Where is Mathematical Modeling in Psychology Headed?

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ABSTRACT. The current state of mathematical psychology is characterized in terms of five paired contrasts: behavioral vs information-processing; linearity vs non-linearity; static vs dynamic; mathematical vs computational; and deterministic vs probabilistic. Each contrast is evaluated in terms of three basic facts of psychology: complexity, irreversibility of experience, and individual differences; and also in terms of three contemporary developments: computational power, brain imaging and recent mathematical developments. From these I draw some (low-confidence) conclusions about where I think the field is headed, and why.

KEY WORDS: computation, future directions, imaging, mathematical psychology, non-linear dynamics

Let me admit two things at the outset. First, I have very little faith in my ability, or that of anyone else, to predict the future. Had I been alive in the late 19th century and making predictions about major social problems of the early 20th century, I suspect one high on my list would have been how to deal more effectively with horse excrement in cities. Automobiles would not have been my solution, if a solution they be.

Second, I was nurtured in the highly successful uses of mathematics in the physical sciences that arose in the 18th, 19th and early 20th centuries, and to this day I maintain a warm feeling for modeling of this character, although it has been colored by some of the axiomatic developments of algebraic systems in this century. This style may well be at variance with many ongoing developments (more below).

So, were I the reader, I would take seriously only what is said about the general nature of contemporary modeling and the factors that seem to be affecting it, but I would pay little attention to any real predictions.

Introduction

As in other sciences, formal models of psychological processes have two major virtues. First, the very act of formulating models forces one to be very

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precise about what is being said, which, all too often, is not true of informally stated theories. This fact is important in making theories falsifiable. Second, modeling increases one's ability to draw accurate predictions and/or explanations from the underlying assumptions. There are two aspects to this. The most obvious is that when such predictions are non-obvious, they are especially telling. The other, less appreciated aspect is that rather spare, almost trivial, assumptions, when combined, can sometimes have far-reaching implications.

The obstacles to successfully carrying out such modeling in psychology are formidable, for at least three reasons:

- The system being modeled is obviously complex, even when one looks at some special subpart such as vision or hearing, and that complexity is reflected in part by some lack of consensus about what the basic variables are, what the major features of the system(s) under study are, and which problems should be the first order of business.
- People's experiences tend to be irreversible in the sense that we do not know how to make them forget a specific experience and then re-experience it.
- People often exhibit great individual differences as well as high degrees of sameness.

The latter two observations have serious implications for evaluating theories using experimental replications. Irreversibility means that one cannot easily subject an individual to independent repeated trials under identical conditions, which makes the study of dynamic processes very difficult indeed. And individual differences mean that thoughtless aggregation of responses from many people, which is certainly facilitated by statistical packages, has a very real potential to be misleading. One has to work around these difficulties by seeking experiences that are not direct replicas but are sufficiently similar that they can be treated as examples of the same process, by seeking those aspects of deeper regularity in behavior that underlie the differences and that appear to be common to all people or to significantly large classes of them, and by using each person as his or her own control in the study.

Often, in lieu of well-accepted models, much of psychology has adopted a number of extremely sophisticated statistical tools: path analysis, multiple regression, loglinear analysis, factor analysis and analysis of variance. This is a complex, not entirely happy, topic which I will not examine here except to note that it does not automatically overcome the problems of irreversibility and individual differences.

My approach here will be to list some general distinctions—formulated as pairs of contrasts—that one can make about models. Many are illustrated in the chapters in Marley (1997).

Within the terms of these distinctions, I will try to indicate what seem to be some of the impacts of three major scientific/methodological forces that I and others believe are relevant:

- growing computational power and software sophistication;
- brain imaging; and
- recent mathematical developments.

These may lead—some claim they already have—to formulations far different from and more powerful than traditional scientific ones. These may well affect deeply the direction of modeling in the early part of the coming century. However, the exact nature of the interplay is probably beyond anyone's ability to predict, and I will not attempt to do very much predicting. Along the way, I make some value judgments, not all of which are shared by other model-builders.

Behavioral vs Information-Processing

The distinction here is whether the individual (or group) is treated as a 'black box' having observable behavioral (phenomenological) properties whose formulation is the theory, or whether in some sense one attempts to 'open' the black box in order to formulate what is going on inside and how these processes give rise to the observed behavior. Of course, behavioral regularities are common to both approaches—they are just dealt with differently.

