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Associative joint receipts

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Abstract

This paper examines the assumption that the joint receipt of alternatives together with a preference order and the status quo as an identity forms an Archimedean ordered group. This model agrees with earlier work (Luce, Fishburn, 1991, Luce, Fishburn, 1995) for gains and losses separately, but it forces a somewhat different representation for mixed joint receipts. The predictions of the associative model accommodate Thaler's (1985) data better than did the earlier model. For gambles with mixed consequences that leads either to a somewhat different representation from that of prospect theory or to replacing the empirically supported linking property of duplex decomposition by a modified version of it or by a generalization of the segregation property that was used earlier for linking joint receipt and gambles of gains. Issues of empirical testing are discussed. © 1997 Elsevier Science B.V.

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1. Introduction to joint receipts

The last half century of research about the utility of goods has been largely dominated by methods based on trade-offs within gambles, i.e., uncertain alternatives. Perhaps the most influential works were Ramsey (1931), Von Neumann, Morgenstern (1947), Savage (1954), and Kahneman and Tversky (1979). Theories of riskless utility were mostly focused on variants of additive conjoint measurement methods (Ramsey, 1931; Keeney, Raiffa, 1976) in which either attributes of goods or types of goods were manipulated and

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their trade-offs studied. Little was done to establish the conditions under which riskless and risky utility measures are the same or even whether riskless utility is a meaningful concept. For example, Arrow, 1951 (p. 425) remarks "First, the utilities assigned are not in any sense to be interpreted as some intrinsic amount of good in the outcome (which is a meaningless concept in any case.)" If the operative word is "intrinsic" then I agree, as I would if we spoke of the "intrinsic amount of mass in an object." In the mass case we measure relative mass, not intrinsic mass. Perhaps the same is possible for goods.

A number of papers have focused attention on the question of whether a useful meaning can be given to utility differences which, if restricted to pure consequences, gives a measure that seems not to bear on risk. Among these papers are Alt (1936), Bouyssou, Vansnick (1988); Dyer, Sarin (1982), and Wakker (1994). The latter is especially concerned about separating the concepts of utility difference from issues having to do with risk. Nonetheless, all of these approaches are basically dependent on uncertain or risky alternatives and never deal with riskless goods themselves.

More recently a way to generate riskless utility has been explored. It is based on the intuition that utility of goods acts a good deal like mass measurement in the following sense: If x and y are valued entities, then having (or receiving) both x and y is also a valued entity. A major difference, however, is that we think of valued entities as having either a positive, a negative, or a neutral quality—of being a gain or a loss relative to a status quo, which is denoted e, or indifferent to e. The analogy to mass holds only for gains (or losses separately), and so the case of mixed gains and losses must be approached in some other way.

Denote the receipt of both x and y by $x \oplus y$, the preference of x over y by $x \succ y$, and indifference by $x \sim y$. Let $\succsim = \succ \cup \sim$. Two tasks facing a theorist are: (i) to uncover behavioral (i.e., empirically testable) properties of $\oplus$ and $\succsim$ their interrelations that are adequate to generate a cardinal (interval or ratio scale) theory of riskless utility; and (ii) to provide an analysis of how this scheme for riskless utility relates to the more familiar risky utility. One seeks behavioral properties that are necessary and sufficient for some sort of tight relationship between the two measures of utility.

The history of work on joint receipt is brief. Theoretical discussions were by Fishburn and Luce, (1991, 1995, 1997) Luce and Fishburn, (1991, 1995), and experimental ones by Cho, Luce (1995), Cho et al. (1994), Linville and Fischer, 1991, Payne, Braunstein (1971), Slovic (1967), Slovic, Lichtenstein (1968), Thaler (1985), and Thaler, Johnson (1990). One reason that it has not been much pursued appears to have been the belief that one can easily dismiss the idea of basing utility measurement on $\oplus$.

One objection to using $\oplus$ goes as follows. (i) If joint receipt is like mass measurement, then it has an additive representation V, $V(x \oplus y) = V(x) + V(y)$. (ii) For money amounts x and y, one anticipates $x \oplus y \sim x + y$. These two statements readily imply: (iii) The additive representation V of (i) is proportional to money, $V(x) = cx$. But since much evidence suggests that the marginal utility of money gains diminishes with amount (e.g., utility is concave over gains), something about this argument must be wrong. Quite so. The only question is exactly what. Many have questioned (i), although to my knowledge not with data, and so dismissed the use in this context of extensive measurement assumptions (e.g., see Krantz et al., 1971, Ch. 3). A few have questioned (ii) (e.g., Thaler, 1985), but direct experimental evidence strongly supports it (Cho, Luce, 1995). Fishburn and I have argued the inconsistency lies elsewhere (Luce and Fishburn, 1991, 1995). Specifically, we claim that an unjustified implicit assumption is involved, namely, that the additive representation V of (i) is identical to a concave utility measure U arising in another context such as for preferences among gambles. Indeed, we have shown that if gambles and $\oplus$ satisfy a certain behavioral, although highly rational, property called segregation [see Eq. (13) below], which has been confirmed empirically (Cho, Luce, 1995; Cho et al., 1994), then U is non-linearly (indeed, concavely) related to the additive V over $\oplus$ and so U is a non-additive representation of $\oplus$.

A second argument often made against basing utility measurement on joint receipt is that some goods are complementary, bullets and a gun being an example, and so do not satisfy the axioms of extensive measurement. This observation strikes me not so much as an objection to basing utility measurement on joint receipt but as a caution about how elementary entities are to be defined in this domain. I think of it as being somewhat on a par to questioning the possibility of basing mass measurement on observations of a pan balance because some pairs of substances, when placed in close proximity on a pan, will result in an explosion. Caution is simply called for in both domains.

2. The Extensive-Conjoint and associative models

2.1. The Extensive-Conjoint model

The model proposed by Luce and Fishburn, 1995, which can be called the *Extensive-Conjoint* (E-C) model, assumes that for gains the operation $\oplus$ is extensive[1] (monotonic, associative, commutative, Archimedean, solvable, and positive) with additive representations on $\mathbb{R}^+$. This is the same model as for masses. Losses are treated similarly with a representation on $\mathbb{R}^-$. The more delicate problem is how mixed gains and losses behave. In the E-C model, mixed joint receipts were assumed to form a conjoint structure leading to the following representation

$$U(x \oplus y) = \begin{cases} U(x) + U(y) - U(x)U(y)/C, & x \succ e,\ y \succ e \\ U(x) + U(y), & x \succsim e \succsim y \\ U(x) + U(y) + U(x)U(y)/K, & x \prec e,\ y \prec e \end{cases} \tag{1}$$

This is purely additive only in the mixed case. For gains and for losses, it is non-linearly related to the additive measures that also exist. A somewhat complicated condition relating gains to mixtures was needed to establish that the utility function U for gains is the same when it is constructed from the conjoint structure and when it is a particular non-additive representation of the extensive one (see Theorem 8 of Luce, 1996).

One disadvantage of this model is that it is not associative for mixed gains and losses. An example may help. Suppose you have shipped three fragile, uninsured items, x, y, and z, and they are in two packages. When opened, you find z is broken, so it is

[1] For formal definitions of the terms extensive, additive conjoint, and Archimedean ordered group, see Chs. 3, 6 and 2, respectively, of Krantz et al. (1971).

decidedly a loss. Does it matter to you if x and y were in one package and z in a separate one or x alone and y and z together? If the packaging is immaterial to you, then $(x\oplus y)\oplus z \sim x\oplus(y\oplus z)$, which is associativity. The E-C model assumes this only for all losses and all gains, but the mixed case is treated differently, as an additive conjoint structure[2] over pairs.

