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Joint receipt and Thaler's hedonic editing rule

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Abstract

We consider a rule of 'hedonic editing' suggested by R.H. Thaler and others to describe how people evaluate the joint receipt of two separate quantities of a real variable x. Let U be a continuous and increasing utility function on x. We refer to $x \geq 0$ as a *gain*, $x \leq 0$ as a *loss*, fix $U(0) = 0$, and denote by $x \oplus y$ the joint receipt of x and y. The *hedonic editing* rule says that

$$U(x \oplus y) = \max\{U(x + y), U(x) + U(y)\}$$

so that $U(x \oplus y)$ is the larger of the utility of the integrated sum of x and y, and the sum of the utilities of x and y considered separately.

The paper explains structures of U constrained by hedonic editing. Two main cases are analyzed. Case (I) assumes that U is concave in gains and convex in losses. Case (II) assumes that U is concave separately in gains and in losses. Each main case divides into six subcases according to the limiting relations among the slopes of U at ± 0 and $\pm\infty$. These partition the behavior of U in the mixed $(x > 0, y < 0)$ joint-receipt region into two subregions of integration and segregation.

The paper also axiomatizes the cases with assumptions about $(\mathbb{R}, \oplus, \geq)$ from which a suitable U can be constructed. Each main case uses a few axioms satisfied by all its subcases. Special axioms are then invoked for the different subcases.

Keywords: Joint receipt; Thaler's editing rule

1. Introduction

Joint receipts are common in daily life: two or more checks in the mail; a check

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and three bills; notices of taxes owed to the IRS and the state. The issue of how people combine joint receipts of money or other similar items may be important and interesting, but it has not been a popular research topic. An early empirical study of the joint receipt of gambles was reported in Slovic and Lichtenstein (1968), and recent theory and experiments involving gambles are described in Luce (1991), Luce and Fishburn (1991) and Cho et al. (1994). As far as we are aware, the first empirical and somewhat theoretical study of the joint receipt of pure sums of money is Thaler (1985). This was followed up and modified in Thaler and Johnson (1990) and Linville and Fischer (1991).

Thaler had the following to say about pure amounts x and y in his original paper (1985, p. 201):[1]

> The question is how does the joint outcome (x, y) get coded? Two possibilities are considered. The outcomes could be valued jointly as $U(x+y)$ in which case they are said to be *integrated*. Alternatively, they may be valued separately as $U(x)+U(y)$ in which case they are said to be *segregated*.[2] The issue to be investigated is whether segregation or integration produces the greater utility.

Let $\oplus$ denote the binary operation of *joint receipt*. Then we interpret Thaler as assuming the rule

$$U(x \oplus y) = \max\{U(x+y), U(x)+U(y)\} . \tag{1}$$

Thaler and Johnson (1990) refer to (1) as *hedonic editing*, and we shall do likewise. Our aim is to explicate the effect of (1) on U under common forms for U on gains ($x \geq 0$) and on losses ($x \leq 0$), namely concavity in gains and either convexity or concavity in losses, with U increasing over $\mathbb{R}$. Concavity and convexity are understood in the strict sense throughout the paper, so cases with U linear over a nondegenerate interval are excluded.

Our focus on (1) rather than other evaluative rules for joint receipt is motivated by the empirical findings of Thaler and others, which show that it has fared reasonably well as a descriptor of joint-receipt evaluation. We note also that it is one of the simplest nontrivial rules that could be considered. When x and y are received jointly, the obvious comparative factors are the imputed or computed utilities of x, y and $x+y$. It seems natural to form $U(x)+U(y)$ on the separate consideration of the two quantities for comparison with the integrated utility $U(x+y)$. The hedonic rule posits that $x \oplus y$ has the larger of $U(x)+U(y)$ and $U(x+y)$ as its utility. An alternative rule, which has a pessimistic cast, is $U(x \oplus y) = \min\{U(x+y), U(x)+U(y)\}$, but this has not been seriously pursued by others and we do not pursue it here. Yet another alternative factors $U(z)+$

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[1] We change Thaler's utility notation from v to U.

[2] Care is needed to distinguish other meanings of the term. In particular, Luce and Fishburn (1991) uses 'segregation' in another sense.

$U(w)$ into the process for other pairs that satisfy $z + w = x + y$, but because such pairs are not at hand it seems unlikely that they would figure in the evaluation of $x \oplus y$. In any event, we believe that it is prudent to understand the inner workings of (1) before alternative possibilities are examined and will therefore devote our attention to the hedonic editing rule.

We view $x \oplus y$ informally as an ordered pair (x, y) that is indifferent to (y, x) and to some quantity z, its certainty equivalent. Because it is natural to assume that preference increases over $\mathbb{R}$, we bypass the preference-indifference step and associate $x \oplus y$ directly with z. The *formal* theory that we use to analyze (1) therefore takes $\oplus$ as a complete binary operation from $\mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$, so $x \oplus y$ is a quantity on the same scale as x, y and $x + y$. Because U increases over $\mathbb{R}$, the hedonic rule implies $x \oplus y \geq x + y$. Strict inequality holds if $U(x) + U(y) > U(x + y)$, and in this case $x \oplus y = U^{-1}[U(x) + U(y)]$, so $x \oplus y$ depends in an essential way on the shape of U. As mentioned, we consider concave utility for gains and either convex or concave utility for losses. We interpret $x = 0$ as the natural zero and assume that $U(0) = 0$. A modified treatment would be needed if we assumed different forms for U below and above a nonzero target amount such as a minimal acceptable gain point or a maximum tolerable loss point. Kahneman and Tversky (1979) and Fishburn and Kochenberger (1979) discuss two-piece utility functions with a general target amount for situations that involve risk or uncertainty, but they do not consider joint receipt. On the other hand, our treatment for (1) pertains solely to pure sums. It has nothing to do with gambles or uncertainty. Some discussion of how to reduce the case of gambles to pure sums through the use of certainty equivalents is found in Luce (1995). The forms for U that we consider derive from our interest in hedonic editing and not from people's attitudes toward risk.