Most of classical physics of the 18th, 19th and early 20th centuries is of the former, behavioral type. Included are Newtonian mechanics, hydrodynamics, thermodynamics, electro-magnetic theory and special and general relativity theory. Fluids were, and for many purposes still are, described in terms of measurable variables, such as pressure, temperature, viscosity, and so on, and not as interacting molecules or atoms. Models of planetary motion treated the planets as gravitationally interacting point masses. Electromagnetic phenomena were first adequately described in terms of macroscopic variables such as resistance, charge, potential, and so on. This is not to say that the idea of internal structure was ignored, but it was not until physics was relatively highly developed that any serious opening of the black boxes occurred, for example the kinetic theory of gases, atomic theory, electron theory of electricity, quantum mechanics. Even so, purely phenomenological models continue to play a major role both as theory, for example general relativity, and in most engineering applications, for example aircraft design.

Only a small fraction of psychological modeling is behavioral in the sense of attempting to capture mathematically some of the behavioral, empirically testable, regularities, and from these to explore what other regularities can be predicted, Sometimes this is effectively summarized by showing that the behavioral properties describe a mathematical system of 'laws' that is isomorphic to some numerical or other well-understood mathematical system. An example is the work on the foundations of measurement (see, e.g., Krantz, Luce, Suppes, & Tversky, 1971; Luce, Krantz, Suppes, & Tversky, 1990; Suppes, Krantz, Luce, & Tversky, 1989). Another, and closely related, example concerns individual decision-making. There the modeling involves a good deal of fairly complex stinulus structure with fairly simple, transparent principles of behavior. Two fairly recent references are Edwards (1990) and Wakker (1989), but for the most recent work one must consult such journals as *Journal of Mathematical Psychology*, *Journal of Organizational Behavior and Human Decision Process*, *Journal of Risk and Uncertainty* and *Mathematical Social Sciences*. Another example is the modeling of overall time allocation among alternative responses when schedules of reinforcement interact with the behavior (see, e.g., Davison & McCarthy, 1988; Loewenstein & Elster, 1992). Still another is the work on probability models of choice that are in some sense equivalent to choices based on comparing the realizations of random variables. This is illustrated in some of the papers in Marley (1997).

In contrast, much modeling that was part of the cognitive revolution of the past 30–40 years has been formulated as information processing within the organism. These models typically entail a structure or architecture of information flow that begins with some challenge perceived by the organism, through information gathering and memory retrieval, to a response, which is then followed by some form of feedback that affects both future behavior and what new information is added to memory by the system. It may be thought of as trying to describe analytically a flow diagram of mental activity based on certain elementary processing stages. A recent example is Estes (1994), and some earlier ones are summarized by Luce (1986) and Townsend and Ashby (1983). The level of internal units postulated is, nonetheless, quite complex, certainly well above single neurons.

Advantages and Disadvantages

Behavioral models have two advantages. First, they do not go beyond what is in principle observable—although making the observations is sometimes difficult in practice for the reasons mentioned in the introduction. Second, their formulation does not tend to come with a number of free parameters that have to be estimated from data. The latter statement needs some amplification. The 'laws' that define the model typically have no free parameters at all, but if a numerical representation is constructed from them, then parameters will arise—sometimes in profusion. For example, in the classical subjective expected utility models the representation had two free functions: a utility function and a subjective probability function. To me this simply means that the scientist has failed to find sufficient behavioral constraints to limit significantly the forms of these functions. In more recent work, constraints have been proposed that limit the utility function to three parameters and the weighting functions to two for gains and two for losses (Gonzalez & Wu, 1999; Luce, in press; Prelec, 1998).

The major weakness of behavioral modeling is our inability to deal effectively with such behaviors as the manipulation of propositions, as in many linguistic and interaction situations, and with the temporal aspects of choices. Some attempts at the latter exist in the operant literature, but I think they are far from acclaimed success, except asymptotically (Davison & McCarthy, 1988).

The information-processing models come in a wide variety of forms, but typically they are quite flexible both in dealing with symbolic systems (Newell, 1990) and in providing rather natural accounts of temporal aspects of decision-making (Luce, 1986; Townsend & Ashby, 1983). This is their great advantage at present over behavioral models.