2.2. The associative model

A.A.J Marley remarked (personal communication) that one really should investigate the fully associative case separately, both empirically and theoretically. So far as I know, no experiments have been performed to decide between the E-C and associative models. However, an examination in Section 2.5 of the representations in terms of Thaler's (1985) data seem to favor the associative one. Of course, it is not difficult to think of direct tests of associativity, and some are underway by a student of mine, G. Fisher.

This paper not only explores the associative (and commutative) model, but develops its possible relations to risky utility.

Assume that the structure $\chi = \langle X, \succsim, \oplus, e\rangle$, where e is interpreted to be the status quo, forms an Archimedean ordered group with identity e. By Hölder's theorem, χ has an additive numerical representation V, i.e.

$$x \succsim y \text{ if and only if } V(x) \geq V(y), \tag{2a}$$

$$V(x \oplus y) = V(x) + V(y). \tag{2b}$$

As is well known, this representation forms a ratio scale in the sense that it is unique up to multiplication by positive constants and these: transformations correspond to automorphisms of the underlying structure.

2.3. A non-additive, ratio-scale representation of the associative model

For positive constants C and K, define

$$U(x) = \begin{cases} C[1 - e^{-v(x)}], & x \succsim e \\ K[e^{v(x)} - 1], & x \prec e \end{cases} \tag{3}$$

Note that $U(e)=0$. Because χ is a group, each element x has an inverse, x^{-1}, with the property $V(x^{-1}) = -V(x)$. So, by Eq. (3), if $x \succsim e$,

$$U(x) = C(1 - \exp[-V(x)]) = C[1 - \exp V(x^{-1})] = -\frac{C}{K} U(x^{-1}).$$

Eq. (3) first arose in the E-C model (see Section 3.1) because the measure U exhibits a fairly simple relation to the usual utility expressions for gambles. Because the gains and losses part of the associative model are the same as in the E-C model, that argument

[2]Axiomatizations of additive conjoint measurement are well known: Krantz et al. (1971), Ch. 6, Michell (1990), and Wakker (1989).

is unchanged. As we shall see, the associativity of $\oplus$, which is certainly simpler than the E-C model axiomatically, results in a slightly more complex representation of both mixed joint receipts and mixed gambles.

We first arrive at the expression for U of mixed gains and losses when $\chi = \langle X, \succsim \oplus, e\rangle$ is an Archimedean ordered group and U is defined by Eq. (3). Including the gains and losses as well so as to parallel Eq. (1) fully, it is

$$U(x \oplus y) = \begin{cases} U(x) + U(y) - U(x)U(y)/C, & x \succ e, y \succ e \\ C([U(x)/C] + [U(y)/K])/[1 + U(y)/K], & x \succsim e \succsim y, x \oplus y \succsim e \\ K([U(x)/C] + [U(y)/K])/[1 - U(x)/C], & x \succsim e \succsim y, x \oplus y \prec e \\ U(x) + U(y) + U(x)U(y)/K, & x \prec e, y \prec e \end{cases} \quad (4)$$

The simple calculation is as follows. There are four distinct cases. Suppose $x \succ e$, $y \succ e$, then using Eq. (2b),

$$U(x \oplus y) = C[1 - e^{-V(x \oplus y)}] = C[1 - e^{-V(x)}e^{-V(y)}]$$
$$= C(1 - [1 - U(x)/C][1 - U(y)/C]) = U(x) + U(y) - \frac{U(x)U(y)}{C}.$$

If $x \succsim e$, $y \succsim e$, then

$$U(x \oplus y) = K[e^{v(x \oplus y)} - 1] = K[e^{v(x)}e^{v(y)} - 1]$$
$$= K\left(\left[1 + \frac{U(x)}{K}\right]\left[1 + \frac{U(y)}{K}\right] - 1\right) = U(x) + U(y) + \frac{U(x)U(y)}{K}.$$

If $x \succsim e \succsim y$ and $x \oplus y \succsim e$, then

$$U(x \oplus y) = C[1 - e^{-V(x \oplus y)}] = C\left[1 - \frac{e^{-V(x)}}{e^{V(y)}}\right] = C\left[1 - \frac{1 - U(x)/C}{1 + U(y)/K}\right]$$
$$= C\frac{\frac{U(x)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}}.$$

And if $x \succsim e \succsim y$ and $x \oplus y \prec e$, then

$$U(x \oplus y) = K[e^{V(x \oplus y)} - 1] = K\left[\frac{e^{V(y)}}{e^{-V(x)}} - 1\right] = K\left[\frac{1 + U(y)/K}{1 - U(x)/C} - \right]$$
$$= K\frac{\frac{U(x)}{C} + \frac{U(y)}{K}}{1 - \frac{U(x)}{C}}.$$

A curious fact about both Eqs. (1) and (4) is that in a sense U, as well as V, forms a ratio scale. Observe that if, for $r > 0$, $U \to rU$ and the constants $C \to rC$ and $K \to rK$, which is reasonable because C and K are dimensional constants with the same unit as U, then Eqs. (1) and (4) are invariant. A major difference between U and V being ratio scales is that the positive multiplicative transformations of V, but not of U, correspond to automorphisms of the underlying structure. This is an important distinction which also occurs in some physical measurement. For example, the usual relativistic measure of velocity is a non-linear mapping of an additive representation, called rapidity, of velocity concatenation. Both measures, velocity and rapidity, form ratio scales in the sense of being invariant under multiplicative transformations (provided the velocity of light is also transformed), but only the multiplicative transformations of rapidity are automorphisms of the velocity concatenation. Those of velocity, however, do correspond to automorphisms of the velocity–distance–time conjoint structure for which $v=d/t$.

2.4. Additive joint receipt of money

It might seem plausible to introduce a third constant k into Eq. (3) so that for $x<e$ one has $U(x)=K[e^{kV(x)}-1]$. Doing so, however, makes the form of $U(x\oplus y)$ and all of the subsequent results a good deal more complex in the mixed cases than is true for Eq. (4). Moreover, this amount of freedom may not really be necessary. Suppose, for example, that one is dealing with money consequences and that for positive constants c and k,

$$V(x)=\begin{cases} cx, & \text{if } x\geq 0 \\ kx, & \text{if } x<0, \end{cases} \tag{5a}$$

then by Eq. (3)

$$U(x)=\begin{cases} C[1-e^{-cx}], & \text{if } x\geq 0 \\ K[e^{kx}-1], & \text{if } x<0. \end{cases} \tag{5b}$$

With three constants (because we may choose $C=1$ with no loss of generality) we have a good deal of freedom in handling the differential growth of utility for gains and losses.

The above expression (5a) for V clearly arises if $x\oplus y=x+y$ holds for gains and losses separately. What does that restriction imply about the mixed case? Suppose, first, that $x\oplus y\succsim 0$, then letting $CE(x\oplus y)$ denote the certainty equivalent of $x\oplus y$, i.e., $CE(x\oplus y)$ is the sum of money for which $CE(x\oplus y)\sim x\oplus y$, we see that

$$cCE(x\oplus y)=V[CE(x\oplus y)]=V(x\oplus y)=V(x)+V(y)=cx+ky.$$

For $x\oplus y\precsim 0$ by a similar argument $kCE(x\oplus y)=cx+ky$. So, $x\oplus y$ is additive for gains and losses separately, and in the mixed case of $x\geq 0\geq y$,

$$x\oplus y\sim CE(x\oplus y)=\begin{cases} x+(k/c)y, & \text{if } x\oplus y\succsim 0 \\ (c/k)x+y, & \text{if } x\oplus y\precsim 0 \end{cases}$$

Thus, simple additivity of money is not expected in the mixed case, even if it is true for gains and losses separately, unless $c=k$. Unfortunately, the data of Cho, Luce (1995) bear only on the gains and losses cases.

