ x_1 \oplus y_2$, again the numerator of the second display is changed to $W(D)$. The additive case is the same as in the RDU case.

In sum, the W expressions for the binary cases of Theorem 5 are identical for RDU and GDU.

3.2.4. Equivalence of convolution and joint receipt for lotteries. For lotteries Cho and Luce (1995) experimentally studied the issue of whether $* \approx \oplus$, where $*$ is given by Eq. (1). They partitioned their informants into two groups, called moderate gamblers and moderate non-gamblers. This was done by collecting choice certainty equivalents for 10 lotteries with small negative expected values. If 7 or more of these lotteries were assigned positive certainty equivalents by an informant, then he or she was classed as a moderate gambler, and for 6 or fewer as a moderate non-gambler. Fourteen of the respondents were of the former type and 26 of the latter. Cho and Luce found that $* \approx \oplus$ appears to be valid for the gamblers, but not for the non-gamblers.

Here we address the implications of $* \approx \oplus$ for the two important special cases RDU and GDU. For events $C, D \in \mathscr{E}_{\mathbf{E}}$, let $\Pr(C)$ be the probability of C, and let $\Pr(C, D)$ be the probability of the event pair (C, D). We say that *RDU holds on lotteries* if RDU, Def. 3, holds and there are strictly increasing functions $W_{\Pr}, W''_{\Pr}\colon [0, 1] \xrightarrow{onto} [0, 1]$ such that $C, D \in \mathscr{E}_{\mathbf{E}}$, $W_{\Pr}[\Pr(C)] = W(C)$, and $W''_{\Pr}[\Pr(C, D)] = W''(C, D)$. We say that *GDU holds on lotteries* if GDU, Def. 4, holds and there is a strictly increasing function $W_{\Pr}\colon [0, 1] \xrightarrow{onto} [0, 1]$ such that for $E(j) \in \mathbf{E}(j+1)$, $j = 1, \ldots, m-1$,

$$W_{\Pr}\left(\frac{\Pr(E(j))}{\Pr(E(j+1))}\right) = W_{\mathbf{E}(j+1)}(E(j)).$$

Under such conditions, the definitions and theorems developed to this point for gambles are valid for lotteries. In particular, they are valid with events replaced by the corresponding event probabilities.

Theorem 6. *Consider a structure of lotteries (money consequences and known probabilities) for which the conditions of Theorem* 1 *and binary segregation are satisfied and either RDU or GDU holds. Then,* $* \approx \oplus$ *iff the representation is actually EU where, for parameters* δ *and* $c \neq 0$, *U is given by one of the following two forms*

$$U(x) = cx,\ c > 0 \qquad \text{if } \delta = 0,$$
$$\delta U(x) = 1 - e^{-cx}, \qquad \text{if } \delta c > 0.$$

We do not have any comparable results for RWU in general, which is not surprising because the RWU class is quite diverse.

3.3. Convolution of mixed gains and losses

The case of mixed gains and losses is considerably more complex (and incomplete) than the case of all gains (and all losses, which is a totally parallel theory to the case of gains). The reason is that, because the utility functions for gains and losses are already determined, it is a question of how they are put together in the mixed case. Under one assumption the situation is largely the same, but under another, more plausible one, the equivalences simply do not hold.

Luce (1997, 2000) assumed that a mixed gamble can be decomposed into a binary gamble whose consequences are the subgamble of gains and the subgamble of losses. Given that assumption, we may restrict our attention to the binary case. To that end, two assumptions about joint receipt of mixed gains and losses have been explored: additive U, i.e., if $x_+ \succsim e \succsim y_-$,

$$U(x_+ \oplus y_-) = U(x_+) + U(y_-) \tag{41}$$

and additive V

$$V(x_+ \oplus y_-) = V(x_+) + V(y_-). \tag{42}$$

From the latter one derives expressions for $U(x_+ \oplus y_-)$ which are nonadditive when U and V are negative exponentially or exponentially related.

Next expressions for $U(x_+, C; y_-)$ are arrived at using these results and either of two assumptions, one a generalization of segregation and the other an empirically suggested form called duplex decomposition. There is no reason to restate them explicitly here.

3.3.1. Additive U. If U is additive, Eq. (41), then, as in Theorem 5, one shows $\oplus \approx \boxplus$. The relation of $\circledast$ to $\boxplus$ depends upon the explicit form for $U[(x_+, C; y_-) \circledast (u_+, D; y_-)]$. For the cases studied in Luce (2000, p. 215, Theorem 6.3.1), one obtains the RWU form for general segregation but not for duplex decomposition as the weights in the representation of the mixed gamble do not add to 1, whereas RWU (in particular, Eq. (12)) requires that they do add to 1. So in the case of general segregation, the proof of Theorem 5 still holds that $\circledast \approx \boxplus$, but it is not valid for duplex decomposition.