The combination of risk attitudes and joint receipt was considered in Luce and Fishburn (1991). That paper proposed a rank- and sign-dependent utility theory for arbitrary finite gambles defined over an algebra of events in which we found it useful to introduce a binary operation of joint receipt over gambles and pure consequences. We were unaware at that time of Thaler's 1985 paper, but interpreted $g \oplus h$ for gambles or consequences as the receipt of both g and h. We argued for a particular form of utility function over $\oplus$ but in practice assumed only the additive form of

$$U(g \oplus h) = U(g) + U(h) . \tag{2}$$

Others, contending that it could not be this simple, point out that it is only reasonable for pure amounts x and y to have

$$x \oplus y = x + y , \tag{3}$$

in which case (2), (3), and the monotonicity of U imply $U(x) = cx$ with $c > 0$.

Thaler (1985) shows, however, that either (2) or (3) alone is less obvious than it may seem. To approach (1), which in general selects neither (2) nor (3) exclusively, Thaler assumed a U form suggested by Kahneman and Tversky's (1979) application of prospect theory to experimental data. He took $U(0) = 0$ with U concave in gains, convex in losses, and $-U(-x) > U(x)$ for $x > 0$ so that the utility magnitude for a loss exceeds that for a gain of the same size. Thaler then divided the (x, y) space for $x \oplus y$ into four regions:

1. multiple gains: $x \geq 0$ and $y \geq 0$;
2. multiple losses: $x < 0$ and $y \leq 0$;
3. mixed net gain: $x > 0$, $y < 0$ (or $x < 0$, $y > 0$) and $x + y \geq 0$;
4. mixed net loss: $x > 0$, $y < 0$ (or $x < 0$, $y > 0$) and $x + y < 0$.

Without providing a detailed argument, he contended that one should find segregated multiple gains, integrated multiple losses and mixed net gain, and either integrated or segregated mixed net loss, i.e.

$$U(x \oplus y) = U(x) + U(y) \text{ in region } 1\,,$$

$$U(x \oplus y) = U(x + y) \text{ in regions 2 and 3}\,,$$

$$\text{either, or some mixture, in region } 4\,.$$

Thaler (1985) supports these contentions with some group data and observes that for regions 1–3 they tally with his earlier U form under hedonic editing. The later papers of Thaler and Johnson (1990) and Linville and Fischer (1991) strongly suggest that multiple losses as well as multiple gains are segregated, thus requiring U to be concave in losses when (1) holds.

We offer no new data and take no side on the multiple losses question, but consider both main cases for multiple losses to see where they lead. The mixed case is complex. A basis for the investigation is provided by the following assumption. Here, and later, $\mathbb{R}^+ = \{x \in \mathbb{R}: x \geq 0\}$ and $\mathbb{R}^- = \{x \in \mathbb{R}: x \leq 0\}$.

Assumption 0. U is a strictly increasing and continuous real-valued function on $\mathbb{R}$ such that

(i) $U(0) = 0$;

(ii) U is concave on $\mathbb{R}^+$;

(iii) either (I) U is convex on $\mathbb{R}^-$, or (II) U is concave on $\mathbb{R}^-$.

Moreover, $\oplus$ is a complete binary operation on $\mathbb{R}$ such that

(iv) $U(x \oplus y) = \max\{U(x + y), U(x) + U(y)\}$ for all $x, y \in \mathbb{R}$.

It is easily checked that (i) and (iv) in conjunction with increasing U imply $x \oplus y = y \oplus x$, $x \oplus 0 = x$, and $x > y \Rightarrow x \oplus z > y \oplus z$. These properties of joint receipt will be an explicit part of our axioms for $(\mathbb{R}, \oplus, \geq)$ that are used later to imply utility functions that agree with Assumption 0.

Our pre-axiomatic analyses for joint receipt under Assumption 0 focus on the one (possibly both) of $U(x+y)$ and $U(x)+U(y)$ that attains the maximum on the right-hand side of (1) or (iv) in various regions of the plane. We use

I for *integrated:*[3] $x \oplus y = x + y, U(x \oplus y) = U(x+y)$;

S for *segregated*: $x \oplus y > x + y$, $U(x \oplus y) > U(x+y)$.

Parts (ii) and (iii) of Assumption 0 determine I versus S unambiguously in the multiple gains and multiple losses regions. To see this, recall that a real-valued function V on an interval J of $\mathbb{R}$ is *concave* on J if

$$V(\alpha x + (1-\alpha)y) > \alpha V(x) + (1-\alpha)V(y) \quad \text{for all } x \neq y \text{ in } J, \text{ all } \alpha \in (0,1),$$

and that V is *convex* on J if $-V$ is concave on J.