2.5. A distinction between the associative and E-C models

Although Eqs. (1) and (4) agree for gains and for losses separately, they differ considerably in the mixed case. We explore this difference. For gains U is concave because, for $x \succ y \succsim e, z \succ e$,

$$U(x \oplus z) - U(y \oplus z) = [U(x) - U(y)][1 - U(z)/C] < U(x) - U(y).$$

Setting $y = e$ and using $U(e) = 0$, we see immediately that U is subadditive in the sense that for $x \succ e$, $z \succ e$,

$$U(x \oplus z) < U(x) + U(z). \tag{6a}$$

For losses a similar argument shows that U is convex and so superadditive, i.e., for $x \succ e$, $z \succ e$,

$$U(x \oplus z) > U(x) + U(z). \tag{6b}$$

How do $U(x \oplus y)$ and $U(x) + U(y)$ relate in the mixed case? For the E-C model they are simply equal, see Eq. (1). For the associative model of Eq. (4) the relationship is more complex. Specifically, if $x \succsim e \succsim y$, then

$$U(x \oplus y) \gtreqless U(x) + U(y) \text{ if and only if } U(x) \gtreqless C - K - U(y). \tag{7a}$$

and

$$x \oplus y \lesseqgtr e \text{ if and only if } U(x) \gtreqless -\frac{C}{K} U(y) = U(y^{-1}). \tag{7b}$$

Before proving these assertions, let us be clear what the mixed case says. The key dividing lines in that case are

$$U(x) = -\frac{C}{K} U(y) = U(y^{-1}), \tag{8a}$$

which has slope $-C/K$ and

$$U(x) = C - K - U(y). \tag{8b}$$

which has slope -1. Thus in a plot of y versus x, when $C < K$ the line corresponding to Eq. (8a) lies entirely above that corresponding to Eq. (8b); when $C = K$, they coincide; and when $C > K$, the line of Eq. (8a) lies entirely below that of Eq. (8b). The several regions for these three cases along with the sign of $U(x \oplus y) - U(x) - U(y)$ are shown in Fig. 1.

We establish Eqs. (7a) and (7b). Suppose $x \succsim e \succsim y$ and that $x \oplus y \succsim e$. Then using the fact that $1 + U(y)/K > 0$,

$$U(x \oplus y) = C \frac{\dfrac{U(x)}{C} + \dfrac{U(y)}{K}}{1 + \dfrac{U(y)}{K}} \gtreqless U(x) + U(y)$$

is equivalent to

$$C\left(\frac{U(x)}{C}+\frac{U(c)}{K}\right) \gtreqless U(x)+U(y)+\frac{U(x)U(y)}{K}+\frac{U(y)^2}{K},$$

which because $U(y)<0$, is in turn equivalent to

$$U(x) \gtreqless C-K-U(y),$$

which is Eq. (7a).

Next, suppose $x \oplus y<e$. Because $1-U(x)/C>0$,

$$U(x \oplus y)=K\frac{\frac{U(x)}{C}+U(y)}{1-\frac{U(x)}{C}} \gtreqless U(x)+U(y)$$

is equivalent to

$$K\left(\frac{U(x)}{C}+\frac{U(y)}{K}\right) \gtreqless U(x)+U(y)-\frac{U(x)^2}{C}-\frac{U(x)U(y)}{C},$$

which, because $U(x)>0$, is also equivalent to Eq. (7a).

The proof of Eq. (7b) follows immediately from Eq. (4).

Thaler (1985) and Thaler, Johnson (1990) conducted experiments based on scenarios involving money consequences in which either $x+y$ was presented or x and y were treated very separately. They spoke of the former case as *integrated*, which is evaluated as $U(x+y)$, and the latter as *segregated*, which is evaluated as $U(x)+U(y)$. The conclusion from Thaler, as summarized by Thaler, Johnson, 1990 (p. 647), was that their subjects prefer to:

1. Segregate gains.
2. Integrate losses.
3. Segregate small gains from larger losses (The 'silver lining' principle).
4. Integrate (cancel) smaller losses with larger gains.

The data from Thaler, Johnson (1990) are more complex than those of Thaler (1985), but they also involve a temporal aspect that is not modeled in the present theory.

If we assume that for money $x \oplus y \sim x+y$, then we see that Thaler's (1985) data are consistent with the pattern derived above and summarized in Fig. 1. In particular, subadditivity for gains, Eq. (6a), and superadditivity for losses, Eq. (6b), correspond to (1) and (2), respectively. And the pattern in the mixed case shown in Fig. 1 is consistent with their statements. These data probably are not sufficiently refined to choose between $C<K$ or $C>K$. However, data from Kahneman and Tversky, 1979 and Tversky, Kahneman (1991), which yielded estimates of the utility functions, strongly suggest $C<K$.

Because the E-C model predicts only additivity, $U(x \oplus y)=U(x)+U(y)$, for the mixed case, Thaler's data appear to recommend the associative model as the better of the two.

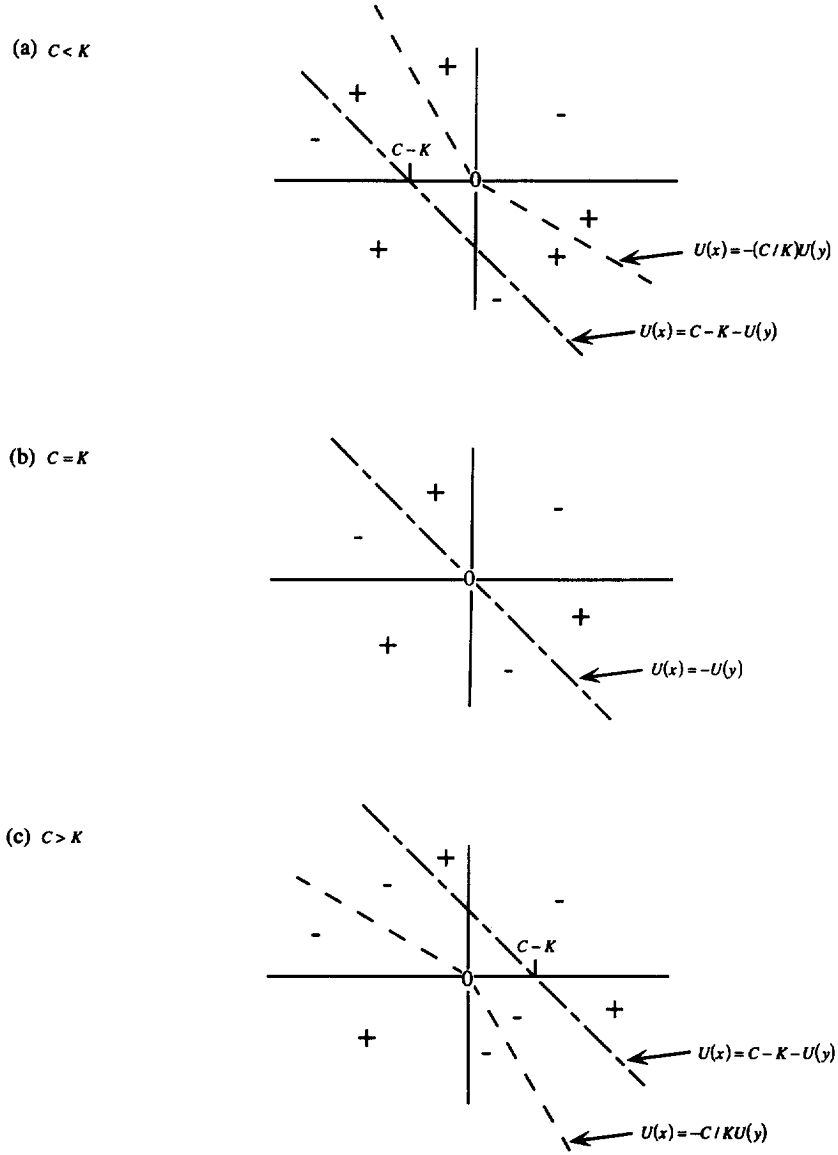

Fig. 1. For the associative model, the sign of $U(x \oplus y) - U(x) - U(y)$ for the several regions depending on the sign of $C-K$.

