Aequationes Math. 58 (1999) 3–15
0001-9054/99/020003-13 $ 1.50+0.20/0

Aequationes Mathematicae

Personal reflections on an unintentional behavioral scientist

R. Duncan Luce

To János Aczél – friend and scientific collaborator – on his 75th birthday

Summary. This article consists of some personal reflections on aspects of János Aczél's role in the development of mathematical behavioral and social sciences; it is not a new contribution to the literature on functional equations, but rather a recounting of some history.

Mathematics Subject Classification (1991). 01A70, 39B03, 39B22.

Keywords. Functional equations, behavioral sciences, János Aczél.

It is unlikely that prior to reaching age 40 János Aczél anticipated any challenging questions about functional equations arising from the behavioral and social sciences. Yet, increasingly, behavioral and social scientists have come up with difficult equations. To be sure, the earliest equations we encountered were largely familiar and were covered in his treatise (Aczél (1961, 1966)). But some after that proved to be much more unusual and challenging.

My focus here is on both our first few years of interaction and on the last six years. Our recent collaboration has been greatly facilitated by the fact that since retirement János has elected to spend winters visiting the Institute for Mathematical Behavioral Sciences at the University of California, Irvine. The in-between period, during which János made substantial contributions to behavioral questions, I do not cover for two reasons: I do not know that research very completely and it is well documented elsewhere (Aczél (1987, 1995, 1997)).

1. Early examples of functional equations in mathematical psychology

1.1. Fechnerian just-noticeable differences. At the Center for Advanced Studies in the Behavioral Sciences, Stanford, California, during its opening year 1954–55, Albert Hastorf, professor of psychology at Stanford University, introduced me to the problem of just noticeable differences as formulated by Fechner (1860). Fechner's mathematical argument, which was repeated in many psychology texts, struck me as peculiar. Luce and Edwards (1958) argued that the posed problem was best formulated as a family of Abel functional equations. The argument went as follows. Let $\delta(x, \pi)$ be the increment in stimulus value x such that the probability of detecting $y = x + \delta(x, \pi)$ as different from x is π, i.e.,

$$\pi = \Pr[x + \delta(x, \pi), x].$$

Then Fechner hypothesized that subjective sensation $u(x)$ is such that the sensation increment due to the change $\delta(x, \pi)$ is a function only of π, not of x, i.e.,

$$u[x + \delta(x, \pi)] - u(x) = k(\pi). \tag{1.1}$$

Fechner had been interested in the special case where $\delta(x, \pi) = \alpha(\pi)x$, which is called Weber's law after the physiologist E. H. Weber who had provided empirical evidence favoring that relationship[1].

Rather than actually solve Eq. (1.1), Fechner carried out a not clearly justified limiting process which led to the differential equation for the logarithm. We pointed out that only in this special case of Weber's law did his approach actually lead to a solution of the Abel family. Based on material in the literature concerning this family of equations, we discussed the solutions.

That experience led me to think functional equations might play a role in the behavioral sciences, but it had not yet led me to an awareness of Aczél. That came with two further functional equations. Both arose from ideas posed in my monograph *Individual Choice Behavior* (Luce (1959a)).

1.2. Invariance principles. One arose from a growing awareness of Stevens' classification (Stevens (1951)) of measurement scales into uniqueness types — the class of transformations that take one numerical measurement representation into another equally good one using the same representational structure, e.g., the multiplicative, ordered, positive real numbers or the additive, ordered real numbers. This struck me as important because it imposed constraints on the possible relations between two types of measurement. My imperfect understanding of this problem was presented in Luce (1959b); an improved, deeper analysis is given in Luce (1990). It asked and answered the following question: What constraints are imposed on a strictly increasing function F that relates two measures x and y, each being one or another of the scale types identified by Stevens, so that an admissible change in the measure x is reflected as an admissible change in y? For example, if both are ratio scales (multiplicative similarity transformations) the principle is simply that for each $\alpha > 0$ there exists a $\beta(\alpha) > 0$ such that for all $x > 0$

$$F(\alpha x) = \beta(\alpha)F(x). \tag{1.2}$$

[1] Later evidence made clear that Weber's law is at best an approximation to the empirical truth. For example, in the study of pure tones a power relation is a better approximation.

Anyone with a passing knowledge of functional equations knows the solution. A number of other simple cases were examined and solved using standard methods. The only mathematical reference I made was to Hamel's work on the Cauchy equation (Hamel (1905)), thus suggesting that I still was unaware of Aczél.