3.3.2. Additive V. The case where U is proportional to V reduces to the one just discussed. If U is nonlinear with V, the situation is more complex. Without going into details, when U is negative exponential with V for gains and exponential for losses with the common parameter κ, the expression for F takes the form for $\zeta \geq 0 > \eta$,

$$F(\zeta, \eta) = \begin{cases} \frac{\zeta+\eta}{1+\eta}, & F(\zeta, \eta) \geq 0 \\ \frac{\zeta+\eta}{1-\zeta}, & F(\zeta, \eta) < 0, \end{cases}$$

and so,

$$H(\zeta, \eta) = F(\zeta, \eta) - \zeta - \eta = \begin{cases} -\eta\frac{\zeta+\eta}{1+\eta}, & F(\zeta, \eta) \geq 0 \\ \zeta\frac{\zeta+\eta}{1-\zeta}, & F(\zeta, \eta) < 0. \end{cases}$$

Consider the question of $\oplus \approx \boxplus$. The line of argument used in the proof of Theorem 5(i)(a) shows that if $\oplus \approx \boxplus$ holds, then $H(\zeta, \eta) = \delta\zeta\eta, \delta \neq 0$, which is inconsistent with the above observation.

It is also not the case that $\circledast \approx \boxplus$ because the expression for $\boxplus$ is, by definition, RWU whereas Luce (1997, 2000, p. 215, Theorem 6.3.1) shows that for both general segregation and duplex decomposition the expression for U of a binary mixed gamble is nonlinear in terms of $U(x_+)$ and/or $U(x_-)$ and so $U[(x_+, C; y_-) \circledast (u_+, D; v_-)]$ cannot be RWU.

So, although we do not fully understand the mixed case, it is clear that it is not nearly as simple as is the gains case. The analogy to convolution of random variables breaks down because of the inherent non-linearities in many of the models for mixed gambles.

4. Summary

The basic utility structure of the paper is the very general rank-weighted form for finite gambles that encompasses several axiomatized and unaxiomatized representations in the literature. For binary gambles, it reduces to the rank-dependent form. That coupled with the concept of joint receipt, $\oplus$, and the linking law of binary segregation leads to the p-additive representation first derived in this fashion by Luce (1991) and Luce and Fishburn (1991). Several results about RWU are reported. Basically this was background for introducing generalizations to arbitrary finite gambles of the concepts of adding random variables, their convolution, and the expected value of a convolution. The three generalizations are based on replacing the usual concepts $+$ by $\oplus$ and of numerical consequences by utilities. The question addressed is the equivalence of these generalizations. The major result, Theorem 5, provides for all gains (or all losses) sufficient (and in some cases necessary) conditions on the RWU representation so that various pairs of operations are equivalent. The exact results differ depending on whether the utility function U is additive or not over $\oplus$, but approximately they say the weights exhibit a multiplicative form analogous to multiplying probabilities. Specializations are shown for the case of convolving binary gambles and assuming the two axiomatized versions of RWU, namely, rank-dependent utility and gains-decomposition utility. Theorem 6 shows that, for lotteries of gains with either a general RDU or a GDU representation and binary segregation holding, then ordinary convolution $*$ and $\oplus$ are behaviorally equivalent if and only if the representation is actually EU with a quite special utility function. The final section points out that the simple equivalences found for gains do not carry over to mixed gains and losses under the assumptions of Luce (2000).

5. Proofs

5.1. Theorem 2

Applying the RWU representation and using the p-additive representation of Theorem 1,

$$\begin{aligned} &U(x_1 \oplus x_m, E_1; \ldots; x_{m-1} \oplus x_m, E_{m-1}; x_m, E_m) \\ &\quad = \sum_{i=1}^{m-1} U(x_i \oplus x_m)(W_{i,m} - W_{i-1,m}) + U(x_m)(1 - W_{m-1,m}) \end{aligned}$$

$$= \sum_{i=1}^{m-1} (U(x_i)[1 - \delta U(x_m)] + U(x_m))(W_{i,m} - W_{i-1,m}) + U(x_m)(1 - W_{m-1,m})$$
$$= \left[\sum_{i=1}^{m-1} U(x_i)(W_{i,m} - W_{i-1,m})\right][1 - \delta U(x_m)] + U(x_m)$$
$$= \left[\sum_{i=1}^{m-1} U(x_i)(W_{i,m} - W_{i-1,m}) + U(e)(1 - W_{m-1,m})\right][1 - \delta U(x_m)] + U(x_m)$$
$$= U(x_1, E_1; \ldots; x_{m-1}, E_{m-1}; e, E_m)[1 - \delta U(x_m)] + U(x_m)$$
$$= U[(x_1, E_1; \ldots; x_{m-1}, E_{m-1}; e, E_m) \oplus x_m].$$

Taking U^{-1} yields general segregation of order m.