Of course, the basic distinction between the associative model and the E-C one lies in whether or not associativity holds for the mixed cases, i. e., for all x, y, and z,

$$x \oplus (y \oplus z) \sim (x \oplus y) \oplus z. \tag{9}$$

More precisely, in the E-C model, with the representation of Eq. (1), instead of associativity x, y, and z are of different signs we have the inequalities

$$x \oplus (y \oplus z) > (x \oplus y) \oplus z \text{ if } x > e, \tag{10a}$$

$$x \oplus (y \oplus z) < (x \oplus y) \oplus z \text{ if } x < e. \tag{10b}$$

To show Eq. (10a), suppose $x > e$. If $y > e$ and $z < e$, then, according to Eq. (1),

$$U[(x \oplus y) \oplus z] = U(x \oplus y) + U(z) = U(x) + U(y) + U(z) - U(x)U(y)/C.$$

In calculating $x \oplus (y \oplus z)$ we must distinguish two cases. If $y \oplus z \succsim e$, then

$$\begin{aligned} U[x \oplus (y \oplus z)] &= U(x) + U(y \oplus z) - \frac{U(x)U(y \oplus z)}{C} \\ &= U(x) + U(y) + U(z) - \frac{U(x)U(y)}{C} - \frac{U(x)U(z)}{C} \end{aligned}$$

and if $y \oplus z < e$

$$U[x \oplus (y \oplus z)] = U(x) + U(y \oplus z) = U(x) + U(y) + U(z).$$

Because $U(x)U(y) > 0$ and $U(x)U(z) < 0$, in both cases we see that the inequality (10a) holds.

If $x > e$ and y, $z < e$, then

$$U(x \oplus (y \oplus z)) = U(x) + U(y \oplus z) = U(x) + U(y) + U(z) + \frac{U(y)U(z)}{K}.$$

If $x \oplus y \succsim e$, then

$$U[(x \oplus y) \oplus z] = U(x \oplus y) + U(z) = U(x) + U(y) + U(z).$$

Because $U(y)U(z) > 0$, the inequality (10a) follows. And if $x \oplus y < e$, then

$$\begin{aligned} U[(x \oplus y) \oplus z] &= U(x \oplus y) + U(z) + \frac{U(x \oplus y)U(z)}{K} \\ &= U(x) + U(y) + U(z) + \frac{U(x)U(z)}{K} + \frac{U(y)U(z)}{K}. \end{aligned}$$

Because $U(x)U(z) < 0$, the inequality (10a) again follows.

Similar calculations for $x < e$ lead to the inequality (10b).

3. The relation to rank- and sign-dependent utility of risky alternatives

3.1. Segregation and rank-dependent utility

Let $(x, E; y)$ denote the uncertain alternative in which the consequence received is determined by running a chance experiment; if the event E occurs the consequence is x

and if not, it is y. The set of consequences is denoted $\mathscr{C}$ and of events, $\mathscr{E}$. We invoke the assumption A3 of Luce (1996) which is repeated here:

A3. $\forall x, y \in \mathscr{C}$, $\forall D, E \in \mathscr{E}$,

1. *Idempotence of gambles*: $x \sim (x, E; x)$.
2. *Null event in gambles*: $y \sim (x, \varnothing; y)$.
3. *Complementarity of gambles*: $(x, E; y) \sim (y, \neg E; x)$.
4. *Monotonicity of consequences*: $(x, E; e) \succsim (y, E; e)$ iff $x \succsim y$.
5. *Monotonicity of events*: $\forall x, y \succsim e$ or $\forall x, y \precsim e$, $(x, D; e) \succ (x, E; e)$ iff $(y, D; e) \succsim (y, E; e)$.
6. *Restricted solvability*:
   If $\bar{x}, \underline{x} \in \mathscr{C}$ are such that $(\bar{x}, E; e) \succsim g \succsim (x, E; e)$, then $\exists x \in \mathscr{C}$ such that $(x, E; e) \sim g$.
   If $\bar{E}, \underline{E} \in \mathscr{E}$ are such that $(x, \bar{E}; e) \succsim g \succsim (x, E; e)$, then $\exists E \in \mathscr{E}$ such that $(x, E; e) \sim g$.

Because the expressions for gains and losses are the same in the associative model as in the E-C model, the relation between utility of joint receipts and utility of gambles is also the same. To state it, even informally, one needs to recall three concepts. U is called *separable* if there exist functions W^+ and W^- on events such that

$$U(x, E; e) = \begin{cases} U(x)W^+(E), & \text{if } x \succsim e \\ U(x)W^-(E), & \text{if } x \prec e \end{cases} \tag{11}$$

The axiomatization of this is, of course, the well known additive conjoint one. U is said to be *rank dependent* (*for gains*) if for $x \succsim e$, $y \succsim e$,

$$U(x, E, y) = \begin{cases} U(x)W^+(E) + U(y)[1 - W^+(E)], & x \succsim y \\ U(x)[1 - W^+(\neg E)] + U(y)W^+(\neg E), & x \prec y \end{cases} \tag{12}$$

Axiomatizations of this form can be found in Luce, Narens (1985) and in numerous other papers summarized in Quiggin (1993). Clearly, if $U(e)=0$, then rank dependence implies separability. Of course, the converse is not true except under additional assumptions. A similar rank-dependent form holds for losses with W^+ replaced by W^-.

The structure χ is said to satisfy *segregation*[3] if for either $x \succsim e$, $y \succsim e$ or $x \precsim e$, $y \precsim e$,

$$(x, E; e) \oplus y \sim (x \oplus y, E; y). \tag{13}$$

This is a highly rational condition in the sense that both sides really mean the same thing: if E occurs, one receives $x \oplus y$ and if not, $y \sim e \oplus y$. The major conclusion of Luce and Fishburn, 1995 was that for gains within a sufficiently rich structure any two of the following properties implies the third:

1. Segregation, Eq. (13), holds.

[3]This use of the term "segregation" has nothing whatsoever to do with Thaler's use of the term. One simply must distinguish them by context.

2. U of a gamble of gains is rank dependent, Eq. (12), and U is weakly subadditive in the sense that $U(x \oplus x) < 2U(x)$.[4]
3. U satisfies separability, Eq. (11), and $U(x \oplus y) = U(x) + U(y) - (U(x)U(y)/C)$.[5]

A similar result holds for losses. One significance of this result is that the rank-dependent model can be axiomatized in terms of $\oplus$ on gains as both satisfying segregation and forming an extensive structure with a representation, as in Eq. (1) or Eq. (4) that is also separable (see the next subsection).

It is obvious how to generalize segregation to non-binary gambles of all gains or of all losses, and by invoking that recursively Luce and Fishburn, 1995 showed how this leads to a general rank-dependent representation. So the theory here, which is only stated for binary gambles, generalizes in exactly the same way.

3.2. Separable utility for gains

The joint imposition in part (3) above of the conditions that U is separable and satisfies Eq. (4) is not a trivial constraint. Aczél, Luce, Maksa (1997) and Luce (1996) established distinct conditions for the separability of U for gains when it arose from the joint receipt of just gains and when it arose from conjoint trade-offs in the mixed case because these two cases were axiomatized separately. Here, because the same U is used for gains in both the gains and the mixed cases, the proof of separability for gains (and for losses) is all that is needed—see Theorems 3 and 4 of Luce (1996) and the solution of the relevant functional equation, Eq. (5), in Aczél et al. (1997). So there is no new mathematical problem here.