1.3. Commutative learning operators. A second, more complex class of functional equations arose as follows. In my monograph (Luce (1959a)) I had arrived at an operator model for learning that was commutative. In such models a probability vector over possible choices is modified by applying operators to the components so that the result is again a probability vector. Which of several operators is applied depends upon the reinforcement that occurs following a choice. Bush and Mosteller (1955) had worked out the linear cases rather fully. Mine was non-linear, but was reasonably simple to study because the operators were commutative. So it was natural to ask about the general class of commutative learning operators, which except for very special cases were necessarily non-linear. That work appeared in Luce (1964) and in a paper by my student A. A. J. Marley (Marley (1967)). By then I had become aware[2] of Aczél (1961) and drew on his classic book, but I was not yet in direct contact with him. That came about as follows.

2. Our first contacts[3]

Based on my three rewarding experiences using functional equations, I urged that Volume 3 of the *Handbook of Mathematical Psychology* (Luce, Bush and Galanter (1965)) include a chapter on uses of functional equations in psychology. We asked Richard Bellman to do it (Bellman (1965)). At some point during this period Aczél became aware of this chapter, and asked Bellman for references, a request passed on to me. By then Aczél was visiting the University of Florida at Gainsville. On March 12, 1964, I sent the draft chapter to him along with a preprint of Luce (1964) and a letter. On March 30, he provided a very detailed, critique of my paper — handwritten in his well-known, crabbed, microscopic scratches on onionskin — bringing to my attention related work of M. Hosszú. His copy of that letter is largely unreadable. Judging by my reply of April 2, he must have announced the forthcoming English edition of his book (which appeared two years later in a series edited by Bellman); and in my reply, I suggested several psychological papers that used functional equations. These were listed in the 1966 English edition. We

[2] According to Marley — about this I have no independent recollection at all — he uncovered Aczél (1961) in the mathematics library at the University of Pennsylvania and brought it to my attention.

[3] I have been greatly helped by Aczél on reconstructing this early history. He has sent me copies of correspondence from that time (mine being buried in the archives of Harvard University).

also had an exchange in which I discussed the work I had done with John Tukey (Luce and Tukey (1964)) on additive conjoint measurement, and he brought to my attention the closely related literature on webs. My letter ends:

> "It has become the custom of many of us who are interested in mathematical psychology to spend our summers at Stanford University, where we have informal working groups, seminars, and so on. It is usually a fairly stimulating atmosphere and a fair amount of research gets done. Although, as far as I know, you have no particular interest in psychology, perhaps the fact that we seem to get involved in functional equations and the attractions of California would tempt you to spend some time with us...."

This came to pass. For me that summer was memorable in two respects: meeting and interacting with Aczél and learning to fly an aeroplane. Out of it came three papers by him: Aczél (1965, 1967) and Aczél and Wallace (1967).

3. 1968–1993: Invariance and aggregation issues

Aczél's major activities on social science problems during 1968–1993 did not much involve me directly. He and Fred S. Roberts (Aczél (1990); Aczél, Roberts and Rosenbaum (1986); Roberts (1990)) took off from the problem of Eq. (1.2) and the other elementary cases that I had considered. They worked out numerous much more difficult cases that involved several independent variables with or without constraints on their admissible transformations. Some of that work is summarized in Sections 3 and 4 of Aczél (1987). Results of a related sort, plus other forms of invariance arising in psychophysics, played a major role in Falmagne (1985).

A second form of invariance arose in aggregation problems, in particular in economics and operations research (Aczél, Ng and Wagner (1984)). One way to describe the problem is that one has two types of independent variables, which form a matrix, and some measure over the cells of the matrix. A question is how to aggregate such information. One way involves aggregation first over all of the columns in a row and then aggregation of these aggregated values over rows; the other is to reverse the order of aggregation. The question addressed is for what rules of aggregation does the order of aggregation not matter. Formalizing that question leads to solving a generalized form of bisymmetry. Again, this is discussed in Section 1 of Aczél (1987) and in Aczél (1995, 1997).

4. Measuring utility in two ways

Although our paths crossed some during that 25 year period and my use of functional equations continued, little of real substance transpired because I was mostly able to solve the equations I evolved. This changed sharply when I began what has turned out to be a prolonged development of new utility models that are generalizations of subjective expected utility (SEU). The need for such generalizations has been forced by the growing body of evidence showing that descriptively SEU is inadequate (for an overview, see Edwards (1992)). My work has been guided by a very simple principle that is most easily illustrated by a simple physical example.

