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Quantification and symmetry: Commentary on Michell, Quantitative science and the definition of measurement in psychology

R. Duncan Luce*
Institute for Mathematical Behavioral Sciences, Social Science Plaza, University of California at Irvine, Irvine, CA 92697-5100, USA

Several of Michell's points are amplified and emphasized and the following additional point is made. Most quantitative attributes can be measured in more than one way, and there are interesting questions about how they relate. Among other things, units of measurement and symmetries of the underlying structure may or may not agree.

Because I agree with almost everything Michell says, my commentary is restricted to some amplification and to an added observation.

1. Quantification and scale type

The British Association for the Advancement of Science Subcommittee on Measurement claimed quantification to be possible only when there is an empirical operation satisfying Hölder's (1901) conditions (see Michell's article). In this they were wrong, although at the time there were no specific examples to prove it. In response S. S. Stevens (1946, 1951, 1975) claimed that what counted was having an interval or ratio scale type. Subsequent research has given meaning to this assertion (see §2), but given his attempts to invoke scale type ideas it is doubtful if he understood it himself. The issue hinges on his meaning of a 'rule' in his famous definition of measurement. Stevens (1975, pp. 46–47) said: 'Measurement is the assignment of numbers to objects or events according to rule (Stevens, 1946). The rule of assignment can be any consistent rule. The only rule not allowed would be random assignment, for randomness amounts in effect to a non-rule'. So, to him, the rule had to do with the person performing a measurement, not with an empirical law involving the attribute being measured. In particular, he must have viewed IQ measurement as forming an interval scale because the scientists involved rescale the counts of questions answered so that IQ is normally distributed. Nothing empirical forces this choice. No measurement theorist I know accepts Stevens' broad definition of measurement. In our view, the British committee got it wrong by being far too narrow and Stevens got it wrong by being far too broad, extending the concept of measurement much beyond the empirical. In our view, the only sensible meaning for 'rule' is empirically testable laws about the attribute.

This aspect of Stevens' views makes it very difficult to understand what is involved in, for example, his method of magnitude estimation. Is it more than ordinal? He confounded two meanings of the word 'ratio' in discussing it. He instructed his participants to preserve subjective ratios and he assumed that the resulting numerals formed a ratio scale, but he did not make at all clear what empirically justified

* Requests for reprints.

that belief. Recently, Narens (1996) has worked out an interpretation of what he might have meant, including a plausible empirical law, which if true indeed justifies believing this is measurement at the level of ratio scales. The subtle argument is, however, very different from any that Stevens gave.

2. Quantification of symmetry

One of the most significant discoveries of contemporary representational measurement theory is that a very broad class of structures can be quantified indirectly (Alper, 1987; Cohen & Narens, 1979; Luce & Narens, 1985; Narens, 1981*a*, *b*). An isomorphism of an ordered empirical structure with itself is called a *symmetry* (or automorphism) of the structure. A symmetry with no fixed point is called a *translation*. All familiar examples of measurement structures are rich in translations—they are *homogeneous* in the sense that any point can be mapped into any other point by a translation. So the elements of the structure are structurally identical. What is remarkable is that if the structure forms a continuum, then the set of translations plus the identity forms a group under function composition, has an order induced from the structural order, and together these form a quantitative system in the classical sense of Hölder. Moreover, the group of translations can be mapped onto the multiplicative positive real numbers. And in the homogeneous case the structure itself can be mapped isomorphically into the translations and so into a numerical representation in which the translations form a multiplicative ratio scale. This enlarges enormously the potential domain of measurable structures.

For example, consider the class of structures involving an ordering and a binary operation (e.g. receiving two things at once, placing two objects on a pan of a pan balance, etc.). For such structures that are homogeneous and continuous, the possible numerical representing operations $\oplus$ are of the form

$$x \oplus y = yf(x/y), \tag{1}$$

where f is strictly increasing and $f(z)/z$ is strictly decreasing. This forms a ratio representation because for $k > 0$,

$$kx \oplus ky = kyf(ky/kx) = k(x \oplus y). \tag{2}$$

The classical case of measurement in physics is the special case where $f(z) = 1 + z$.

This development opens many possibilities which simply have not been explored by empirical scientists. It enlarges much beyond additive systems ones that, in a principled way, can be viewed as quantifiable. Of course, it does not extend to the non-empirical generalizations of Stevens.

3. Units and symmetries

Implicitly Michell seems to assume that an attribute either is or is not quantifiable and he does not discuss the possibility that it might be quantifiable in several ways. In that case, which in fact is common, the question can be raised in what sense do the quantifications agree.

For example, in classical physics many attributes are quantifiable both extensively and conjointly. A case in point is mass where the conjoint structure involves manipulating mass by varying the volumes and substances forming the masses. For this and the usual extensive measure to agree, as they do in classical physics, a certain type of interlocking distribution law must be satisfied, as indeed it is. The details of this, which are somewhat complex, can be found in Krantz, Luce, Suppes & Tversky (1971) and Luce, Krantz, Suppes & Tversky (1990). Also, as we saw in the previous section, the extensive ones are quantifiable both in terms of the structural operation and the group of translations. These too agree. This means that changes of units in the representation are the same as the translations of the representation.

Relativistic velocity is a similar but interestingly different kind of example. Constant velocities can be concatenated—a person walking in a train that is moving relative to an observer. The velocity of the walker relative to the observer is the concatenation of the walker's velocity relative to the train and the train's velocity relative to the observer. Such a velocity–concatenation structure is extensive and so has additive representations, called *rapidity*; in that measure the speed of light maps to ∞. Of course, velocities form a conjoint structure with elapsed distance, s, and time, t, so that $v = s/t$. This velocity measure is not rapidity but rather a transform of it which has the well-known representation of concatenation

$$u \oplus v = \frac{u+v}{1+uv/c^2}, \tag{3}$$

where c denotes the velocity of light in the same units. Note that both the rapidity and v are invariant under multiplication by positive constants. The former corresponds to the translations of the extensive structure whereas the latter does not; it is just a change of units. Such changes of units, however, do correspond to translations of the two components of the conjoint structure. It is simply the case that the two groups of translations are not the same.

A somewhat similar example has arisen recently in utility theory. For half a century utility has been studied using uncertain alternatives (gambles) for which the representation is a weighted average of numerical utilities over pure consequences—the famous subjective expected utility model and its modern generalizations. Luce & Fishburn (1991, 1995) introduced a second way to measure utility based on a binary operation $\oplus$ of joint receipt, i.e. receiving two valued objects at the same time. They assumed for gains relative to a *status quo* that it forms an extensive structure. Assuming an interlock called *segregation*, which is highly rational and has been sustained empirically (Cho & Luce, 1995; Cho, Luce & von Winterfeldt, 1994), they show that the usual utility function U derived from gambling behaviour has the following non-additive representation of $\oplus$:

$$U(x \oplus y) = U(x) + U(y) - \frac{U(x)U(y)}{C}, \tag{4}$$

where C is the asymptotic maximum of the utility function. Note that this representation is invariant under change of unit, but that like the velocity case these transformations do not correspond to translations of the extensive structure although they do of the gambling structure.

I believe that these two cases suggest that if we examine sensory intensity measurement from both the perspective of conjoint structures and extensive ones based on the physical operation of concatenation, we may find an interlock leading to somewhat similar non-additive, bounded representations.

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