For completeness, the relevant behavioral condition, called *joint receipt decomposability (for gains)*, is: For all $x \succ e$ and events E, there exists an event $D = D(x, E)$ such that for all $y \succ e$,

$$(x \oplus y, E; e) \sim (x, E; e) \oplus (y, D; e). \tag{14}$$

I know of no experimental evidence on this property.

3.3. Duplex decomposition

The only interesting difference between the E-C and the associative models concerns mixed consequences. For the case of the E-C model, the relation to risky utility is dealt with in Theorem 6 of Luce (1996). For the associative representation of Eq. (4) we replace that theorem with the following closely related result.

[4]As noted above, separability is easily axiomatized. I do not know of any axiomatization of weak subadditivity in this context. In Luce (1996) I spoke of the latter property as weak concavity, but the present term is really better.

[5]The property of separability is behavioral, and the form postulated for U arises if $\oplus$ and $\succsim$ satisfy the axioms of extensive measurement.

Theorem 1. *Suppose a structure of binary gambles satisfies assumption A3 and there exist mappings W^+ and W^- from the set of events onto $[0, 1]$ and an order preserving U, from binary gambles onto the open real interval $]-K, C[$, $C>0$, $K>0$, for which $U(e)=0$. Then any two of the following properties imply the third where $x \succsim e \succsim y$ and E is an event*:

(*i*) *Duplex decomposition*:

$$(x, E; y) \sim (x, E'; e) \oplus (e, E''; y), \tag{15}$$

where E' and E'' denote the event E occurring in two independent realizations of the underlying experiment.

(*ii*) *U on mixed binary gambles has the form*

$$U(x, E; y) = C \frac{\frac{U(x)}{C} W^+(E) + \frac{U(y)}{K} W^-(\neg E)}{1 + \frac{U(y)}{K} W^-(\neg E)}, \text{ if } (x, E; y) \succsim e. \tag{16a}$$

$$U(x, E; y) = K \frac{\frac{U(x)}{C} W^+(E) + \frac{U(y)}{K} W^-(\neg E)}{1 + \frac{U(x)}{C} W^-(\neg E)}, \text{ if } (x, E; y) \prec e. \tag{16b}$$

(*iii*) *U on binary gambles is separable, Eq. (11), and over joint receipt U satisfies Eq. (4), for all x, y, $x \succsim e \succsim y$, for which there exist an event E and consequences $u \succsim e$ and $v \succsim e$ such that on independent realizations of the underlying experiment, denoted ′ and ″, $x \sim (u, E'; e)$ and $y \sim (e, E''; v)$.*

Proof. (i) and (ii)→(iii). Suppose x, y, u, v, and E satisfy the conditions stated in property (iii). Then, by (i) and the monotonicity of $\oplus$,

$$x \oplus y \sim (u, E'; e) \oplus (e, E''; v) \sim (u, E; v).$$

So, using Eq. (16a) and separability

$$U(x \oplus y) = U(u, E; v) = C \frac{\frac{U(u)}{C} W^+(E) + \frac{U(v)}{K} W^-(\neg E)}{1 + \frac{U(v)}{K} W^-(\neg E)}$$

$$= C \frac{\frac{U(u, E; e)}{C} + \frac{U(e, E; v)}{K}}{1 + \frac{U(e, E; v)}{K}} = C \frac{\frac{U(x)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}},$$

which is the relevant part of Eq. (4).

For $x \oplus y \succsim e$, the argument is similar.

(i) and (iii)→(ii). If $x \succ e \succ y$ and $(x, E; y) \succsim e$,

$$U(x, E; y) = U[(x, E'; e) \oplus (e, E''; y)] = C \frac{\dfrac{U(x, E'; e)}{C} + \dfrac{U(e, E''; y)}{K}}{1 + \dfrac{U(e, E''; y)}{K}}$$

$$= C \frac{\dfrac{U(x)}{C} W^+(E) + \dfrac{U(y)}{K} W^-(\neg E)}{1 + \dfrac{U(y)}{K} W^-(\neg E)}.$$

And if $(x, E; y) \prec e$,

$$U(x, E; y) = U[(x, E'; e) \oplus (e, E''; y)] = K \frac{\dfrac{U(x, E'; e)}{C} + \dfrac{U(e, E''; y)}{K}}{1 - \dfrac{U(x, E'; e)}{C}}$$

$$= K \frac{\dfrac{U(x)}{C} W^+(E) + \dfrac{U(y)}{K} W^-(\neg E)}{1 - \dfrac{U(x)}{C} W^+(E)}.$$

(ii) and (iii)→(i). Simply reverse the previous argument. ■

Note that these expressions for the utility of mixed gambles are somewhat more complex than those of the Luce, Fishburn (1991, 1995) version of prospect theory[6] (see Eq. (18) below). In particular, the denominators in Eq. (16) are unusual.

As with Theorem 6 of Luce (1996), we have a proof of Eq. (4) in the mixed case only when x and y are such that $u \succ e \succ v$ and E can be found such that $x \sim (u, E'; e)$ and $y \sim (e, E''; v)$. Because of the boundedness of the representation, this construction need not cover all (x, y) pairs. I still do not know a way around this limitation.

So, in summary, if the simpler axiomatization for $\oplus$ is correct, the representation for mixed gambles is somewhat more complex than in prospect theory. The prospect theory version plus duplex decomposition forces the more complex $\oplus$ of the E-C model that is non-associative for the mixed case. A third alternative is to replace duplex decomposition by a different, but similar condition, so that both $\oplus$ is associative and the prospect theory representation holds for the mixed case. We turn to that in Section 3.5, but first we take up an issue related to theorem 1.

3.4. Testing the gambling representation of Eq. (16)

A possible way to test the representation of Eq. (16) versus the weighted average representation of E-C model (Luce and Fishburn, 1991, 1995) involves three steps

1) For monetary gambles in which 0 exchange is taken to be the status quo, use riskless data on gains and losses, separately, to estimate the utility function. One way to

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[6]This representation was also described by Tversky, Kahneman (1992) and axiomatized in a different way from our use of joint receipt by Wakker, Tversky (1993).

do this is to estimate the three parameters of the special case mentioned earlier where V is linear with money and so

$$U(x) = \begin{cases} C[1 - e^{cx}], & \text{if } x \geq 0 \\ K[e^{kx} - 1], & \text{if } x < 0. \end{cases}$$

2) For $x>0>y$, determine experimentally the following certainty equivalents $CE(x, E; y)$, $CE(x, E; 0)$, and $CE(0, E; y)$.

3) According to Eq. (16) and noting that the denominators are <1, we see that for $C>K$ and $CE(x, E; y)>0$ and for $C<K$ and $CE(x, E; y)<0$,

$$\begin{aligned} U[CE(x, E; y)] &= U(x, E; y) > U(x, E; 0) + U(0, E; y) \\ &= U[CE(x, E; 0)] + U[CE(0, E; y)]. \end{aligned}$$

According to the C-E representation, this should be an equality. So these data provide a possible test once U has been estimated as in step (1) above.