4.1. A physical example. One reason that functional equations arise in scientific measurement, which is what utility theory is, results from the simple fact that more often than not we are able to measure the same attribute in two or more distinct ways. For example, masses on a pan balance can be manipulated both by concatenating them (placing two masses on the same pan) or by varying the volumes of homogeneous substances. Let the two measurement structures be denoted $\langle X, \circ, \succsim \rangle$ and $\langle V \times S, \succsim \rangle$, where V is the set of volumes, S the set of substances, $X = V \times S$, $\succsim$ is the ordering established using the pan balance (in a vacuum if necessary), and $\circ$ is the concatenation operation. Well known axioms assure us that the former structure has an order-preserving mass measure, m, unique up to $m \to \alpha m, \alpha > 0$, for which $m(x \circ y) = m(x) + m(y)$; and the latter has a family of order-preserving mass measures $m^* = vs$, unique up to $m^* \to \alpha(m^*)^\beta$. Because a single dependent attribute, massiveness in this example, is involved we anticipate that we should be able to prove for some β that $m = \alpha(m^*)^\beta$, but all we really know at this point is that there is a strictly increasing function Φ for which $m = \Phi(m^*)$. If, however, an appropriate empirical law can be found that links the two structures — in this case, a distribution law — then that law reflects itself as Φ satisfying a functional equation. In this case, the equation is easy to solve (Krantz, Luce, Suppes and Tversky (1971); Luce, Krantz, Suppes and Tversky (1990)).

4.2. Rank-dependent utility. Let me turn now to the utility representation, called rank-dependent, that has been the focus of a good deal of my recent work. It arose in a series of papers during the 1980s, summarized by Quiggin (1993), and it can be described in the binary case as follows. Let $(x, C; y, E\backslash C)$ denote an uncertain alternative (for short, a gamble) in which a chance experiment with universal set E is run and if the outcome lies in the event C, then the consequence to the decision maker is x, and if it does not lie in C, then the consequence is y. We abbreviate this to $(x, C; y)$ when it is clear which experiment is involved. Over gambles and consequences we have a preference order $\succsim$ that is assumed to be a weak order. The kind of theory that has arisen asserts that there is a numerical utility U over gambles and consequences and a weighting function W over events into $[0, 1]$ such that if $\overline{C} = E\backslash C$

$$U(x, C; y) = \begin{cases} U(x)W(C) + U(y)[1 - W(C)] & \text{if } x \succsim y \\ U(x)[1 - W(\overline{C})] + U(y)W(\overline{C}) & \text{if } x \prec y \end{cases} \tag{4.1}$$

is order preserving. The earlier subjective expected utility (SEU) model is the special case in which the preference order over the consequences does not matter which is equivalent to assuming $W(C) + W(\overline{C}) = 1$ holds for all events C.

I began to become interested in these problems from the perspective of algebraic measurement theory during my third sabbatical year at the Center for Advanced Study in the Behavioral Sciences in 1987–88 when work was being completed on Volumes II and III of the *Foundations of Measurement* (Suppes, Krantz, Luce and Tversky (1989); Luce, Krantz, Suppes and Tversky (1990)). My first paper on the topic was Luce (1988). The kinds of questions involved have included: What properties give rise to Eq. (4.1), and can we say anything about the mathematical forms of the functions U and W, at least in the case of money consequences and events with known probabilities of occurring?

4.3. Joint receipt of gambles. The first example of two ways of measuring utility was Luce (1991) and Luce and Fishburn (1991, 1995). What we did initially was accept the binary rank-dependent model, Eq. (4.1). The novelty — from the perspective of utility theory — of our approach was to add to this structure a commutative binary operation $\oplus$, where $g \oplus h$ means holding (or receiving) both alternatives g and h. There is also a special consequence e that is an identity of $\oplus$ and is interpreted as the status quo. Any alternative $g \succsim e$ is interpreted as a gain and any $g \precsim e$ as a loss.

Here we have two structures, and we suggested that they may be linked by a property called *segregation:* For all gains x, y

$$(x, C; e) \oplus y \sim (x \oplus y, C; y). \tag{4.2}$$

This property is "rational" in the sense that the two sides really denote the same thing: If C occurs, then the consequence is $x \oplus y$ and if not it is y. Assuming U is onto a real interval $[0, a[$ and W is onto $[0, 1]$, Eqs. (4.1) and (4.2) lead to a functional equation for U over $\oplus$. We were able to reduce it to solving the Cauchy equation which led to

$$U(x \oplus y) = U(x) + U(y) - \delta U(x)U(y), \tag{4.3}$$

where $\delta \in \mathbb{R}$. This implies the existence of an additive representation V of $\oplus$ and that U is either proportional to, or exponentially related to, or negative exponentially related to V. For this reason U is called the *polynomial-additive,* or *p-additive,* representation of joint receipt.