Lemma 1. *Suppose V is a strictly increasing and concave real-valued function on an interval $J \subseteq \mathbb{R}$ that has 0 as an end point, and $V(0)=0$. Then for all nonzero $x, y \in J$ for which $x+y \in J$,*

(i) $V(x)+V(y) > V(x+y)$;

(ii) $|V(x+y) - V(x)|$ *strictly decreases in* x;

for all nonzero x in J,

(iii) $V(x)/x$ *strictly decreases in* x;

and for all $x, y \in J$,

$$\text{(iv)} \quad \frac{V(y)-V(x)}{y-x} < \frac{V(y)}{y} \quad \textit{if } 0<x<y,$$
$$\frac{V(y)-V(x)}{y-x} > \frac{V(y)}{y} \quad \textit{if } y<x<0.$$

Convexity of V on J replaces V by $-V$ in Lemma 1(i), so convexity implies $V(x+y) > V(x)+V(y)$. We omit the proofs of Lemma 1 and Lemma 2 (see below), which are typical examples of familiar results in elementary convex analysis.

Now suppose that Assumption 0 holds. Then (ii) of Assumption 0 and Lemma 1(i) give $U(x)+U(y) > U(x+y)$, so

$$U(x \oplus y) = U(x) + U(y) \quad \text{whenever } x>0 \text{ and } y>0 .$$

Here S obtains for strictly positive multiple gains. Similarly, for (iii) of Assumption 0 we have I for case (I) and S for case (II):

(I) if U is convex on $\mathbb{R}^-$ then $U(x+y) > U(x)+U(y)$ for negative x and y, so

$$U(x \oplus y) = U(x+y) \quad \text{whenever } x<0 \text{ and } y<0 ;$$

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[3] To avoid confusion between the I for integrated and case (I), the latter always and the former never is enclosed in parentheses.

(II) if U is concave on $\mathbb{R}^-$ then $U(x)+U(y)>U(x+y)$ for negative x and y, so

$$U(x\oplus y)=U(x)+U(y) \quad \text{whenever } x<0 \text{ and } y<0 .$$

The situation for multiple gains and multiple losses is therefore clear. What is unclear is the behavior of $U(x\oplus y)$, and consequently of I and S, in the mixed region

$$Q=\{(x,y): x>0 \text{ and } y<0\}$$

that equals the union of Thaler's regions 3 and 4 for mixed net gain and mixed net loss.[4] We therefore focus on Q in our further analyses of Assumption 0. The following lemma will assist in the analysis of Q.

Lemma 2. *Suppose V is a strictly increasing and continuous real-valued function on $\mathbb{R}^+$ with $V(0)=0$. If V is either concave or convex on $\mathbb{R}^+$ then* $\lim_{x\to\infty} V(x)/x$ *is well defined (perhaps as $+\infty$ in the convex case), and for all $\delta>0$,*

$$\lim_{x\to\infty}\frac{V(x+\delta)-V(x)}{\delta}=\lim_{x\to\infty}\frac{V(x)}{x} .$$

Cases (I) and (II) of (iii) are substantially different, and we treat them separately. Case (I) for convex U on $\mathbb{R}^-$ is discussed in Section 2; case (II) for concave U on $\mathbb{R}^-$ is discussed in Section 4. We shall see that each main case divides into six subcases that describe different behaviors in region Q. The six subcases correspond to possible orders of four quantities: the limiting slopes of U on gains at $+0$ and $+\infty$, and the limiting slopes of U on losses at -0 and $-\infty$.

Fig. 1 illustrates the types of utility functions and constant-utility curves (indifference curves) in the (x,y) plane for $x\oplus y$ that are involved for cases (I) and (II). More detailed pictures of the subcases of each main case appear later. The utility-curve illustrations show the four limiting slopes for U at ± 0 and $\pm\infty$. The curves of constant $U(x\oplus y)$ value in the lower figures show the effects of concave gains and convex (I) or concave (II) losses in the $(+,+)$ and $(-,-)$ quadrants. The curves there are symmetric about the 45° $(x=y)$ line and are straight lines for I. Those for S are concave from above, or convex from below. The behaviors in the $(+,-)$ quadrant of Q show an S region and an I region separated by a common boundary at which $U(x+y)=U(x)+U(y)$.

Sections 2 and 4 are followed in turn by sections that present axioms for $(\mathbb{R},\oplus,\succeq)$ which are necessary and sufficient for the existence of utility functions that satisfy Assumption 0 and the possible behaviors in Q implied by that assumption. Section 3 deals with case (I) and Section 5 with case (II).

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[4] As we shall see, Thaler's partition of the mixed case into regions 3 and 4 plays no role in describing the results for case (I), which is the one that interested him, but it does affect case (II).