5.2. Theorem 3

(i) We show the result by induction using gains decomposition and the following observation about Eq. (20) for $0 < i < k$,

$$W_{i,k-1} W_{\mathbf{E(k)}}[E(k-1)] = \prod_{j=i}^{k-2} W_{\mathbf{E(j+1)}}[E(j)] W_{\mathbf{E(k)}}[E(k-1)]$$
$$= \prod_{j=i}^{k-1} W_{\mathbf{E(j+1)}}[E(j)] = W_{i,k}.$$

It is trivial to verify that GDU for $k = 2$ is binary RDU. So, turning to the induction and using the property just shown, we have, using Eq. (25),

$$U(g^{(m)}) = U[g^{(m-1)}, E(m-1); x_m, E_m]$$
$$= U(g^{(m-1)}) W_{\mathbf{E(m)}}[E(m-1)] + U(x_m)(1 - W_{\mathbf{E(m)}}[E(m-1)])$$
$$= \sum_{i=1}^{m-1} U(x_i)(W_{i,m-1} - W_{i-1,m-1}) W_{\mathbf{E(m)}}[E(m-1)] + U(x_m)(W_{m,m} - W_{m-1,m})$$
$$= \sum_{i=1}^{m} U(x_i)(W_{i,m} - W_{i-1,m}),$$

which is RWU. The argument the other way just reverses the above calculation.

(ii) Suppose U satisfies both RDU and GDU. Consider the case of $m = 3$ and set $x_2 = x_3 = e$. So we have from RDU

$$U(x_1, E_1; e, E_2; e, E_3) = U(x_1) W_{\mathbf{E(3)}}[E(1)],$$

and from GDU

$$U(x_1, E_1; e, E_2; e, E_3) = U(x_1)W_{\mathbf{E(2)}}[E(1)]W_{\mathbf{E(3)}}[E(2)].$$

These are equal iff $W_{\mathbf{E(3)}}[E(1)] = W_{\mathbf{E(2)}}[E(1)]W_{\mathbf{E}}[E(2)]$ which, on setting $C = E(1)$, $D = E(2)$, $E = E(3)$, is Eq. (26).

Given Eq. (26) we see from Eq. (20) that the GDU weights satisfy

$$\begin{aligned}
W_{i,m} &= W_{\mathbf{E(i+1)}}[E(i)]W_{\mathbf{E(i+2)}}[E(i+1)]W_{\mathbf{E(i+3)}}[E(i+2)]\cdots W_{\mathbf{E(m)}}[E(m-1)] \\
&= W_{\mathbf{E(i+2)}}[E(i)]W_{\mathbf{E(i+3)}}[E(i+2)]\cdots W_{\mathbf{E(m)}}[E(m-1)] \\
&\quad \ldots \\
&= W_{\mathbf{E(m-1)}}[E(i)]W_{\mathbf{E(m)}}[E(m-1)] = W_{\mathbf{E(m)}}[E(i)],
\end{aligned}$$

which is the RDU weight.

5.2.1. *Corollary to Theorem 3.* Using gains decomposition and binary RDU

$$\begin{aligned}
U(g^{(m)}) - U(x_m) &= U\Big(g^{(m-1)}, E(m-1); x_m, E_m\Big) - U(x_m) \\
&= U(g^{(m-1)})W_{\mathbf{E(m)}}[E(m-1)] \\
&\quad + U(x_m)[1 - W_{\mathbf{E(m)}}[E(m-1)]] - U(x_m) \\
&= \Big[U(g^{(m-1)}) - U(x_m)\Big]W_{\mathbf{E(m)}}[E(m-1)].
\end{aligned}$$

5.3. *Theorem 4*

For $g^{(3)} = (x_1, E_1; x_2, E_2; x_3, E_3)$ the two decomposition assumptions say

$$(x_1, E_1; (x_2, E_2; x_3, E_3), E_2 \cup E_3) \sim ((x_1, E_1; x_2, E_2), E_1 \cup E_2; x_3, E_3).$$

Applying binary RDU successively on both sides, suppressing the subscript **E(3)** when it arises, and collecting terms, we have

$$\begin{aligned}
0 &= U(x_1)[W(E_1) - W(E_1 \cup E_2)W_{\mathbf{E_1 \cup E_2}}(E_1)] \\
&\quad + U(x_2)(W_{\mathbf{E_2 \cup E_3}}(E_2)[1 - W(E_1)] - [1 - W_{\mathbf{E_1 \cup E_2}}(E_1)]W(E_1 \cup E_2)) \\
&\quad + U(x_3)([1 - W_{\mathbf{E_2 \cup E_3}}(E_2)][1 - W(E_1)] - [1 - W(E_1 \cup E_2)]).
\end{aligned} \tag{43}$$

First, setting $x_2 = x_3 = e$ yields the choice axiom, Eq. (26).