The following converse problem was not so easy. Could it be that Eqs. (4.2) and (4.3) imply Eq. (4.1)? Not quite. One had also to assume among the hypotheses that there is in addition a weighting function W for which the following property of *separability* is satisfied:

$$U(x, C; e) = U(x)W(C). \tag{4.4}$$

It is easy to give qualitative conditions that give rise to separability, namely the axioms of additive conjoint measurement (Krantz, Luce, Suppes and Tversky (1971), Ch. 6). The key Thomsen condition is assured if one imposes a special case of a behavioral condition called *event commutativity,* namely, for all $x, y \succsim e$ and all events C, D arising from independent experiments, that

$$((x, C; y), D; y) \sim ((x, D; y), C; y). \tag{4.5}$$

The special case is when $y = e$, which we call *status-quo event commutativity.* And, of course, it is equally easy to give qualitative conditions that give rise to an additive representation V of $\oplus$ and so a p-additive one. But there is no reason to assume that the two functions arising from separability and p-additivity are, in fact, the same. Luce (1996) showed that a necessary condition for this to be true is the property, called *joint-receipt decomposability*, that for each gain x and each event C there exists an event $D = D(x, C)$ such that for all gains y

$$(x \oplus y, C; e) \sim (x, C; e) \oplus (y, D; e). \tag{4.6}$$

The task was to reverse direction and to show that if there is a separable representation (u, w), a p-additive one u^*, and Eq. (4.6) holds, then there are U and W such that both (U, W) forms a separable representation of $\langle X, \succsim \rangle$ and U is p-additive. Since both u and u^* preserve the same order there is a strictly increasing function relating them. From those assumptions and through some transformations I was led to the following functional equation:

$$\begin{aligned} F^{-1}[F(X) + F(Y) - F(X)F(Y)]Z = F^{-1}[&F(XZ) + F[YP(X, Z)] \\ &- F(XZ)F[YP(X, Z)]], \end{aligned} \tag{4.7}$$

where F is strictly increasing on $[0, 1[$ onto $[0, 1[$, $X, Y \in [0, 1[$, $Z \in [0, 1]$, and $P : [0, 1[\times [0, 1] \to [0, 1]$ is strictly decreasing in the second variable.

I did not have a clue how to solve this functional equation. Indeed, I also did not know how to attack two other functional equations, which I do not detail here, that arose from somewhat related problems. So, I contacted Aczél by e-mail for help. After some back and forth about the exact assumptions, he contacted G. Maksa in Hungary and ultimately the two were able to solve Eq. (4.7) under the added, and from my perspective not easily justified, assumption that F and F^{-1} are both differentiable. The solution is: for some $a > 0, b > 0$,

$$\begin{aligned} P(X, Z) &= Z\frac{(1 - X^b)^{\frac{1}{b}}}{(1 - X^b Z^b)^{\frac{1}{b}}}, \quad X \in [0, 1[, Z \in [0, 1], \\ F(X) &= 1 - (1 - X^b)^a, \qquad X \in]0, 1[. \end{aligned}$$

This and the solutions to the other two equations were published in Aczél, Luce and Maksa (1996).

This result permitted me to prove what I needed for the utility problem except that the differentiability assumption was not a part of the original problem. I urged an effort be made to solve it only under strict monotonicity, and the challenge was widely announced in functional equations circles, in particular at the 1996 meeting in Wisla–Jawornik, Poland. Progress was made toward showing the same solution under only strict monotonicity in Graz, Austria, 1997, with ultimate solution in Brno, Czech Republic, a year later through the concerted efforts of Aczél, Maksa and Páles (1999a).

This solution meant that we had an axiomatization of binary rank-dependent utility that began with an additive measure of joint receipts plus some behavioral properties. Of these, only joint receipt decomposition has not be studied empirically.