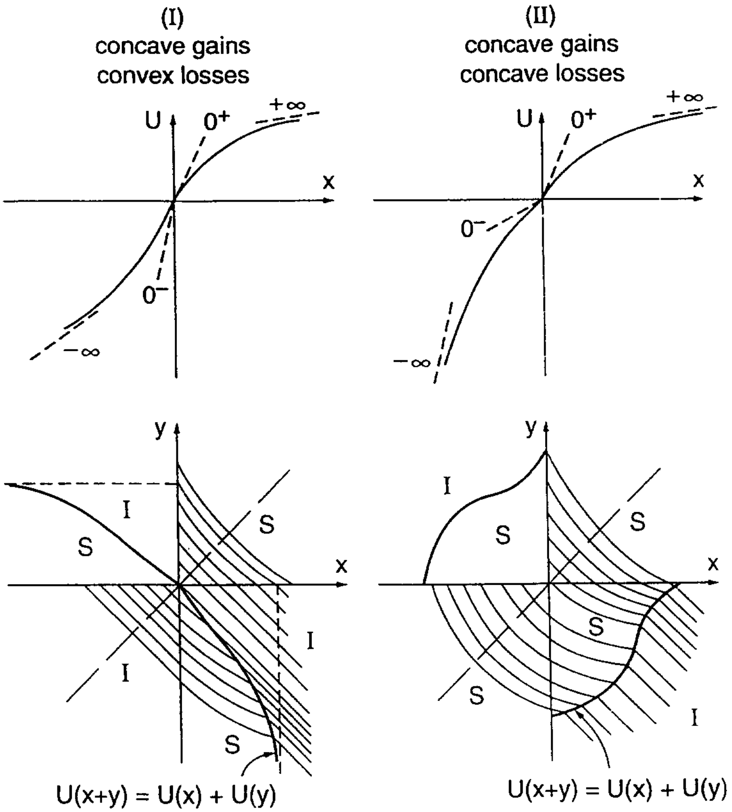

Fig. 1. Constant utility curves of joint receipt.

2. Case (I): Convex utility for losses

We assume throughout this section that Assumption 0 *holds with U convex on* $\mathbb{R}^-$.

It is convenient to have notation for utilities that distinguishes gains from losses. We take

$$\begin{aligned} u(x) &= U(x) && \text{for all } x \geq 0\,, \\ w(x) &= -U(-x) && \text{for all } x \geq 0\,. \end{aligned}$$

Both u and w are defined on $\mathbb{R}^+$: $u(x)$ is the utility of gain x, and $w(x)$ is the disutility of a loss of magnitude x. A picture of the graph of w is obtained by rotating the curve of $U(x)$ for $\mathbb{R}^-$ 180° around the origin as the fixed point. This is illustrated on the top of Fig. 2. The lower part of the figure identifies $U(x \oplus y)$ by (1) for case (I) in each of the four regions noted in the paragraph following (3).

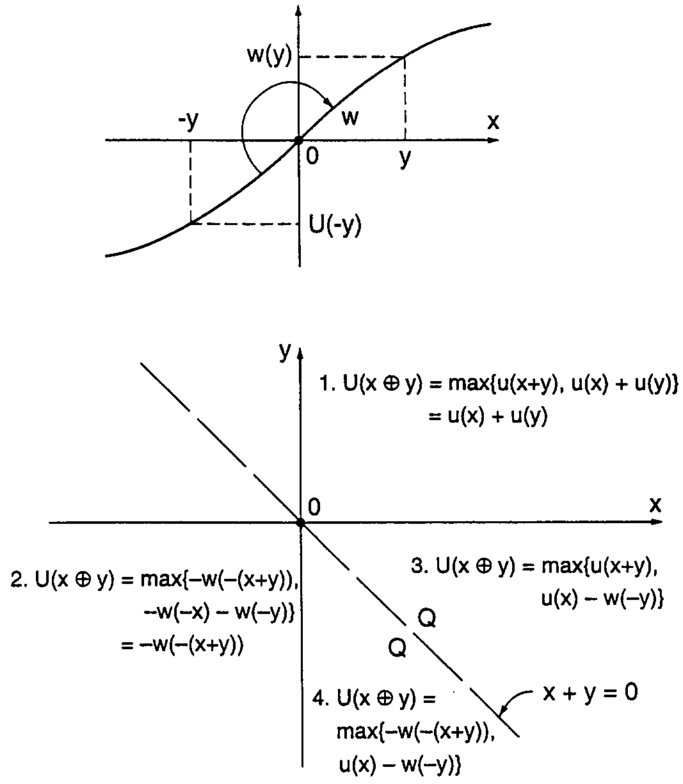

Fig. 2. Case (I).

2.1. Limiting slopes

In quadrant $Q = \{(x, y) \colon x > 0, y < 0\}$ we have

$$U(x \oplus y) = \begin{cases} \max\{u(x+y), u(x) - w(-y)\} & \text{if } x + y \geq 0 , \\ \max\{-w(-(x+y)), u(x) - w(-y)\} & \text{if } x + y \leq 0 . \end{cases}$$

The behavior of $\oplus$ or of $U(x \oplus y)$ in this quadrant is governed by the limiting slopes of u and w at 0 and ∞. We denote these limits by u'_0 u'_∞, w'_0 and w'_∞. Their definitions are

$$u'_0 = \lim_{x \downarrow 0} u(x)/x ,$$

$$u'_\infty = \lim_{x \to \infty} u(x)/x ,$$

$$w'_0 = \lim_{x \downarrow 0} w(x)/x = \lim_{x \uparrow 0} U(x)/x ,$$

$$w'_\infty = \lim_{x \to \infty} w(x)/x = \lim_{x \to -\infty} U(x)/x .$$

The monotonicity and curvature properties of U ensure that the limits exist and are non-negative. It is possible to have $u'_0 = \infty$ or $w'_0 = \infty$, but u'_∞ and w'_∞ must be finite. If the utility of gains approaches saturation, i.e. $U(x)$ for $x > 0$ approaches a finite upper bound as x gets large, then $u'_\infty = 0$. But we can also have $u'_\infty = 0$ when saturation does not occur, as with $u(x) = \sqrt{x}$ or $u(x) = \log(x + 1)$. Similar remarks apply to $w'_\infty = 0$ for the disutility of losses.