Second, using that to get rid of the first term and setting $x_3 = e$ yields

$$W_{\mathbf{E_2 \cup E_3}}(E_2)[1 - W(E_1)] = [1 - W_{\mathbf{E_1 \cup E_2}}(E_1)]W(E_1 \cup E_2). \tag{44}$$

Applying the choice axiom to both sides

$$\frac{W(E_2)}{W(\overline{E}_1)}[1 - W(E_1)] = \frac{W(E_2)}{W(E_2 \cup E_3)}[1 - W(E_1)] = W(E_1 \cup E_2) - W(E_1).$$

Letting $f(E_1) = \frac{1-W(E_1)}{W(E_1)}$, we may rewrite this as

$$W(E_1 \cup E_2) - W(E_1) = W(E_2)f(E_1). \tag{45}$$

By the commutativity of $\cup$ we have

$$W(E_1) + W(E_2)f(E_1) = W(E_1 \cup E_2) = W(E_2 \cup E_1) = W(E_2) + W(E_1)f(E_2).$$

Rewriting

$$\frac{W(E_1)}{1 - f(E_1)} = \frac{W(E_2)}{1 - f(E_2)}. \tag{46}$$

Because E_1 and E_2 can be chosen independently (subject to constraints), these ratios must be some non-zero constant $1/\rho$. Solving for $f(E_1)$ and substituting in Eq. (45) we see that

$$W(E_1 \cup E_2) = W(E_1) + W(E_2) - \rho W(E_1)W(E_2), E_1 \cap E_2 = \varnothing,$$

which is Eq. (28).

To show the converse, it is sufficient to prove Eq. (43) is satisfied. By the choice axiom, Eq. (26), the coefficient of x_1 in Eq. (43) is zero. Next, reversing the argument just given yields Eq. (44), so the coefficient of x_2 in Eq. (43) is zero. Finally, observe that using Eqs. (44) and (26)

$$\begin{aligned}[1 - W_{\mathbf{E_2 \cup E_3}}(E_2)][1 - W(E_1)] &= 1 - W(E_1) - [1 - W_{\mathbf{E_1 \cup E_2}}(E_1)]W(E_1 \cup E_2) \\ &= 1 - W(E_1) - \left[1 - \frac{W(E_1)}{W(E_1 \cup E_2)}\right]W(E_1 \cup E_2) \\ &= 1 - W(E_1 \cup E_2),\end{aligned}$$

which says the coefficient of the x_3 term is also 0.

5.3.1. Corollary to Theorem 4. The corollary follows immediately from the fact that $\cup$ is associative and Eq. (28).

5.4. Theorem 5

(i) Assume U is not additive.

(a) Suppose, first, that U is p-additive with $\delta \neq 0$, which is easily seen to be a special case of this assumption. Then,

$$\begin{aligned}
1 - \delta U(g \oplus h) &= [1 - \delta U(g)][1 - \delta U(h)] \\
&= \left(1 - \delta \sum_{i=1}^{m} U(x_i) S_{i,m}\right)\left(1 - \delta \sum_{k=1}^{n} U(y_k) S'_{k,n}\right) \\
&= \left(\sum_{i=1}^{m}[1 - \delta U(x_i)] S_{i,m}\right)\left(\sum_{k=1}^{n}[1 - \delta U(y_k)] S'_{k,n}\right) \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}[1 - \delta U(x_i)][1 - \delta U(y_k)] S_{i,m} S'_{k,n} \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}[1 - \delta U(x_i \oplus y_k)] S_{i,m} S'_{k,n} \\
&= 1 - \delta \sum_{i=1}^{m}\sum_{k=1}^{n} U(x_i \oplus y_k) S_{i,m} S'_{k,n} = 1 - \delta U(g \boxplus h).
\end{aligned}$$

Thus, since $\delta \neq 0, \boxplus \approx \oplus$.

Conversely, we assume $\boxplus \approx \oplus$ and prove the p-additivity of U with $\delta \neq 0$. Using the function F, Eq. (33), and defining

$$H(\zeta, \eta) := F(\zeta, \eta) - \zeta - \eta, \tag{47}$$

we have

$$U(g \oplus h) = F[U(g), U(h)] = U(g) + U(h) + H[U(g), U(h)],$$

and so

$$\begin{aligned}
U(g \boxplus h) &= \sum_{i=1}^{m}\sum_{k=1}^{n} U(x_i \oplus y_k) S_{i,m} S'_{k,n} \\
&= \sum_{i=1}^{m}\sum_{k=1}^{n}(U(x_i) + U(y_k) + H[U(x_i), U(y_k)]) S_{i,m} S'_{k,n} \\
&= U(g) + U(h) + \sum_{i=1}^{m}\sum_{k=1}^{n} H[U(x_i), U(y_k)] S_{i,m} S'_{k,n}.
\end{aligned}$$

So, equating these,

$$H[U(g), U(h)] = \sum_{i=1}^{m}\sum_{k=1}^{n} H[U(x_i), U(y_k)] S_{i,m} S'_{k,n}.$$