4.4. An axiomatization of RIU. Another approach to the binary rank-dependent representation rests also on knowing how to measure an attribute in two ways. When the event C is held fixed, the gamble $(x, C; y)$ involves a trade-off between x and y. Thanks to Wakker (1991, 1993) we know qualitative conditions such that the rank-dependent case leads to functions Φ, Φ_1 and Φ_2 so that for $x \succsim y \succsim e$,

$$\Phi(x, C; y) = \Phi_1(x, C, E) + \Phi_2(y, C, E). \tag{4.8}$$

Equally well, as noted above, we understand the qualitative conditions such that u and w form a separable representation of $(x, C; e)$ as in Eq. (4.4). In Luce (1998) I mistakenly assumed that one could prove that the *same* separable representation holds for $(e, C; y)$, which is true for the SEU model because one can freely use $(e, C; y) \sim (y, \overline{C}; e)$ to get separability on the second component. But that is not possible for the more general representation. So, for now, accepting this assumption of rank independence, the functional equation (4.8) becomes

$$\Phi(v) = \Phi(vw) + \Phi[vQ(w)], \tag{4.9}$$

where Φ from $[0, k[$ onto $\mathbb{R}_+$ is strictly increasing, $v \in [0, k[$, $w, Q : [0, 1] \overset{onto}{\longrightarrow} [0, 1]$, and Q is strictly decreasing.

Again, help was needed. Without making any monotonicity assumptions Aczél, Ger and Járai (1999) showed that either Φ is a constant, which monotonicity rules out, or for some $\alpha > 0, \beta > 0$

$$\Phi(v) = \alpha v^\beta \text{ and } Q(w)^\beta = 1 - w^\beta. \tag{4.10}$$

From that, the rank-independent (RIU) version of equation (4.1) follows readily.

4.5. An axiomatization of a new representation. After A. A. J. Marley pointed out that what had been accomplished was to axiomatize SEU not binary RDU, he and I worked further on the general problem. We pursued three somewhat different approaches each of which led to a new functional equation.

4.5.1. *Assuming the second term is also separable.* The first approach, worked out in Luce and Marley (1999), assumes separability of the $(y, \overline{C})$ term, but not necessarily using the same utility and weighting functions as before. If we let that utility be denoted u^* and define strictly increasing g by $u^* = g(u)$, then we are led to

$$\Phi(v) = \Phi(vw) + \Phi\left(g^{-1}[g(v)Q(w)]\right), \tag{4.11}$$

where $g : [0, k[\to [0, k[$ is strictly increasing. Equation (4.9) is clearly the special case of Eq. (4.11) where g the identity. Aczél first posed the problem to C. T. Ng who solved it under differentiability assumptions. After that Aczél, Maksa, Ng and Páles (1999) did so without any differentiability assumption. The solution is that for $q > 0, \gamma > 0, \mu > 0$

$$\begin{aligned} \Phi(v) &= \frac{\ln(1+\mu v^q)}{\ln(1+\mu)}, \\ g(x)^\gamma &= \frac{(1+\mu)x^q}{1+\mu x^q}, \\ Q(w)^\gamma &= 1 - w^q. \end{aligned} \tag{4.12}$$

Moreover, the solution continues to hold in the limit as $\mu \to 0$. From this Marley and I showed that the gamble representation is for $x \succsim y \succsim e$,

$$U(x, C; y) = \frac{U(x)W(C) + U(y)[1 - W(C)] + \mu U(x)U(y)W(C)}{1 + \mu U(y)W(C)}, \tag{4.13}$$

which we called *ratio rank-dependent utility* (RRDU), where “ratio” refers to the function being a ratio of polynomials.

4.5.2. *Additivity, separability of first term, and event commutativity.* The second approach, also developed in Luce and Marley (1999) is more satisfactory. We are really only able to justify from rank-dependent additivity, Eq. (4.8), and separability of the (x, C) pair, Eq. (4.4), that

$$U(x, C; y) = \Phi_1[u_1(x)w_1(C)] + \Phi_2(y, C, E). \tag{4.14}$$

In addition, we have good empirical reasons to suppose that U satisfies event commutativity, Eq. (4.5). Putting these together and calling Φ_1 just Φ, we obtain the functional equation

$$\begin{aligned} &\Phi\left(\Phi^{-1}[\Phi(XW) + \Phi(Y) - \Phi(YW)]Z\right) - \Phi(YZ) \\ &\quad = \Phi\left(\Phi^{-1}[\Phi(XZ) + \Phi(Y) - \Phi(YZ)]W\right) - \Phi(YW). \end{aligned} \tag{4.15}$$

Assuming Φ is twice differentiable Aczél and Maksa (in preparation) have shown that this has four possible solutions, of which we ruled out two by requiring that $\Phi(0) = 0$ and that Φ be strictly increasing. The remaining ones are

$$\Phi(v) = \frac{1}{\beta q}\ln(1+\mu v^q), \tag{4.16}$$

$$\Phi(v) = \frac{1}{\alpha q} v^q. \tag{4.17}$$

Again, this leads to the RRDU representation of Eq. (4.13). I am hoping, once again, that monotonicity will suffice.