Concavity of U on $\mathbb{R}^+$ and its convexity on $\mathbb{R}^-$ say that both u and w are concave, so

$$u'_0 > u'_\infty \quad \text{and} \quad w'_0 > w'_\infty .$$

These restrictions allow the following six orderings of $\{u'_0, u'_\infty, w'_0, w'_\infty\}$:

A. $w'_\infty \geq u'_0$, i.e. $w'_0 > w'_\infty \geq u'_0 > u'_\infty$;
B. $u'_\infty \geq w'_0$, i.e. $u'_0 > u'_\infty \geq w'_0 > w'_\infty$;
C. $w'_0 \geq u'_0 > u'_\infty \geq w'_\infty$;
D. $w'_0 \geq u'_0 > w'_\infty > u'_\infty$;
E. $u'_0 > w'_0 > u'_\infty \geq w'_\infty$;
F. $u'_0 > w'_0 > w'_\infty > u'_\infty$.

The following theorem tells how these six orderings govern the behavior of $\oplus$ within Q. The segretated region in Q is

$$S_Q = \{(x, y) \in Q: x \oplus y > x + y, \text{ i.e. } U(x) + U(y) > U(x + y)\} .$$

The integrated region in Q is $Q \backslash S_Q = \{(x, y) \in Q: x \oplus y = x + y\}$.

Theorem 1. *Let Assumption* 0 *hold with* U *convex on* $\mathbb{R}^-$. *Then* $A \Leftrightarrow S_Q = \emptyset$ *and* $B \Leftrightarrow S_Q = Q$. *For each of* C, D, E *and* F *there is a continuous and strictly decreasing function* g *from* $(t, 0]$ *with* $t < 0$ *onto an interval* $K \subseteq \mathbb{R}^+$ *such that*

$$S_Q = \{(x, y) \in Q: y \leq t, \text{ or } t < y < 0 \text{ and } 0 < x < g(y)\}$$

with t *and* K *as follows*:

C. $-\infty \leq t$, $K = \mathbb{R}^+$;
D. $t = -\infty$, $K = [0, s)$, $0 < s < \infty$;
E. $-\infty \leq t$, $K = [s, \infty)$, $0 < s < \infty$;
F. $t = -\infty$, $K = [s_1, s_2)$, $0 < s_1 < s_2 < \infty$.

Fig. 3 illustrates U for subcases A–F, with a sketch of g in Q when applicable. The curve of g asymptotes to $y = t$ in C or E when t there is finite, which can happen only if $u'_\infty > w'_\infty$. Subcase A fully agrees with Thaler's (1985) assumptions, and D does also with the possible exception that $-U(-x) > U(x)$ might fail in the mid-range of $x > 0$.

The proof of Theorem 1 is found in Appendix A.

2.2. *Two examples*

A very common assumption, one that is sometimes derived from other considerations (see, for example, Luce and Fishburn, 1991), is that U is composed of two power functions, i.e.

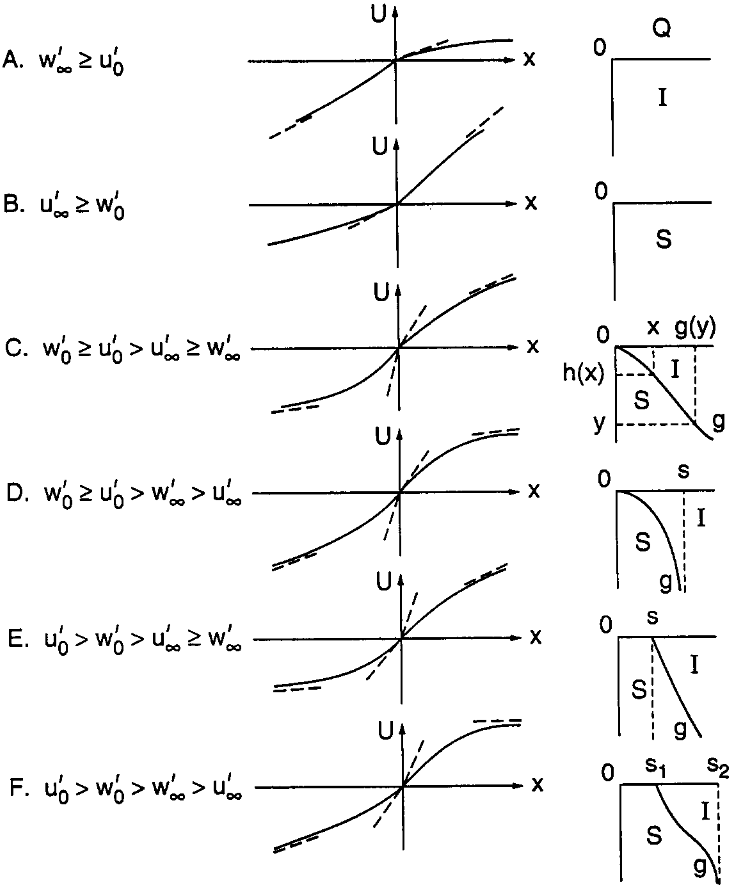

Fig. 3. Theorem 1.

$$U(x) = \begin{cases} cx^\alpha, & x \geq 0,\, c > 0,\, 0 < \alpha < 1\,, \\ -k(-x)^\beta, & x < 0,\, k > 0,\, 0 < \beta < 1\,. \end{cases}$$

As is easily verified, this is concave for gains and convex for losses, $u'_\infty = w'_\infty = 0$ and $u'_0 = w'_0 = \infty$, which is subcase C. One can set up equations to determine g; however, they cannot be solved explicitly.