They exhibit two weaknesses:

- The postulated mental architectures are very hypothetical, and a great deal of data are required to distinguish among various hypotheses about them. Gradually, these inference problems are being worked out, but such research tends to be laborious and taxes one's (or, at least, my) patience.
- These models, especially when they go beyond two stimuli/two response designs proliferate great numbers of free parameters whose empirical meanings are usually not very firm. What is worse, often they do not remain invariant when relatively small changes are made in the experimental design. For the most part, we cannot, once and for all, estimate the relevant parameters from experiments designed to do just that, and then predict the outcome of other, usually more complex, experiments. Increasingly, cognitive psychologists are aware of this issue and evaluate the quality of their model-building in part on the basis of parameter invariance.

Trends

I'll go out on a limb and suggest that the impacts of both computational power and brain imaging will have only subtle and not especially revolutionary impacts on behavioral modeling of individuals.

To be sure, those of us developing such models do and will continue to make considerable use of computational power in evaluating them, but I do not anticipate that such modeling will be transformed into computer programs. In part this prediction reflects the fact that current programs do not seem able to carry out the kinds of mathematical reasoning needed to understand axiomatic systems. Should that change, then so should the prediction.

Computers have had, and can be expected to continue to have, a major impact on one type of behavioral models, those describing social interactions. Such network models tend to be quite complex mathematically, and increasingly they are structured visually using computer algorithms. Wasserman and Faust (1994) demonstrate the successful interplay of powerful mathematics and powerful computers. Another type of related modeling, which is under active development and which I expect will continue for some time, are the so-called 'evolutionary' models often associated with the Santa Fe Institute school of global modeling, but that are also being pursued by many others. However, it appears to be very difficult to confront these models empirically in a serious way (see the discussion of non-linear dynamics below).

Since, by definition, behavioral models concern unopened black boxes, on the face of it brain imaging cannot be expected to play a very direct role in their formulation. Indirectly, it may. For example, if brain imaging leads to an awareness of features of observable behavior previously overlooked, then that could impact behavioral modeling.

Until recently, the information-processing models were based entirely on hypothetical mechanisms, but these hypotheses are now being compared to a flow of empirical information about brain activity arising from the development of (relatively) passive methods of observation (EEG, PET, fMRI). These observations provide information concerning human beings which, in some measure, supplements the earlier animal studies of single-unit recordings of neural activity and the behavioral impact of ablation of local regions. Brain imaging provides data that are increasingly detailed and accurate at some level of aggregated neural electrical and blood-flow activity. It is currently far from clear to me whether the level of aggregation natural to the various imaging techniques and those postulated in information-processing models are the same; I rather doubt it. Nonetheless, brain imaging is a very 'hot' area and almost certainly a strong interaction will continue between modelers and imagers, especially as the latter's methods become more refined. (An example of the interplay is LaBerge, 1995.)

In all of this, computational power is critical. One cannot at all analyze the data recorded at the scalp without it, and often the implications of the information-processing models seem mathematically intractable and can only be understood via computations. Indeed, in some cases the models do not exist except as computer programs.

An intermediate direction has begun to appear in the literature on decision-making that to a degree exhibits some of the virtues and avoids many of the faults of both behavioral and information-processing models. Recall that behavioral models often give rise to a numerical representation somewhat like the laws of physics giving rise to differential equations that characterize behavior of, for example, the motion of an object. Although no one would impute to the object computations to determine its motion, many psychologists have been willing to do the comparable thing for people. This is reinforced by the fact that people do, indeed, do various conscious analyses of complex decisions. So attempts are beginning to be made to model this sort of decision-making in terms of what Gigerenzer and Goldstein (1996) and Gigerenzer, Todd and the ABC Group (1999) call 'fast and frugal heuristics'. These principles are structured somewhat like the steps postulated in information-processing models, but at a far more observable, behavioral level. Moreover, these models seem to avoid the proliferation of free parameters typical of information processing. However, if, as seems to be the case, there are families of distinct fast and frugal heuristics, then selection of which to use amounts to having as many free parameters as there are heuristics. Whether this will ultimately seem to be more efficient, and whether the approach can be extended to a wide variety of cognitive processes, remains to be seen.