4.5.3. *Forcing RDU.* Despite the odd asymmetries of Eq. (4.13), especially in the denominator, finding plausible behavioral arguments that force the RDU representation, i.e., force $\mu = 0$, has been more difficult than I expected. Initially I had thought that event commutativity, Eq. (4.5), would do it, but to my surprise RRDU satisfies that property. One property that does the job, called "comonotonic consistency," had been used earlier in Wakker (1989) to justify Eq. (4.1). I do not describe it in detail in part because, in my opinion, the property is not very intuitive. Luce and Marley (1999) pursued two other conditions. One was to impose a ranked version of bisymmetry, but again this does not seem very behaviorally compelling. The most interesting one that we have discovered that forces RDU is segregation, Eq. (4.2). This is a highly rational property, one that "should" be satisfied, and there is some experimental support for it (Cho, Luce and von Winterfeldt (1994)).

4.6. A direct axiomatization of RDU. The third approach taken in Marley and Luce (1999) rests on some familiar assumptions but also one new one called *gains partition*: Let $\mathcal{E}$ denote the family of chance events and let $\succsim_{\mathcal{E}}$ be the weak order over $\mathcal{E}$ induced in the natural way (a consistency axiom is required). Then, we assumed there exists a 1-1 function $M : \mathcal{E} \overset{onto}{\longrightarrow} \mathcal{E}$ that inverts the order $\succsim_{\mathcal{E}}$ such that for $x, x^{'}, y, y^{'} \in \mathcal{B}_0$, with $x \succsim y$, $x^{'} \succsim y^{'}$, and $C, C' \in \mathcal{E}$, if

$$(x, C; e) \sim (x^{'}, C'; e) \quad \text{and} \quad (y, M(C); e) \sim (y^{'}, M(C'); e),$$

then

$$(x, C; y) \sim (x^{'}, C'; y').$$

Coupling this with event commutativity and other weak conditions, led us to the functional equation

$$Z\gamma^{-1}[Z\gamma(P))] = P\varphi^{-1}[\varphi(Z)\psi(P)], \tag{4.18}$$

where $P, Z \in [0,1]$, $\varphi :]0,1[\to]0,\infty[$, $\psi :]0,1[\to]1,\infty[$, and $\gamma :]0,1[\to]0,\infty[$, and the three unknown functions γ, φ, ψ are strictly increasing and onto. Again, help was needed and Aczél, Maksa and Páles (1999b) provided it yielding the solution that for some constants $A > 0, k > 0, c < 0$,

$$\begin{aligned} \gamma(P) &= \frac{(1-P^k)^{\frac{1}{k}}}{P}, \\ \varphi(Z) &= A\left(\frac{1-Z^k}{Z^k}\right)^c, \\ \psi(P) &= P^{kc}. \end{aligned} \tag{4.19}$$

Marley and I were able to use this to show the RDU representation Eq. (4.1).

5. Conclusions

During the nearly 50 years I have been doing research in mathematical behavioral sciences, it has become increasingly clear that functional equations are very useful for us[4]. It is also clear that the trajectory has gone from problems involving well known equations — Abel, Cauchy, Pexider, bisymmetry — to problems that give rise to functional equations that are strikingly different and that have been quite difficult to solve. Of course, easy ones still play a role, as they did in my work with Fishburn. Recent examples include my reformulation of Prelec's derivation (Prelec (1998)) of a particular mathematical form for the weighting function W over probabilities (Luce (1999)) and work on fully associative joint receipts that leads to interesting non-bilinear models for mixed gains and losses (Luce (1997)).

János Aczél has played a crucial role in aiding those of us who know how to arrive at functional equations from behavioral properties but who have substantially less prowess in solving them than do experts like himself and some of his, often Eastern European, colleagues. For this reason, many behavioral and social scientists — especially me — are grateful that János has found such equations worthy of attention. I do recall, however, that at some point he said, in some exasperation, something to the effect that until then he had never encountered a functional equation he did not love, but that this particular one, Eq. (4.7) if I recall correctly, taxed the accuracy of this proposition.

References

[1] J. Aczél (1961), *Vorlesungen über Funktionalgleichungen und ihre Anwendungen*, Birkhäuser Verlag, Basel, 1961.