Select g, h so that $x_i = e$ for $i > 1$ and $y_k = e$ for $k > 1$. Recall that $H(0, \eta) = H(\zeta, 0) = 0$, so setting $U(x_1) = \zeta, U(y_1) = \eta, S_{1,m} = \varpi, S'_{1,n} = \rho$ we obtain the functional equation

$$H(\zeta\varpi, \eta\rho) = H(\zeta, \eta)\varpi\rho. \tag{48}$$

Setting $\zeta = 1, \eta = 1$, we see that

$$H(\varpi, \rho) = H(1, 1)\varpi\rho. \tag{49}$$

We first prove $H(1, 1) \neq 0$. Suppose, on the contrary, $H(1, 1) = 0$. First consider the case $\zeta < 1, \eta < 1$. Then since $\varpi, \rho \in]0, 1[$, setting $\varpi = \zeta, \rho = \eta$ in Eq. (49) gives $H(\zeta, \eta) = 0$ for $\zeta < 1, \eta < 1$. For $\zeta > 1, \eta > 1$, set $\varpi = 1/\zeta, \rho = 1/\eta$ and so by Eq. (48)

$$0 = H(1, 1) = \frac{H(\zeta, \eta)}{\zeta\eta},$$

and so $H(\zeta, \eta) = 0$ for $\zeta > 1, \eta > 1$. Now suppose $\zeta \geq 1 \geq \eta$, then we may choose ϖ and ρ so that $\zeta\varpi < 1, \eta\rho < 1$, whence by the first part and Eq. (48)

$$0 = H(\zeta\varpi, \eta\rho) = H(\zeta, \eta)\varpi\rho,$$

and so $H(\zeta, \eta) = 0$ for $\zeta \geq 1 \geq \eta$. Since $\oplus$ is commutative, H is symmetric and so the conclusion holds for the other inequality. And so $H(\zeta, \eta) \equiv 0$, i.e., U is additive, contrary to assumption. Therefore, $\delta = H(1, 1) \neq 0$.

Now retracting our steps but with $H(1, 1)$ replaced by δ rather than 0, the first two cases immediately yield that the only solution is

$$H(\zeta, \eta) = \delta\zeta\eta. \tag{50}$$

For $\zeta \geq 1 \geq \eta$ (by the symmetry of H we do not need to consider the opposite ordering), we may choose ϖ and ρ so that $\zeta\varpi < 1, \eta\rho < 1$, whence by Eq. (48) and the first part of this section of the proof, i.e., Eq. (50), we obtain

$$H(\zeta, \eta) = \frac{H(\zeta\varpi, \eta\rho)}{\varpi\rho} = \frac{\delta\zeta\varpi\eta\rho}{\varpi\rho} = \delta\zeta\eta.$$

Thus Eq. (50) holds for all $\zeta, \eta \in [0, k[$ and so U is p-additive with $\delta \neq 0$.

(b) The sufficiency of Eq. (35) follows immediately from the RWU form and the definitions of $\boxplus$ and $\circledast$.

So we turn to the necessity. Let g and h be gambles based, respectively, on the partitions $\{C_1, \ldots, C_i, \ldots, C_m\}$ with consequences $x_i \succ x_j$ iff $i > j$ and $\{D_1, \ldots, D_k, \ldots, D_n\}$ with consequences $y_k \succ y_l$ iff $k > l$. Let (C_i, D_k) denote that event C_i occurs in the first realization of **E** and that event D_k occurs in the second realization of **E**. For any pair of integers q, $1 \le q \le m$, r, $1 \le r \le n$, choose gambles with $x_q = x$ and $y_r = y$. Because the utility function is onto an interval and there are only finitely many inequalities to be satisfied, we may and do choose the x_i and y_k so that no indifference of the form $x_i \oplus y_k \sim x_j \oplus y_l$ holds except when $i = j$ and $k = l$.

For g, h, let $\Phi_{g,h}$ denote the mapping defined by Eq. (34).

For the selected integers q, $1 \le q \le m$, and r, $1 \le r \le n$, consider elements $a_{qr} \succ e$, $b_{qr} \succ e$, and define gambles g' and h' by

$$g' = \begin{cases} x_i, & \text{if } C_i \text{ occurs}, i \neq q \\ x_q \oplus a_{qr}, & \text{if } C_q \text{ occurs} \end{cases}.$$

and

$$h' = \begin{cases} y_j, & \text{if } D_j \text{ occurs}, j \neq r \\ y_r \oplus b_{qr}, & \text{if } D_r \text{ occurs} \end{cases}.$$

Thus, g' agrees with g except on component q, and h' agrees with h except on component r.

Once again, because U is onto an interval and there are only finitely many inequalities to be satisfied, it is possible to select the elements $a_{qr} \succ e$, $b_{qr} \succ e$ such that for $i, j \in \{1, \ldots, m\}$, $k, l \in \{1, \ldots, n\}$,

$$\Phi_{g,h}(i,k) < \Phi_{g,h}(j,l) \Leftrightarrow \Phi_{g,h'}(i,k) < \Phi_{g,h'}(j,l)$$
$$\Leftrightarrow \Phi_{g',h}(i,k) < \Phi_{g',h}(j,l) \Leftrightarrow \Phi_{g,'h'}(i,k) < \Phi_{g',h'}(j,l).$$

Thus, the functions $\Phi_{g,h}$, $\Phi_{g,h'}$, $\Phi_{g',h}$, $\Phi_{g,'h'}$ give the same mapping, so in the following we simply write Φ for that common mapping.