3.5. Modified duplex decomposition

An alternative approach to the mixed case is to retain the simple expression found in prospect theory and to hold to the representation of joint receipts given in Eq. (4), and then ask what link between the risky and riskless cases must replace duplex decomposition. This can be formulated as follows:

Theorem 2. *Suppose a structure of binary gambles satisfies assumption A3 and there exist mappings W^+ and W^- from the set of events onto* [0, 1] *and an order preserving utility function U over the binary gambles onto the open real interval* $]-K,C[$, $C>0$, $K>0$, *for which* $U(e)=0$. *For* $x \succsim e \succsim y$ *and E an event, then any two of the following imply the third.*

(*i*) *Modified duplex decomposition. There exist events D and D* such that*

$$(x, E; y) \sim (x, D; e) \oplus (e, D^*; y), \tag{17a}$$

where D and D are realized in independent runs of the underlying experiment and are related to E as follows: for* $x \oplus y \succsim e$

$$W^+(D) = \frac{W^+(E)}{1 - \frac{U(y)}{C} W^-(\neg E)} \text{ and } W^-(\neg D^*) = \frac{W^-(\neg E)}{\frac{C}{K} - \frac{U(y)}{K} W^-(\neg E)} \tag{17b}$$

and for $x \oplus y < e$

$$W^+(D) = \frac{W^+(E)}{\frac{K}{C} + \frac{U(x)}{C} W^+(E)} \text{ and } W^-(\neg D^*) = \frac{W^-(\neg E)}{1 + \frac{U(x)}{K} W^+(E)}. \tag{17c}$$

(*ii*) *For mixed gambles, the prospect theory representation holds, i.e.,*

$$U(x, E; y) = U(x)W^+(E) + U(y)W^-(\neg E). \tag{18}$$

(iii) U is separable and satisfies Eq. (4) for all x, y, $x \succsim e \succsim y$, for which there exist an event E and consequences $u \succsim e$ and $v \precsim e$ such that on independent realizations of the underlying experiment $x \sim (u, D; e)$ and $y \sim (e, D^*; v)$, where D and D^* are related to u, v, and E as in Eqs. (17b) and (17c).

Proof. Assume $x \oplus y \succsim e$. First, (i) and (iii) imply (ii). Using Eqs. (17a) and (4), separability, and Eq. (17b) in that order,

$$U(x, E; y) = U[(x, D; e) \oplus (e, D^*; y)] = C \frac{\frac{U(x)}{C} W^+(D) + \frac{U(y)}{K} W^-(\neg D^*)}{1 + \frac{U(y)}{K} W^-(\neg D^*)}$$

$$= U(x)W^+(E) + U(y)W^-(\neg E),$$

which proves Eq. (18).

Second, (ii) and (iii) imply (i). Given x, y, and E, define D and D^* by Eq. (17b). Solving for the E terms,

$$W^+(E) = \frac{W^+(D)}{1 + \frac{U(y)}{K} W^-(\neg D^*)} \text{ and } W^-(\neg E) = \frac{C}{K} \frac{W^-(\neg D^*)}{1 + \frac{U(y)}{K} W^-(\neg D^*)}. \qquad (17b')$$

So, substituting these into Eq. (18) and using separability yields

$$U(x, E; y) = U(x)W^+(E) + U(y)W^-(\neg E)$$

$$= C \frac{\frac{U(x)}{C} W^+(D)}{1 + \frac{U(y)}{K} W^-(\neg D^*)} + C \frac{\frac{U(y)}{K} W^-(\neg D^*)}{1 + \frac{U(y)}{K} W^-(\neg D^*)}$$

$$= C \frac{\frac{U(x, D; e)}{C} + \frac{U(e, D^*; y)}{K}}{1 + \frac{U(e, D^*; y)}{K}} = U[(x, D; e) \oplus (e, D^*; y)],$$

and (i) follows from the fact U is order preserving.

Third, (i) and (ii) imply (iii). Let x and y be of the form stated in (iii). Then, using (i), (ii), and Eq. (17b′) in that order

$$U(x \oplus y) = U[(u, D; e) \oplus (e, D^*; y)] = U(u, E; v)$$

$$= U(u)W^+(E) + U(v)W^-(\neg E) = \frac{U(u)W^+(D) + \frac{C}{K} U(v)W^-(\neg D^*)}{1 + \frac{U(v)}{K} W^-(\neg D^*)}$$

$$= C \frac{\frac{U(u, D; e)}{C} + \frac{U(e, D^*; v)}{K}}{1 + \frac{U(e, D^*; v)}{K}} = C \frac{\frac{U(x)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}},$$

which establishes (iii).

The case $(x, E; y) \prec e$ is similar. ■

Note that if $K \geq C$ and $x \oplus y \succsim e$, then because $1+(U(y)/KW^{-}(\neg D^*)<1$, Eq. (17b′) implies $W^{+}(E) \geq W^{+}(D)$ and $W^{-}(\neg E) \geq W^{-}(\neg D^*)$, i.e., $D \precsim{}' E \precsim{}' D^*$, where $\precsim{}'$ is the inferred likelihood ordering of events. For $K \geq C$ and $x \oplus y \prec e$, solving Eq. (17c) for the E-terms yields

$$W^{+}(E) = \frac{K}{C} \frac{W^{+}(D)}{1 - \frac{U(x)}{C} W^{-}(\neg D^*)} \text{ and } W^{-}(\neg E) = \frac{W^{-}(\neg D^*)}{1 - \frac{U(x)}{C} W^{-}(\neg D^*)}. \tag{17c′}$$

Again the denominator is <1 and so the same inequalities hold. For $K<C$, no inequalities seem to follow; however, empirically $K \geq C$ appears to be what happens.

It may well be difficult to devise a sufficiently sensitive experiment to choose between duplex decomposition, which has received some empirical support (Cho et al., 1994; Payne, Braunstein, 1971; Slovic, 1967; Slovic, Lichtenstein, 1968), and the modified version, Eq. (17).

3.6. General segregation

Still another way to proceed in the mixed case is to generalize the concept of segregation, Eq. (13), to cover these cases. Define $x \ominus y \sim u$ if and only if $x \sim u \oplus y$. We say that *general segregation* holds if for all $x \succsim y$,

$$(x, E; y) \sim \begin{cases} (x \ominus y, E; e) \oplus y, & \text{if } (x, E; y) \succsim e \\ x \oplus (e, E; y \ominus x), & \text{if } (x, E; y) \precsim e. \end{cases} \tag{19}$$

Note that Eq. (13) is a special case of Eq. (19) where $x \succsim e$ and $y \succsim e$, then with $u = x \ominus y \succsim e$, the top limb reads $(u, E; e) \ominus y \sim (x, E; y) \sim (u \ominus y, E; y)$, which except for a change in notation is Eq. (13). The case $x \precsim e$ and $y \precsim e$ is similar, using the bottom limb.

The first case that we examine is what happens to Theorem 1 when we replace duplex decomposition by general segregation.

Theorem 3. *Suppose a structure of binary gambles satisfies assumption A3 and there exist mappings W^{+} and W^{-} from the set of events onto* [0, 1] *and an order preserving utility function U over the binary gambles onto the open real interval* $]-K, C[$, $C>0$, $K>0$, *for which* $U(e)=0$. *Then any two of the following properties imply the third where $x \succsim e \succsim y$ and E is an event*:

(*i*) *General segregation*, Eq. (19).