[2] J. Aczél (1965), *A remark on functional dependence*, J. Math. Psych. *2* (1965), 125–127.

[3] J. Aczél (1966), *Lectures on Functional Equations and Their Applications*, Academic Press, New York, 1966. (Translation and updating of Aczél, 1961.)

[4] J. Aczél (1967), *On rational quasi-multiplicative families of operators*, J. Math. Psych. *4* (1967), 335–340.

[5] J. Aczél (1987), *A Short Course on Functional Equations Based on Applications to the Social and Behavioral Sciences*, Riedel–Kluwer, Dordrecht–Boston–Lancaster–Tokyo, 1987.

[6] J. Aczél (1990), *Determining merged relative scores*, J. Math. Anal. Appl. *150* (1990), 20–40.

[7] J. Aczél (1995), *Some recent applications of functional equations to the social and behavioral sciences. Further problems*, Aequationes Math. *50* (1995), 38–49.

[8] J. Aczél (1997), *Bisymmetry and consistent aggregation: Historical review and recent results*, in: A. A. J. Marley (1997), 225–233.

[9] J. Aczél, R. Ger and A. Járai (1999), *Solution of a functional equation arising from utility that is both separable and additive*, Proc. Amer. Math. Soc. (1999), in press.

[10] J. Aczél, R. D. Luce and G. Maksa (1996), *Solutions to three functional equations arising from different ways of measuring utility*, J. Math. Anal. Appl. *204* (1996), 451–471.

[4] Of course, many other mathematical approaches, probably the most important being stochastic processes, are widely used in the behavioral and social sciences.

[11] J. Aczél and G. Maksa, in preparation.
[12] J. Aczél, G. Maksa, C. T. Ng and Z. Páles (1999), *A functional equation arising from ranked additive and separable utility*, submitted (1999).
[13] J. Aczél, G. Maksa and Z. Páles (1999a), *Solution of a functional equation arising from different ways of measuring utility, J. Math. Anal. Appl.* (1999), in press.
[14] J. Aczél, G. Maksa and Z. Páles (1999b), *Solution of a functional equation arising in an axiomatization of the utility of binary gambles*, Proc. Amer. Math. Soc. (1999), in press.
[15] J. Aczél, C. T. Ng and C. Wagner (1984), *Aggregation theorems for allocation problems*, SIAM J. Algebra Discr. *5* (1984), 1–8.
[16] J. Aczél, F. S. Roberts and Z. Rosenbaum (1986), *On scientific laws without dimensional constants*, J. Math. Anal. Appl. *119* (1986), 389–416.
[17] J. Aczél and A. D. Wallace (1967), *A note on generalizations of transitive systems of transformations*, Colloq. Math. *17* (1967), 29–34.
[18] R. Bellman (1965), *Functional equations*, in: Luce, Bush and Galanter (1965), 487–513.
[19] R. R. Bush and F. Mosteller (1955), *Stochastic Models for Learning*, Wiley, New York, 1955.
[20] Y. Cho, R. D. Luce and D. von Winterfeldt (1994), *Tests of assumptions about the joint receipt of gambles in rank- and sign-dependent utility theory*, J. Exper. Psych.: Human Percept. Perform. *20* (1994), 931–943.
[21] W. Edwards (Ed.) (1992), *Utility Theories: Measurements and Applications*, Kluwer, Boston, 1992.
[22] J.-C. Falmagne (1985), *Elements of Psychophysical Theory*, Oxford University Press, New York, 1985.
[23] G. T. Fechner (1860), *Elemente der Psychophysik*, Breitkopf und Hartel, Leipzig, 1860.
[24] G. Hamel (1905), *Eine Basis aller Zahlen und die unstetigen Lösungen der Funktionalgleichung* $f(x+y) = f(x) + f(y)$, Math. Ann. *60* (1905), 459–462.
[25] D. H. Krantz, R. D. Luce, P. Suppes and A. Tversky (1971), *Foundations of Measurement*, Vol. I, Academic Press, New York, 1971.
[26] R. D. Luce (1959a), *Individual Choice Behavior*, Wiley, New York, 1959.
[27] R. D. Luce (1959b), *On the possible psychophysical laws*, Psychol. Rev. *66* (1959), 81–95.
[28] R. D. Luce (1964), *Some one-parameter families of commutative learning operators*, in: R. C. Atkinson (Ed.) *Studies in Mathematical Psychology*, Stanford University Press, Stanford, CA, 1964.
[29] R. D. Luce (1988), *Rank- and sign-dependent linear utility models for finite first-order gambles*, J. Risk Uncertainty *4* (1988), 29–59.
[30] R. D. Luce (1990), *"On the possible psychophysical laws" revisited: Remarks on cross-modal matching*, Psychol. Rev. *97* (1990), 66–77.
[31] R. D. Luce (1991), *Rank- and sign-dependent linear utility models for binary gambles*, J. Econom. Theory *53* (1991), 75–100.
[32] R. D. Luce (1996), *When four distinct ways to measure utility are the same*, J. Math. Psych. *40* (1996), 297–317.
[33] R. D. Luce (1997), *Associative joint receipts*, Math. Social Sci. *34* (1997), 51–74.
[34] R. D. Luce (1998), *Coalescing, event commutativity, and theories of utility*, J. Risk Uncertainty *16* (1998), 87–114.
[35] R. D. Luce (1999), *Reduction invariance and Prelec's weighting functions*, J. Math. Psych. (1999), in press.
[36] R. D. Luce, R. R. Bush and G. Galanter, (Eds.) (1965), *Handbook of Mathematical Psychology*, Vol. 3, Wiley, New York, 1965.
[37] R. D. Luce and W. Edwards (1958), *The derivation of subjective scales from just noticeable differences*, Psychol. Rev. *65* (1958), 222–237.
[38] R. D. Luce and P. C. Fishburn (1991), *Rank- and sign-dependent linear utility models for finite first-order gambles*, J. Risk Uncertainty *4* (1991), 25–59.
[39] R. D. Luce and P. C. Fishburn (1995), *A note on deriving rank-dependent utility using additive joint receipts*, J. Risk Uncertainty *11* (1995), 5–16.
[40] R. D. Luce, D. H. Krantz, P. Suppes and A. Tversky (1990), *Foundations of Measure-*