Now assume that $\boxplus \approx \circledast$. To simplify the notation we let $a = a_{qr}$, $b = b_{qr}$. Then we have

$$\begin{aligned} 0 = U(g \circledast h) - U(g \boxplus h) &= \sum_{i=1}^{m} \sum_{k=1}^{n} [U(x_i \oplus y_k)][S''_{\Phi(i,k),mn} - S_{i,m} S'_{k,n}]. \\ &= U[(x_q \oplus y_r)][S''_{\Phi(q,r),mn} - S_{q,m} S'_{r,n}] \\ &\quad + \sum_{i \neq q}^{m} \sum_{k \neq r}^{n} [U(x_i \oplus y_k)][S''_{\Phi(i,k),mn} - S_{i,m} S'_{k,n}] \\ &\quad + \sum_{k \neq r}^{n} [U(x_q \oplus y_k)][S''_{\Phi(q,k),mn} - S_{q,m} S'_{k,n}] \\ &\quad + \sum_{i \neq q}^{m} [U(x_i \oplus y_r)][S''_{\Phi(i,r),mn} - S_{i,m} S'_{r,n}]. \end{aligned} \tag{51}$$

Similarly,

$$\begin{aligned}
0 &= U(g' \circledast h') - U(g' \boxplus h') \\
&= U((x_q \oplus a) \oplus (y_r \oplus b))[S''_{\Phi(q,r),mn} - S_{q,m}S'_{r,n}] \\
&\quad + \sum_{i \neq q}^{m} \sum_{k \neq r}^{n} [U(x_i \oplus y_k)][S''_{\Phi(i,k),mn} - S_{i,m}S'_{k,n}] \\
&\quad + \sum_{k \neq r}^{n} [U((x_q \oplus a) \oplus y_k)][S''_{\Phi(q,k),mn} - S_{q,m}S'_{k,n}] \\
&\quad + \sum_{i \neq q}^{m} [U(x_i \oplus (y_r \oplus b))][S''_{\Phi(i,r),mn} - S_{i,m}S'_{r,n}], \qquad (52)
\end{aligned}$$

and

$$\begin{aligned}
0 &= U(g' \circledast h') - U(g' \boxplus h) = \sum_{i \neq q}^{m} \sum_{k \neq r}^{n} [U(x_i \oplus y_k)][S''_{\Phi(i,k),mn} - S_{i,m}S'_{k,n}] \\
&\quad + \sum_{i \neq q}^{m} [U(x_i \oplus y_r)][S''_{\Phi(i,r),mn} - S_{i,m}S'_{r,n}] \\
&\quad + \sum_{k=1}^{n} [U((x_q \oplus a) \oplus y_k)][S''_{\Phi(q,k),mn} - S_{q,m}S'_{k,n}], \qquad (53)
\end{aligned}$$

and

$$\begin{aligned}
0 &= U(g \circledast h') - U(g \boxplus h') \\
&= \sum_{i \neq q}^{m} \sum_{k \neq r}^{n} [U(x_i \oplus y_k)][S''_{\Phi(i,k),mn} - S_{i,m}S'_{k,n}] \\
&\quad + \sum_{k \neq r}^{n} [U(x_q \oplus y_k)][S''_{\Phi(q,k),mn} - S_{q,m}S'_{k,n}] \\
&\quad + \sum_{i=1}^{m} [U(x_i \oplus (y_r \oplus b))][S''_{\Phi(i,r),mn} - S_{i,m}S'_{r,n}]. \qquad (54)
\end{aligned}$$

Subtracting the sum of Eqs. (53) and (54) from the sum of Eqs. (51) and (52), and recalling the choice $x_q = x$, $y_r = y$, we have

$$\begin{aligned}
0 &= [S''_{\Phi(q,r),mn} - S_{q,m}S'_{r,n}] \times [U(x \oplus y) + U((x \oplus a) \oplus (y \oplus b)) \\
&\quad - U((x \oplus a) \oplus y) - U(x \oplus (y \oplus b))].
\end{aligned}$$

Thus, provided for some sufficiently small a, b,

$$U(x \oplus y) + U((x \oplus a) \oplus (y \oplus b)) - U((x \oplus a) \oplus y) - U(x \oplus (y \oplus b)) \neq 0, \qquad (55)$$

we obtain that

$$S''_{\Phi(q,r),mn} = S_{q,m}S'_{r,n}.$$

So, assume the contrary assumption that for all x, y and sufficiently small a, b,

$$U(x \oplus y) + U((x \oplus a) \oplus (y \oplus b)) - U((x \oplus a) \oplus y) - U(x \oplus (y \oplus b)) = 0 \quad (56)$$

holds, and then show this implies U is additive, contrary to assumption. Let $\zeta = U(x)$, $\eta = U(y)$, $\alpha = U(x \oplus a) - \zeta$, $\eta' = U(y \oplus b)$, then, given Eq. (33), Eq. (56) is equivalent to

$$F(\zeta + \alpha, \eta) - F(\zeta, \eta) = F(\zeta + \alpha, \eta') - F(\zeta, \eta').$$