A second example, which is also concave for gains and convex for losses is

$$U(x) = \begin{cases} c \log(1 + x), & x \geq 0,\; c > 0\,, \\ -k \log(1 - x), & x < 0,\; k > 0\,. \end{cases}$$

It is easy to verify that $u'_\infty = w'_\infty = 0$, $u'_0 = c$, and $w'_0 = k$. Thus, if $k \geq c$, it falls under subcase C and if $k < c$, it falls under subcase E.

3. Axioms for case (I)

This section presents axioms for $(\mathbb{R}, \oplus, \succsim)$ that are necessary and sufficient for the existence of $U: \mathbb{R} \to \mathbb{R}$ that satisfies Assumption 0 with U convex on $\mathbb{R}^-$. Because subcases A through F have different behaviors in $Q = \{(x, y): x > 0, y < 0\}$, different axioms will be used for different subcases. We say that *Assumption* 0(I) *holds for a particular subcase or set of subcases if Assumption* 0 *holds with U convex on* $\mathbb{R}^-$ *and the limiting slope conditions that define the subcase or subcases are satisfied by U.* As in the preceding section, u and w on $\mathbb{R}^+$ relate to U by $U = u$ on $\mathbb{R}^+$ and $U(-x) = -w(x)$ for all $x \in \mathbb{R}^+$.

Three axioms apply to all six subcases.

Axiom 1. $(\mathbb{R}, \oplus, \succsim)$ satisfies the following conditions for all $x, y, z \in \mathbb{R}$:

$$x \oplus y \in \mathbb{R},$$
$$x \oplus y = y \oplus x,$$
$$x > y \Rightarrow x \oplus z > y \oplus z,$$
$$x \oplus 0 = x.$$

Axiom 2. For all $x, y \in \mathbb{R}$,

(a) $x \oplus y \succsim x + y$,

(b) $x > 0 \Rightarrow \exists \epsilon > 0$ such that $x \succsim \epsilon \oplus \epsilon$,

(c) $(x, y \geq 0, x \neq y) \Rightarrow (x + y)/2 \oplus (x + y)/2 > x \oplus y$,

(d) $x, y < 0 \Rightarrow x \oplus y = x + y$.

In Axiom 3, $(\mathbb{R}^+, \oplus, \succsim)$ denotes the restriction of $(\mathbb{R}, \oplus, \succsim)$ to $\mathbb{R}^+$. The definition of a *closed extensive structure* is given on p. 73 of Krantz et al. (1971).

Axiom 3. $(\mathbb{R}^+, \oplus, \succsim)$ is a closed extensive structure.

Lemma 6 of Appendix B notes that Axioms 1–3 are implied by Assumption 0(I). It is well known (p. 74 of Krantz et al., 1971) that Axiom 3 implies the existence of $u: \mathbb{R}^+ \to \mathbb{R}^+$ such that u preserves $\succsim$ with $u(x \oplus y) = u(x) + u(y)$. Axiom 2(b) ensures that u on $\mathbb{R}^+$ is continuous at 0, and 2(c) implies concavity and continuity. We thus have u over $\mathbb{R}^+$ for which

$$u(x \oplus y) = u(x) + u(y) > u(x + y).$$

Moreover, u is unique up to multiplication by a positive constant. The matter is summarized in Lemma 7 of Appendix B.

We let $U = u$ on $\mathbb{R}^+$ henceforth. We have not yet defined U on $\mathbb{R}^-$, with $U(x) = -w(-x)$, although any continuous and increasing concave w on $\mathbb{R}^+$ with $w(0) = 0$ would suffice for joint losses and satisfy Axiom 2(d). We need more than this, however, for the definition of w must mesh with u in a way that yields an admissible situation in the mixed region Q.

This requires consideration of subcases. We begin with A and B since they do not involve g, and will then look at C–F together. As noted in the preceding section, A has $x \oplus y = x + y$ and B has $x \oplus y > x + y$ throughout Q. The following summarizes the desired behaviors on Q:

	A	B
$x + y \geq 0$	$u(x + y) \geq u(x) - w(-y)$	$u(x) - w(-y) > u(x + y)$
$x + y \leq 0$	$-w(-(x + y)) \geq u(x) - w(-y)$	$u(x) - w(-y) > -w(-(x + y))$

3.1. *Subcase A*

We use two axioms in addition to Axioms 1–3 to complete our characterization of subcase A.

Axiom A1. $x \oplus y = x + y$ for all $(x, y) \in Q$.

Axiom A2. There exist positive z and ϵ such that

$$z > \lambda z \oplus (1 - \lambda)\epsilon \quad \text{for all } 0 \leq \lambda < 1 .$$

Axiom A1 has the apparent effect of inducing I throughout Q. To do this it is necessary to allow w to be defined so that $w'_0 > w'_\infty \geq u'_0$, and this is accomplished through Axiom A2 which implies that u'_0 is finite. Because A2 allows z to be large with ϵ near 0, it is a very plausible condition.