Linear vs Non-linear Systems

The defining idea of a linear system is *superposition*: if one knows the response of the system to an input A and separately to an input B, then the response of the system to both A and B is simply the 'sum' in some suitable sense of the two responses with no interaction taking place. This is the basis of any Fourier analysis, which is widely used in vision and audition. Any non-linear system fails this property. The distinction is highly asymmetric, with the class of possible linear systems being vanishingly small among the class of all possible systems.

Advantages and Disadvantages

Linear systems typically exhibit highly regular behavior, such as periodic oscillations, and are comparatively easy to analyze.[1] Linear systems go very far beyond linear functions. All of the classical non-linear functions of mathematical physics—trigonometric, exponential, Bessel, gamma, and on and on—arose from studying linear systems. When a linear model applies, the overwhelming mathematical advantage is the chance of its being very well understood. The difficulty is that we suspect that a good deal of human behavior is fundamentally not linear.

Non-linear systems typically are not only less regular, but sometimes quite discontinuous in character. This has increasingly been investigated in the physical sciences by making heavy use of computational power, and we have all seen the remarkable fractal plots corresponding to apparently very simple non-linear processes. Applications in the behavioral sciences range from what I think is best called analogical to the serious (e.g. Saari, 1994a, 1996).

Trends

Here the trends will rest primarily on further developments in the study of non-linear systems and their selective importation into psychology. I doubt that brain imaging will have any direct impact; rather the direction should be the other way. To the degree that the underlying processes that give rise to what we observe and measure at the scalp are non-linear, our mathematical understanding of the physics and biology involved will impact our ability to draw sound inferences from observations. (The complexity of this inference issue seems rather under-appreciated by enthusiasts of imaging methods. For example, if multiple but close regions are active, some inference schemes may infer a single source located where, in fact, none exists.) The impact of the computer is already seen in the 'empirical' exploration of systems we have great difficulty in approaching analytically. Indeed, it was via computations that some of the attractor aspects of some non-linear systems were first discovered. And, here, we may anticipate that new mathematical developments may make a big difference. I am thinking of developments comparable to the fairly recent one of the theory of wavelets as often providing a more suitable approach to transients than Fourier analyses.

Static vs Dynamic

Static models deal with relations among variables when time is not a factor; dynamic ones entail temporal changes. For example, the widely used signal detection model that characterizes the trade-off between errors of commission and omission is static. It becomes dynamic when one includes a learning process describing how that trade-off and/or the sensitivity is altered with experience and/or training.

Advantages and Disadvantages

The major advantage of static models is their greater simplicity; of course, that means we know rather less about what is going on than often we would like. Static theories almost always predate dynamic ones. The disadvantage is that static models rarely apply in psychology except approximately. For example, the custom in psychophysics is to record data only after subjects have been thoroughly trained, which often involves hundreds or even thousands of trials, until stable, asymptotic behavior is attained. These data are then treated as if the process were static—not just stochastically stationary. But whenever we look carefully at the asymptotic data we find quite significant sequential effects belying its supposedly static quality. (A good example is shown on Luce, 1986, p. 260.) These sequential effects presumably are evidence of the underlying dynamic mechanisms which were clearly manifest before the behavior became asymptotic, and do not simply vanish at asymptote. One effect of such dynamics is that estimated quantities are subject to an unwanted source of variance, namely those fluctuations due to the learning process. The reason we do what we do centers mainly on the difficulty of rerunning the dynamic process over and over to see just what it looks like, but we should never forget that the approximation involved may be potentially misleading.

Trends: Non-linear Dynamic Models

In talking about dynamic models in the psychological sciences, one is often also talking about non-linearity. The rich and complex behavior of even quite simple non-linear dynamic systems has attracted the attention of many social and behavioral scientists who suspect that some of the apparent chaos and instability seen in both individuals and social systems may be the result of non-linear dynamics. Their advantage is their evident potential to explain important, complex behavior.