ment, Vol. III, Academic Press, San Diego, 1990.

[41] R. D. Luce and A. A. J. Marley (1999), *Separable and additivity representations of binary gambles of gains*, submitted (1999).

[42] R. D. Luce and J. W. Tukey (1964), *Simultaneous conjoint measurement: A new type of fundamental measurement*, J. Math. Psych. *1* (1964), 1–27.

[43] A. A. J. Marley (1967), *Abstract one-parameter families of commutative learning operators*, J. Math. Psych. *4* (1967), 414–429.

[44] A. A. J. Marley (Ed.) (1997), *Choice, decision, and measurement: Essays in honor of R. Duncan Luce*, Lawrence Erlbaum Associates, Mahwah, NJ, 1997.

[45] A. A. J. Marley and R. D. Luce (1999), *A simple axiomatization of binary rank-dependent expected utility of gains (losses)*, submitted (1999).

[46] D. Prelec (1998), *The probability weighting function*, Econometrica *66* (1998), 497–527.

[47] J. Quiggin (1993), *Generalized Expected Utility Theory: The Rank-Dependent Model*, Kluwer Academic Publishers, Boston, 1993.

[48] F. S. Roberts (1990), *Merging relative scores*, J. Math. Anal. Appl. *147* (1990), 30–52.

[49] D. Schmeidler (1989), *Subjective probability and expected utility without additivity*, Econometrica *57* (1989), 571–587.

[50] S. S. Stevens (1951), *Mathematics, measurement, and psychophysics*, in: S. S. Stevens (Ed.) *Handbook of Experimental Psychology*, pp. 1–49, Wiley, New York, 1951.

[51] P. Suppes, D. H. Krantz, R. D. Luce and A. Tversky (1989), *Foundations of measurement*, Vol. II, Academic Press, San Diego, 1989.

[52] A. Tversky and D. Kahneman (1992), *Advances in prospect theory: Cumulative representation of uncertainty*, J. Risk Uncertainty *5* (1992), 204–217.

[53] P. P. Wakker (1989), *Additive Representations of Preferences: A New Foundation of Decision Analysis*, Kluwer Academic Publishers, Dordrecht, The Netherlands, 1989.

[54] P. P. Wakker (1991), *Additive representations on rank-ordered sets. I. The algebraic approach*, J. Math. Psych. *35* (1991), 501–531.

[55] P. P. Wakker (1993), *Additive representations on rank-ordered sets. II. The topological approach*, J. Math. Econom. *22* (1993), 1–26.

R. D. Luce
Social Science Plaza
University of California
Irvine, CA 92697-5100
USA
e-mail: rdluce@uci.edu

Manuscript received: January 5, 1999 and, in final form, April 27, 1999.