(*ii*) *U on mixed binary gambles has the form, for* $(x, E; y) \succsim e$,

$$U(x, E; y) = U(x)W^{+}(E) + C \frac{U(y)/K}{1 + U(y)/K} [1 - W^{+}(E)], \tag{20a}$$

and for fixed y,

$$\text{as } U(x) \to C, \text{ then } U(x \oplus y) \to C. \tag{20b}$$

And for $(x, E; y) \prec e$,

$$U(x, E; y) = K\frac{U(x)/C}{1 - U(x)/C}[1 - W^-(\neg E)] + U(y)W^-(\neg E), \tag{20c}$$

and for fixed x,

$$\text{as } U(y) \to -K, \text{ then } U(x \oplus y) \to -K. \tag{20d}$$

(*iii*) *U is separable, Eq. (11), and the representation of Eq. (4) holds.*

Proof. Because $x \gtrsim y$, $u \sim x\ominus y \gtrsim e$ and $v \sim y\ominus x \lesssim e$. So, if Eq. (4) holds

$$U(x) = U(u \oplus y) = C\frac{\frac{U(u)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}},$$

and solving for $u = x\ominus y$,

$$U(x\ominus y) = U(u) = U(x)\left[1 + \frac{U(y)}{K}\right] - C\frac{U(y)}{K}. \tag{21a}$$

Similarly,

$$U(y\ominus x) = U(v) = U(y)\left[1 - \frac{Ulx)}{C}\right] - K\frac{U(x)}{C}. \tag{21b}$$

Assuming (iii), we show (i) is equivalent to (ii). First, suppose $(x, E\colon y) \gtrsim e$, then by Eq. (4), separability, and Eq. (21a),

$$u[(x\ominus y, e; E) \oplus y] = C\frac{\frac{U(x\ominus y, E; e)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}} = C\frac{\frac{U(x\ominus y)W^+(E)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}}$$

$$= U(x)W^+(E) + C\frac{U(y)/K}{1 + U(y)/K}[1 - W^+(E)].$$

Thus, (i) is equivalent to Eq. (20a). Note that by taking limits in Eq. (4), Eq. (20b) follows. The case where $(x, E; y) \prec e$ is similar.

Next, we show that (i) and (ii) imply (iii). The proof is similar to that of Theorem 1 of Luce and Fishburn, 1991. The separability is obvious from (ii) and $U(e) = 0$. To show Eq. (4) for the case $x \oplus y \gtrsim e$, consider Eq. (19) in the form

$$(x \oplus y, E; y) \sim (x, E; e) \oplus y \gtrsim e,$$

Applying U to this, writing $F(x,y) = x\oplus y$, and using Eq. (20a)

$$UF(U^{-1}[U(x)W^+(E)], y] = UF[(x, E; e), y] = U[F(x, y), E; y]$$

$$= U[F(x, y)]W^+(E) + C\frac{U(y)/K}{1 + U(y)/K}[1 - W^+(E)].$$

Introduce the notation: $X=U(x)$, $Y=U(y)$, $W=W^{+}(E)$, and $H(X,Y)=UF[U^{-1}(X), U^{-1}(Y)]$, then the functional equation becomes

$$H(XW, Y) - C\frac{Y/K}{1+Y/K} = \left[H(X, Y) - C\frac{Y/K}{1+Y/K} \right] W.$$

Because of the limitation $(x, E; e) \oplus y \succsim e$, we see that this holds only for $H(XW, Y) \geq 0$. To extend it to the full range of X on the interval $[0, C[$, Y on $]-K, 0]$, and W on $[0, 1]$, we need only extend the relevant forms of Eqs. (4), (19), (20a) with no limits on the combined variables (of course the values in the extended regions do not correspond to anything observable). Because H is strictly increasing in each variable, we know from Aczél (1966) that for some function α,

$$H(X, Y) = X\alpha(Y) + C\frac{Y/K}{1+Y/K}.$$

To determine α, we take the limit as $X \to C$ and invoke Eq. (20b), yielding

$$C = C\alpha(Y) + C\frac{Y/K}{1+Y/K'}$$

whence

$$\alpha(Y) = \frac{1}{1+Y/K'}$$

and thus

$$U(x \oplus y) = UF(x, y) = H(X, Y) = C\frac{\frac{X}{C} + \frac{Y}{K}}{1 + \frac{Y}{K}} = C\frac{\frac{U(x)}{C} + \frac{U(y)}{K}}{1 + \frac{U(y)}{K}},$$

which is the appropriate part of Eq. (4).

The case $x \oplus y < e$ is similar. ■

It is perhaps worth noting that general segregation and associativity imply a slightly more general form of segregation, namely, for $x \succsim z \succsim y$,

$$(x, E; y) \sim (x \ominus z, E; y \ominus z) \oplus z.$$

A natural question to raise is: Just how far wrong is duplex decomposition if, in fact, general segregation holds? Consider the case where $(x, E; y) \succsim e$ and $(x, E'; e) \oplus (e, E''; y) \succsim e$. By separability and Eq. (4) we see

$$U[(x, E'; e) \oplus (e, E''; y)] = C\frac{\frac{U(x)}{C}W^{+}(E) + \frac{U(y)}{K}W^{-}(\neg E)}{1 + \frac{U(y)}{K}W^{-}(\neg E)}.$$

To the degree that $U(y)/K$ is small relative to 1 and $W^{-}(\neg E) = 1 - W^{+}(E)$, we see that this agrees with Eq. (20a) for $U(x, E; y)$, in which case duplex decomposition is predicted approximately. Thus, if we expect to find a difference it will have to be when the magnitude of y is relatively large and, to keep the first term from being negligible, $W^+(E)$ is not too far from 1.

3.7. General segregation and prospect theory

Continuing with the assumption of general segregation, an alternative approach is to suppose prospect theory, Eq. (18), holds for mixed gambles and to ask what happens to the expression for $U(x\oplus y)$ when $x \succsim e \succsim y$.

Define $\mathscr{C}^+=\{x|x \succsim e)\}$ and $\mathscr{C}^-=\{x/x \precsim e\}$.

Theorem 4. *Suppose for a structure of binary gambles there exist a strictly increasing mapping W^+ and a strictly decreasing mapping W^- from the set of events onto* [0, 1] *and an order preserving utility function U over gambles onto the open interval* $]-K, C[$, *where $C>0$ and $K>0$, for which prospect theory for mixed gambles, Eq. (18), holds. If, in addition, there is a monotonic operation $\oplus$ over gambles and pure consequences for which general segregation, Eq. (19), holds, then*

$$W^+(E) + W^-(\neg E) = 1. \tag{22}$$

Define ϕ: $\mathscr{C}^+ \to \mathscr{C}^-$ by, for all $x \in \mathscr{C}^+$, $x\oplus\phi(x) \sim e$; and θ: $[0, C[\to]-K, 0]$ by, for all $X \in [0, C[$, $\theta(X) = U\phi U^{-1}(X)$, then for some strictly increasing function h

$$U(x\oplus y) = U(x) + U(y) + U(X)U(y)\begin{cases} h[\theta^{-1}U(y)] - 1, & x\oplus y \succsim e \\ h[U(x)] - 1, & x\oplus y \precsim e. \end{cases} \tag{23}$$

Proof. As in the proof of Theorem 3 we extend the expression in Eq. (19) for $(x, E; y) \succsim e$ to the entire region of mixed gambles. Applying U to the general segregation expression with z playing the role of x,

$$U[(z\ominus y, E; e)\oplus y] = U(z_i\ E; y) = U(z)W^+(E) + U(y)W^-(\neg E). \tag{24}$$

Let $x \sim z\ominus y$, so $z \sim x\oplus y$. Note that since $(z, E; y) \succsim e$, by segregation $(z\ominus y, E; e) \succsim e$, whence by consequence monotonicity of gambles $x \sim z\ominus y \succsim e$. Denote $x\oplus y = G(x, y)$ and for $X\in[0, C[$, $Y\in]-K, 0]$, $H(X, Y) = UG[U^{-1}(X), U^{-1}(Y)]$. So

$$U(x\oplus y) = UG(x, y) = UG[U^{-1}U(x), U^{-1}U(y)] = H[U(x), U(y)]. \tag{25}$$

Using this notation, Eq. (24) becomes

$$\begin{aligned} H[U(x, E; e), U(y)] &= H[U(x)W^+(E), U(y)] \\ &= H[U(x), U(y)]W^+(E) + U(y)W^-(\neg E). \end{aligned} \tag{26}$$

Let $X=U(\mathrm{x})\in[0, C[$, $Y=U(y)\in]-K, 0]$, $W=W^+(E)\in[0, 1]$, and $F(W)=W^-(\neg E)\in[0, 1]$, where F is strictly decreasing, because W^+ is strictly increasing and W^- is strictly decreasing, and $F(1)=0$ and $F(0)=1$. We see that the functional Eq. (26) becomes

$$H(XW, Y) = H(X, Y)W + YF(W). \tag{27}$$

The following lemma and proof are due to J. Aczél (personal communication).