Theorem 2. *Assumption* 0(I) *holds for subcase* A *if and only if Axioms* 1–3, A1, *and* A2 *hold.*

The proof is given in Appendix B.

On the uniqueness of w. Lemma 7 of Appendix B notes that u on gains is unique up to the choice of scale unit. However, the proof of Theorem 2 shows that w is not similarly unique for subcase A: any w that is increasing, concave, and has $w(0) = 0$ and $w'_\infty \geq u'_0$ suffices. The reason that w is not further constrained is that subcase A has no S region within Q, hence has no tradeoff between u and w as required when $U(x \oplus y) = U(x) + U(y) = u(x) - w(-y)$. Each of the other five subcases for (I) has a substantial S region in Q, and in each of these five w is uniquely determined when u is fixed.

3.2. Subcase B

We next consider subcase B which, because it entails S throughout Q, has an axiomatization that is only slightly more involved than that for subcase A. Three axioms in addition to Axioms 1–3 are used for subcase B. We say that $\oplus$ *induces an additive conjoint structure over a region R in the plane* if there exist real-valued functions U_1 on $R_1 = \{x: (x, y) \in R \text{ for some } y\}$ and U_2 on $R_2 = \{y: (x, y) \in R \text{ for some } x\}$ such that, for all (x, y), $(x', y') \in R$,

$$x \oplus y > x' \oplus y' \Leftrightarrow U_1(x) + U_2(y) > U_1(x') + U_2(y') . \tag{4}$$

In Axiom B2 that follows, $R_1 = \mathbb{R}^+$, $R_2 = \mathbb{R}$, and $R = R_1 \times R_2$. Specific axioms for additive conjoint measurement on a rectangular subset of the plane are described in Fishburn (1970), Krantz et al. (1971) and Wakker (1989). They usually include conditions which imply that U_1 and U_2 are continuous in the relative usual topologies of R_1 and R_2, respectively, and when we assume this, we say that $\oplus$ induces a *continuous* additive conjoint structure over R. We do not postulate continuity in B2 since it will follow from other axioms, but continuity is assumed later in subcases C–F.

Axiom B1. $x \oplus y > x + y$ for all $(x, y) \in Q$.

Axiom B2. $\oplus$ induces an additive conjoint structure over $\mathbb{R}^+ \times \mathbb{R}$.

Axiom B3. For all $x, x', x^* \in \mathbb{R}$ and all $y, y' \in \mathbb{R}^-$ for which $y \neq y'$,

$$x \oplus y = x' \oplus y' = x^* \oplus \frac{y + y'}{2} = 0 \Rightarrow x^* \oplus x^* > x \oplus x' .$$

These axioms address the behavior of $\oplus$ in Q and its closure (B3). In addition, Axiom B2 makes an intimate connection with Axiom 3 for a closed extensive structure in the multiple gains region. Our construction of w, or of U on $\mathbb{R}^-$, follows from this connection and Lemma 7. Axiom B3 then implies that w is concave, or that U on $\mathbb{R}^-$ is convex.

Theorem 3. *Assumption* 0(I) *holds for subcase* B *if and only if Axioms* 1–3, B1, B2, *and* B3 *hold.*

The proof is given in Appendix B.

3.3. Subcases C–F

We treat subcases C through F together in this subsection and then conclude the section with remarks on their differences. Their common feature is the continuous curve g that separates S from I in Q: see Theorem 1 and Fig. 3. Their differences involve the domain and codomain K of g in Theorem 1, or their orderings of limiting slopes of U at ± 0 and $\pm\infty$.

We begin our extension of Axioms 1–3 for subcases C–F with a direct approach to g. An indirect approach that derives g from other axioms is possible but is considerably less transparent. For convenience we take $(-\infty, 0)$ as the domain of g for Q, rather than $(t, 0]$ as in Theorem 1, and account for $y \le t$ by allowing $g(y)$ to equal ∞. For each $y < 0$ let

$$S_y = \{x : x > 0 \text{ and } x \oplus y > x + y\} .$$

Then in the notation of Theorem 1, $S_Q = \bigcup_{y<0} S_y$. Axiom 2(a) gives $x \oplus y = x + y$ for all $(x, y) \in Q \backslash S_Q$.

Axiom 4. There exists an interval $(t, 0)$ with $t < 0$ and possibly $t = -\infty$, and a mapping $g\colon (-\infty, 0) \to [\mathbb{R}^+ \backslash \{0\}] \cup \{\infty\}$ such that

(i) $g(y)$ is finite for $t < y < 0$, and $g(y) = \infty$ for $y \le t$;
(ii) $g(y)$ is continuous and decreasing on $(t, 0)$;
(iii) $S_y = (0, g(y))$ for all $y \in (-\infty, 0)$.

A direct consequence of Theorem 1, which is stated as Lemma 9 of Appendix B, is that Axiom 4 holds in subcases C–F of Assumption 0(I).

Axiom 4 can be viewed as a modification of Axiom B1 for subcases in which part but not all of Q is segregated. Our next three axioms modify Axioms B2 and B3. We presume S_Q as characterized through Axiom 4 and let $\mathrm{c}(S_Q)$ denote the closure of S_Q. Thus, in addition to S_Q, $\mathrm{c}(S_Q)$ includes the negative ordinate, the origin, a finite interval to the right of the origin on the abscissa in subcases E and F, and the points on the g curve for $t < y < 0$.