The major challenge facing the discipline is to figure out just which non-linear processes are actually involved, and so far this is only very incompletely developed in any psychological subfield. Part of the reason arises from the difficulties mentioned in the introduction: the considerable individual differences among people and the usual irreversibility of people's experiences. The former makes averaging over individuals problematic and the latter makes repeated observations on single individuals largely impossible. So considerable cleverness is required to study dynamic psychological processes directly. Put another way, the issues of empirically testing the adequacy of specific non-linear dynamic models of behavior are very complex and very ill understood. One can anticipate great efforts being placed on this issue of testability in the coming years. One fact to remember in this connection is that crude empirical descriptions of asymptotic, chaotic behavior do not have strong implications about the detailed dynamics leading to it. The mathematics and simulations go from the specific dynamics to patterns, not the other way round.

The following caution has been expressed in the 'Mathematics Review Report', dated May 12, 1997, for the Natural Sciences and Engineering Research Council of Canada: 'Notions such as quasi-periodic, chaotic, or stable behaviors are now widely used. Sometimes mathematicians feel that they are not used with the rigor that would be necessary to draw solid conclusions' (p. 16).

Of the three forces at work, new mathematics and computation power are clearly paramount in the theoretical study of dynamic systems. The major limiting factor will, I believe, prove to be difficulty in obtaining adequate data on the dynamic processes themselves. Until the temporal resolution of passive mental imaging is vastly improved—from at best seconds to milliseconds—I rather doubt that it can or will play a very serious role in uncovering the dynamics of human performance.

Mathematical vs Computational

Until about 30 years ago, virtually all formal models in all fields were mathematical or they simply did not exist. With the advent of modern computers, the possibility of exploring formal processes not expressed in mathematical form became a real possibility and has been extensively developed. Of course computers are also widely used to explore explicit mathematical models that prove to be analytically difficult or intractable. Much of modern engineering is of this character, but it is based on physical models often cast mathematically as partial differential equations.

Two major classes of computational models are of relevance to psychology. One is neural nets and distributed processing models in which stored information is distributed across the net and various learning rules are assumed to modify the net's functioning as an information storage device. These have a non-symbolic, analogical character despite being implemented digitally. Such nets seem to mesh well with the developments of parallel processing in computers. The other includes algorithmic models that formulate a kind of computer code with input assertions and conditional constructions telling the model system what to do next. These simulations are symbolic in nature. Included among the latter are 'genetic algorithms'. Such models appear to provide considerable insight into self-organization and growth in complexity.

Mathematical modeling continues in parallel with these newer developments, and in some areas remains the dominant form of theorizing. Some models are reminiscent of those in classical physics, for example the work in vision and audition, in that analytic functions and linear spaces play a major role. Others are of a different character, looking more like axiomatic algebraic or geometric systems such as the axiomatic measurement models mentioned earlier (Luce et al., 1990; Saari, 1994b, 1995).

Advantages and Disadvantages

The great advantage of a mathematically explicit system is the depth of understanding that can often be achieved by mathematical methods. Everyone agrees that when such analysis is possible, it is far more satisfactory than numerical simulations. And indeed, for some modeling in terms of axiomatic systems, such as occurs in a good deal of the work on decision theory, no one knows how to deduce the results except mathematically.

The problem, of course, is that for some types of models, especially the information-processing ones, formal analysis is sometimes beyond current mathematical methods, and so we have no choice but to study them computationally. More important for psychology is the difficulty one has in mathematically formulating certain types of complex symbolic systems, some of which are quite naturally set up as computer programs.

Typically, both classes of computational models—nets and symbolic—are very difficult to characterize as mathematical systems—although some general theoretical properties have been discovered—and so they are largely understood via computations. A major problem with such models is how to confront and evaluate them with data. Complex issues of empirical validation exist for all these computational models, and they are far from resolved to the satisfaction of all, including me.

Trends

The major trend, apparent to everyone, is the growth of computational power and the wide familiarity among younger scientists with sophisticated programming. Less striking, but nevertheless important, are the mathematical developments—both of new results and of new importations into psychology. To me the most troubling one is some lack of concern about how complex computer models are to be evaluated empirically. Often the mere demonstration that a program is capable of something human-like is deemed sufficient. The criterion for success often seems one more appropriate to artificial intelligence and engineering than to psychology, which, after all, is concerned with actual human behavior and capabilities.