Lemma. (*Aczél*). *The general solution to Eq. (27) with* $F(1)=0$ *and* $F(0)=1$ *is*

$$F(W) = 1 - W, \ (W \in [0, 1]), \tag{28a}$$

and for some arbitrary function g *on* $]-K, 0]$

$$H(Z, Y) = Zg(Y) + (1 - Z)Y, \quad (Z \in [0, C[, Y \in]-K, 0]). \tag{28b}$$

Proof of lemma. Select $X=B \in]0, \min(1, C)[$, write $Z=BW$, and let $h(Y)=H(B, Y)/B$, and $G(Z)=F(Z/B)=F(W)$. Then, Eq. (27) yields

$$H(Z, Y) = Zh(Y) + G(Z)Y, \quad (Z \in [0, B], Y \in]-K, 0]. \tag{29}$$

Insert this back into Eq. (27) to get

$$h(Y)ZW + YG(ZW) = h(Y)ZW + YWG(Z) + YF(W),$$

yielding

$$G(ZW) = G(Z)W + F(W), \ Z \in [0, B], W \in [0, 1].$$

From the fact $G(ZW)=G(WZ)$, we may interchange the roles of W and Z in the last expression to get

$$G(W)Z + F(Z) = G(Z)W + F(W), \ (Z, W \in [0, B]). \tag{30}$$

Choose $W=B$ in Eq. (30) and solve for $F(Z)$:

$$F(Z) = BG(Z) + a + bZ, \ a = F(B), b = G(B).$$

Substitute this back into Eq. (30) to get

$$G(Z)(W - B) + G(W)(B - Z) + b(W - Z) = 0.$$

Choose W to be some constant $W_0 < B$ and solve for $G(Z)$:

$$G(Z) = \gamma - \delta Z, \ \gamma = \frac{G(W_0)B + bW_0}{B - W_0}, \delta = \frac{G(W_0) + b}{B - W_0}. \tag{31}$$

Because by assumption $F(W)=G(BW)$, $F(0)=1$, and $F(1)=0$, we see that $\gamma=1$. $\delta=1/B$, and $F(W)=1-W$, which is Eq. (28a). Note from Eq. (31) that, in this notation, $G(Z)=1-Z/B$, which on substitution into Eq. (29) yields

$$H(Z, Y) = Zh(Y) + \left(1 + \frac{Z}{B}\right)Y,$$

Setting $g(Y)=h(Y)+(1-(1/B))Y$, we see that

$$H(Z, Y) = Zg(Y) + (1 - Z)Y, \tag{32}$$

which is Eq. (28b) but with Z restricted to $[0, B]$. So it must be extended to $[0, C[$. For any $Z \in [0, C[$ there exists a $W \in [0, 1]$ such that $ZW \in [0, B]$. So by Eqs. (27) and (31)

$$H(Z, Y)W + YF(W) = H(ZW, Y) = ZWg(Y) + (1 - ZW)Y.$$

Rearranging and substituting Eq. (28a) yields Eq. (28b), proving the lemma.

Returning to the proof of the Theorem, by a similar argument for $(z, E; y) < e$ and $K > 1$, for some function f on $[0, C[$

$$H(Z, Y) = Z(1 - Y) + f(Z)Y. \tag{33}$$

Now, consider $z \oplus \phi(z) \sim e$, then by the monotonicity of $\oplus$ we see that both Eq. (28) and Eq. (33) hold with $H(Z, Y) = 0 = H[Z, \theta(Z)]$, and so equating them and simplifying

$$\frac{f(Z) - 1}{Z} = \frac{g[\theta(Z)] - 1}{\theta(z)}.$$

Thus, for some function h,

$$f(Z) = Zh(Z) + 1 \text{ and } g(Y) = Yh[\theta^{-1}(Y)] + 1.$$

Substituting into the two expressions for $H(Z, Y)$ and changing back into $U(x)$ and $U(y)$ we have Eqs. (23) and (24). ■

Eq. (28a), of course, defeats the purpose of having different weighting functions for gains and losses, which does not recommend combining prospect theory for mixed gambles with general segregation.

Observe that if, as in Theorem 3, we assume for y fixed and $z \oplus y \gtrsim e$ that $U(z) \to C$ implies $U(z \oplus y) \to C$, and for z fixed and $z \oplus y < e$ that $U(y) \to -K$ implies $U(z \oplus y) \to -K$, then taking limits in Eqs. (28b) and (33) we have

$$f(Z) = \frac{1 + K}{K} Z - 1 \text{ and } g(Y) = 1 - \frac{1 - C}{C} Y,$$

and so

$$U(z \oplus y) = U(z) + U(y) + U(z)U(y) \times \begin{cases} -1/C, & \text{if } z \oplus y \gtrsim e \\ 1/K, & \text{if } z \oplus y \lesssim e. \end{cases}$$

This simply extends to the mixed cases the expressions found for gains and losses separately, stopping along the curve ϕ defined by $z \oplus \phi(z) \sim e$. The difficulty with this is that the two expressions for that curve do not agree except if $-C = K$, which is impossible because C and K are both positive constants. Indeed, since the term $U(z)U(y) < 0$ and C and K are positive, there is a discontinuity in U as y passes through $\phi(z)$. Thus, the assumptions of mixed prospect theory, general segregation, monotonicity of $\oplus$, and the limit expressions as U approaches its maximum and minimum forces U to be discontinuous, which seems fairly implausible.

I do not at present have any other condition to limit the functions f and g or equivalently, if monotonicity of $\oplus$ is assumed, h.

Given the above theorems for the mixed case, an important empirical task is to try to distinguish between duplex decomposition, generalized duplex decomposition, and general segregation.

4. Conclusions

The assumption of associative joint receipts surely seems plausible, although it has not yet been empirically verified, and it certainly leads to a simpler numerical representation of $\oplus$ than did the earlier E-C model. For example, there is no issue of how the utility in the mixed case relates to that of gains and of losses. A nice feature of the theory is its predictions about the sign of $U(x \oplus y) - U(x) - U(y)$ which seem to accord well with the empirical results of Thaler (1985).

One consequence, however, is that either (1) the representation for gambles with mixed consequences is more complex than that postulated by prospect theory, Eq. (18), or (2) the empirical linkage, duplex decomposition, Eq. (15), between joint receipts and mixed gambles has to be modified. Duplex decomposition has some empirical support, but it is not clear at present whether the data are adequate to distinguish between it and the modified version, Eq. (17), that preserves the prospect theory form. Nor is it clear that one can distinguish between duplex decomposition and the general segregation property, Eq. (19), that leads to the representation of Eq. (20) for the utility of mixed binary gambles.

So two empirical problems need to be attacked: For mixed consequences, is joint receipt associative and, if so, can we decide between duplex decomposition, its modification, or general segregation?

It appears, overall, that the simplest theory results if joint receipts are everywhere associative and if general segregation, Eq. (19), holds. In this case, the theory of binary gambles is rank dependent for gains and losses and has the form of Eq. (20) for the mixed case. From the point of view of classical economic theory, this structure has the advantage that all of the axioms have a strongly rational flavor.

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