Dividing by α and taking the limit as $\alpha \searrow 0$ yields for some differentiable function L and some $\eta^* > \eta$,

$$\frac{\partial F(\zeta, \eta)}{\partial \zeta} = \frac{\partial F(\zeta, \eta')}{\partial \zeta} = L'(\zeta), \qquad \eta' \in [\eta, \eta^*[. \quad (57)$$

Because U is onto an interval $[0, K[$, we may choose the rational numbers in that interval as a countable set of η_i's, i rational, the union of their corresponding intervals for which Eq. (57) holds, is $[0, K[$, and every point in $]0, K[$ lies in at least two intervals. Thus, for all $\zeta, \eta \in]0, K[$,

$$\frac{\partial F(\zeta, \eta)}{\partial \zeta} = L'(\zeta).$$

A similar argument on the second variable yields for some differentiable function M that for all $\zeta, \eta \in]0, K[$,

$$\frac{\partial F(\zeta, \eta)}{\partial \eta} = M'(\eta).$$

Thus, for all $\zeta, \eta \in]0, K[$,

$$F(\zeta, \eta) = L(\zeta) + M(\eta). \quad (58)$$

By the differentiability condition, F is continuous at 0 for either argument, and then from $F(0, \eta) = \eta$, $F(\zeta, 0) = \zeta$, $F(0, 0) = 0$, it follows immediately that for all ζ, η

$$F(\zeta, \eta) = \zeta + \eta,$$

thus violating the nonadditivity assumption. This contradiction proves the result.

(ii) Suppose U is additive, i.e., $U(g \oplus h) = U(g) + U(h)$.

(a) Observe that

$$U(g \boxplus h) = \sum_{i=1}^{m} \sum_{k=1}^{n} U(x_i \oplus y_k) S_{i,m} S'_{k,n} = \sum_{i=1}^{m} \sum_{k=1}^{n} [U(x_i) + U(y_k)] S_{i,m} S'_{k,n}$$

$$= \sum_{i=1}^{m} U(x_i) S_{i,m} + \sum_{k=1}^{n} U(y_k) S'_{k,n} = U(g) + U(h) = U(g \oplus h).$$

Thus, $\oplus \approx \boxplus$.

(b) As just shown,

$$U(g \boxplus h) = \sum_{i=1}^{m} \sum_{k=1}^{n} [U(x_i) + U(y_k)] S_{i,m} S'_{k,n}.$$

Also,

$$U(g \circledast h) = \sum_{i=1}^{m} \sum_{k=1}^{n} U(x_i \oplus y_k) S''_{\Phi(i,k),mn} = \sum_{i=1}^{m} \sum_{k=1}^{n} [U(x_i) + U(y_k)] S''_{\Phi(i,k),mn}.$$

Thus, $\boxplus \approx \circledast$ iff

$$0 = \sum_{i=1}^{m} \sum_{k=1}^{n} [U(x_i) + U(y_k)][S''_{\Phi(i,k),mn} - S_{i,m} S'_{k,n}].$$

Using this result, it is routine to show that if Eq. (36) holds, then $\boxplus \approx \circledast$. Conversely, assume that U is additive and $\boxplus \approx \circledast$. Then subtracting Eq. (51) from Eq. (53) we obtain

$$0 = \sum_{k=1}^{n} U(a)[S''_{\Phi(q,k),mn} - S_{q,m} S'_{k,n}]$$

which with $U(a) > 0$, yields

$$S_{q,m} = \sum_{k=1}^{n} S''_{\Phi(q,k),mn},$$

i.e., the first equality in Eq. (36). The second equality in Eq. (36) follows in a similar manner by subtracting Eq. (51) from Eq. (54).

5.5. Theorem 6

Given the assumptions of either RDU or GDU and binary segregation, Theorem 1 implies that U is p-additive and so U is either additive or strictly nonadditive.

By assumption, $\Pr(C) = \Pr(C')$ iff $W(C) = W(C')$. Thus, with a slight abuse of notation, we write $W(p) = W(C)$ if $p = \Pr(C)$. Because (C, D) denotes the compound event of C and D occurring on independent realizations of the underlying chance experiment, then for $p = \Pr(C), q = \Pr(D), \Pr(C, D) = \Pr(C)\Pr(D) = pq$. Also by assumption, we have $\Pr(C, D) = \Pr(C', D')$ iff $W''(C, D) = W''(C', D')$, so with a similar slight abuse of notation, we write $W''(pq) = W''(C, D)$ if $pq = \Pr(C, D)$.