Some of these issues were discussed at a recent Symposium on Methods for Model Selection, August 3–4, 1997, at the University of Indiana. These papers will appear in due course in a special issue of the *Journal of Mathematical Psychology.*

Deterministic vs Probabilistic

Whether the model is mathematical or computational, it may be either deterministic, in the sense that once some external conditions are specified, the model unfolds in a unique way, or probabilistic, in the sense that chance in various ways determines what happens. Because the data never appear to be deterministic—that is, what seems to be the same situation typically leads to different responses—one can question why anyone bothers with deterministic models. There are at least three reasons:

- Sometimes one can view the deterministic model as characterizing some average behavior of the actual stochastic process, and that average behavior may, in fact, be deterministic.

- One can formulate certain complex relations among variables deterministically, and often these relations are the essence of what we are studying. More often than not, it is very difficult to incorporate this type of structure into an explicit probability framework. An example is the well-developed deterministic models of individual decision-making in which the stimuli are quite structured alternatives in which uncertainty and valued outcomes both play a role and there is an operation of combining several such uncertain alternatives (joint receipt). So far, it has proved impossible to recast our understanding of choices among such combinations of uncertain alternatives in terms of choice probabilities without sacrificing the structural aspects. This acknowledged limitation of our modeling needs to be remedied.
- Once one turns to non-linear dynamic processes, highly complex behavior occurs which, although completely deterministic, looks superficially as if it were generated by randomness. Perhaps what we see and interpret as evidence of chance mechanisms is simply complex non-linear dynamics that we do not understand.

So, given our uncertainties about psychological processes, it is probably wise to investigate both kinds of models.

Advantages and Disadvantages

Since probabilistic models must, in some sense, contain deterministic ones as special cases, their advantage is clear. Everything else equal, there never would be any question about which was preferred because the probabilistic models provide a principled way to analyze noisy data.

But everything else is not equal. We have been singularly unsuccessful in melding the kind of complex structure we can formulate deterministically with the randomness that we know how to formulate probabilistically. One aspect of the difficulty is our total lack of qualitative models of randomness—we really can only deal with it at the numerical level of random variables. To have a fully qualitative understanding of behavior, such a qualitative theory of randomness appears to be essential. Once in hand, the goal would be to seek qualitative behavioral patterns whose natural representation would be as random variables rather than numbers.

Trends

Here I anticipate continued development of both types of models, including non-linear dynamic ones, as a possible explanation for what appears to be randomness. I hope, but certainly do not feel in a position to predict if and when, random and structural aspects will be melded into a single framework rather than continuing with the current bifurcation. As noted above, doing so may well rest upon our working out a qualitative concept of randomness.

This is a major intellectual problem of far greater scope than psychology. It pervades statistics, which combines the two facets by assuming artificially simple, and often inappropriate, structural equations that more often than not are not examined empirically.

Conclusions

Psychology attempts to characterize what are arguably some of the most complex phenomena in the natural sciences. The complex nature of human experience makes it difficult to construct precise models of even the most basic psychological processes without forfeiting insights provided by less precise methods regarding these processes.

Despite these difficulties, some success has been achieved and many researchers feel that the potential for modeling truly complex phenomena behaviorally has been vastly enhanced by the new developments, largely in mathematics, physics and biology, of non-linear dynamic systems. With increasing frequency, such tools are being tried out on various psychological issues. A major research concern continues to be how to evaluate their adequacy as theories of behavior rather than as analogues of behavior, for example intelligence vs artificial intelligence. Of course, growing computational power surely will be put to good effect on these models.

In the short run, the competition between deterministic non-linear dynamics and probabilistic models will surely favor the latter, but I think that may be less clear as a long-term prediction. If probabilistic modeling is to prevail, we will have to crack the issue of combining, more successfully than we have, our understanding of complex structure with randomness.

For the information-processing approach, currently more dominant than the behavioral one, the mathematical developments of non-linear dynamics seem, so far, to be having less impact than either raw computational power or the growing potential for better understanding of the underlying architecture arising from neural imaging. The great advantage of such models is that much can be addressed, such as symbolic reasoning, that we do not know how to deal with behaviorally at present.

I nonetheless suspect that both approaches will be pursued well into the next century.

Note

1. This is truly a comparative statement. The classical mathematics of linear systems can be quite formidible for those who have limited mathematical backgrounds.

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