Suppose $* \approx \oplus$. Note that this means, for $x, y \in \mathscr{C}, x \oplus y \sim x + y$ which, along with the above remarks about probabilities, implies that the right-hand side of Eq. (1) is the same as the right-hand side of Eq. (30). Thus, $* \approx \circledast$, which with $* \approx \oplus$ yields $\circledast \approx \oplus$. In the additive case, by Theorem 5(ii)(a) $\oplus \approx \boxplus$, which with $\circledast \approx \oplus$ yields $\boxplus \approx \circledast$. Therefore, by Theorem 5(ii)(b), Eq. (36) holds. In the nonadditive case, the p-additivity of U with Theorem 5(i)(a) gives $\oplus \approx \boxplus$, which with $\circledast \approx \oplus$ again implies $\boxplus \approx \circledast$. Thus, by Theorem 5(i)(b), Eq. (35) holds.

Next, as noted in Section 3.2.3 for the nonadditive case we have Eqs. (37), (38), and (39) for both RDU and GDU. Observe that for $p = \Pr(C), 1 = \Pr(E)$, the first yields

$$\begin{aligned} W''(p) &= W''[\Pr(C)\mathrm{P}(E)] = W''[\Pr(C, \Omega)] \\ &= W''(C, \Omega) = W(C)W(\Omega) \\ &= W[\Pr(C)]W[\Pr(\Omega)] = W(p), \end{aligned}$$

so $W'' = W$. Also note that

$$\begin{aligned} \Pr[(C, D) \cup (C, \overline{D})] &= pq + p(1 - q) = p \\ \Pr[(C, D) \cup (C, \overline{D}) \cup (\overline{C}, D)] &= p + (1 - p)q = p + q - pq. \end{aligned}$$

Thus, Eqs. (37), (38), and (39) reduce to

$$\begin{aligned} W(pq) &= W(p)W(q), \\ W(p) &= W(p) \text{ or } W(q) = W(q), \\ W(p + q - pq) &= W(p) + W(q) - W(p)W(q), \end{aligned}$$

where the choice of the tautological middle equation depends on the order of the second and third terms in Eq. (30). The first equation is a Cauchy one for which the only strictly increasing solution with $W(0) = 0$ and $W(1) = 1$ is $W(p) = p^{\beta}$, for some $\beta > 0$. Substituting this into the third equation shows that $\beta = 1$. Thus, EU holds in the nonadditive case.

In the additive case, one obtains Eq. (38) and Eq. (40), and the latter says

$$W(p + q - pq) = W(p) + W(q) - W(pq), \tag{59}$$

which is known in the literature as Hosszu's equation. Initially, regular solutions were found and then those conditions were dropped. A good deal of the literature is referenced in Lajkó (1974). The result closest to our needs is Światak (1968) who proved $W(p) = ap + b$ on the assumption of continuity at 0 or at 1. The following is an apparently new and quite direct proof of that fact using only continuity at 0. Define $F(u, v) = W(u - v) + W(v)$, then the functional equation is equivalent to

$$F(u, q) = F[u, (u - q)q], 0 \leq q \leq u \leq 1.$$

Because W is strictly increasing and onto [0, 1], it is continuous, in particular at 0, and so F is also. Consider the following sequence beginning with $q_1 < u$

$$q_n = (u - q_{n-1})q_{n-1}.$$

This is a descending sequence bounded from below by 0 and so has a limit $L \geq 0$. Thus, $L = (u - L)L$, and so either $L = 0$ or $L = u - 1 < 0$. Because the latter is impossible, we conclude $L = 0$. Also, by induction, $F(u, q_n) = F(u, q_1)$. So by the continuity of F,

$$F(u, q_1) = \lim_{n\to\infty} F(u, q_n) = F(u, \lim_{n\to\infty} q_n) = F(u, 0) = f(u).$$

Thus we have the so-called Pexider equation

$$f(u) = W(u - v) + W(v),$$

from which we know $W(u) = au + b$. From $W(0) = 0, b = 0$ and from $W(1) = 1$, $a = 1$. Thus, again we have EU.

Now we derive the form for U. Recall that U is p-additive. This with $x \oplus y = x + y$ yields

$$U(x + y) = U(x \oplus y) = U(x) + U(y) - \delta U(x)U(y).$$

If $\delta = 0$, this is Cauchy's equation and $U(x) = cx$. If $\delta \neq 0$, we see that

$$\ln[1 - \delta U(x + y)] = \ln[1 - \delta U(x)] + \ln[1 - \delta U(y)],$$

and so, again by Cauchy's equation, $\delta U(x) = 1 - e^{-cx}$, where $\delta c > 0$.

Conversely, suppose that EU holds with one of these utility forms. We first show that $x \oplus y \sim x + y$. Again recall that U is p-additive. Therefore, when $U(x) = cx$,

$$c(x \oplus y) = U(x \oplus y) = U(x) + U(y) = cx + cy,$$

yielding the result. For the second form and using p-additivity,

$$e^{-c(x\oplus y)} = 1 - \delta U(x \oplus y) = [1 - \delta U(x)][1 - \delta U(yx)] = e^{-cx}e^{-cy} = e^{-c(x+y)},$$

and the result follows by taking logarithms. Finally, using $x \oplus y \sim x + y$ in Eq. (30) and replacing the events by probabilities, we see that Eq. (1) = Eq. (30) and so $* \approx \circledast$.

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