Axiom 5. $\oplus$ induces a continuous additive conjoint structure over $\mathrm{c}(S_Q)$.

In subcases C and D, the only point in both $\mathbb{R}^+ \times \mathbb{R}^+$ and $\mathrm{c}(S_Q)$ is the origin. We therefore use only $\mathrm{c}(S_Q)$ in the conjoint assumption, Axiom 5, noting that its counterpart for $\mathbb{R}^+ \times \mathbb{R}^+$ is already presumed by Axiom 3. Although $\mathrm{c}(S_Q)$ is not rectangular, it is the union of the set of overlapping rectangles whose top edges are the intervals $[(0, y), (g(y), y)]$ for $t < y < 0$ and which are unbounded below, plus the nonpositive ordinate, so that Axiom 5 can be thought of as the conjunction of additive conjoint structure assumptions for these rectangles.

Our separation of Axiom 5 from $\mathbb{R}^+ \times \mathbb{R}^+$ requires an axiom that makes a connection between $\mathbb{R}^+ \times \mathbb{R}^+$ and $\mathrm{c}(S_Q)$ so that we can obtain the same utility function, namely u, for $\mathbb{R}^+ \times \mathbb{R}^+$ and the U_1 part of (4) in $\mathrm{c}(S_Q)$ over the part of the non-negative abscissa where both are defined, i.e. over

$$\mathrm{c}(S_Q)_1 = \bigcup_{y<0} [0, g(y)] .$$

This x-interval equals $\mathbb{R}^+$ for subcases C and E, but is finite for D and F (see Fig. 3). The necessary connection is made by Axiom 6, where $x_i \oplus y_j$ comes from $c(S_Q)$, and $x_i \oplus x_j$ from $\mathbb{R}^+ \times \mathbb{R}^+$.

Axiom 6. For all $x_3 > x_2 > x_1 \geq 0 \geq y_1 > y_2$, if (x_2, y_1), $(x_3, y_2) \in c(S_Q)$, then

$$[x_1 \oplus y_1 = x_2 \oplus y_2, x_2 \oplus y_1 = x_3 \oplus y_2] \Rightarrow [x_1 \oplus x_3 = x_2 \oplus x_2] .$$

Given that $\oplus$ is commutative, Axiom 6 amounts to saying that the Thomsen condition of additive conjoint measurement holds outside the region $c(S_Q)$ as well as within it, which, in turn, amounts to saying indirectly that $\oplus$ is associative in the region where segregation holds. If associativity actually held universally, and if one could freely compose terms under $\oplus$, then Axiom 6 would be satisfied. Of course, one cannot freely compose terms (without leaving the region of segregation) and that is why indirection is necessary.

Axiom 7. For all $x, x', x^* \geq 0$ and all $y, y' \leq 0$, $y \neq y'$, if (x, y), (x', y'), $(x^*, (y + y')/2) \in c(S_Q)$, then

$$x \oplus y = x' \oplus y' = x^* \oplus \frac{y + y'}{2} \Rightarrow x^* \oplus x^* > x \oplus x' .$$

Axiom 7, like Axiom B3, ensures concavity for w.

We complete our axioms for C–F with a connection between Q and $\mathbb{R}^+ \times \mathbb{R}^+$ that involves $(g(y), y)$ at the right boundary of $c(S_Q)$. The connection is used for part (iv) of Assumption 0 when $t < y < 0$ and $g(y) + y > 0$.

Axiom 8. For all $y \in (t, 0)$, if $g(y) + y > 0 = x \oplus y$, then $(g(y) + y) \oplus x = g(y)$.

At first glance Axiom 8 seems quite obscure, but it is not difficult to outline why it must hold. Recall that the point $(y, g(y))$ lies on the boundary between segregation and integration. Because U is continuous, one has in essence both $g(y) \oplus y = g(y) + y$ and $U[g(y) \oplus y] = U[g(y)] + U(y)$. The assumption that $g(y) + y > 0$ means that

$$U[(g(y) + y) \oplus x] = U[g(y) + y] + U(x) = U[g(y)] + U(y) + U(x) .$$

However, $(x, y) \in S_Q$ and by assumption $x \oplus y = 0$, so $U(x) + U(y) = 0$. Substituting, we see that $(g(y) + y) \oplus x = g(y)$. This outline is formalized in Lemma 11 in Appendix B.

Theorem 4. *Assumption* 0(I) *holds for a subcase in* $\{\mathrm{C}, \mathrm{D}, \mathrm{E}, \mathrm{F}\}$ *if and only if Axioms* 1–8 *hold.*

The proof is given in Appendix B.

3.4. Subcase differences

Because Axioms 1–8 make no distinctions among subcases C, D, E and F, we summarize their differences here. The following characterizations, to be understood in the setting of Theorem 4, follow from Theorem 1 and the definitions. g is understood to be defined on $(t, 0)$ in the notation of Axiom 4.

$$\text{C or D holds} \Leftrightarrow w_0' \geq u_0' \Leftrightarrow g(y)\downarrow 0 \text{ as } y\uparrow 0,$$
$$\text{E or F holds} \Leftrightarrow u_0' > w_0' \Leftrightarrow \lim g(y) > 0 \text{ as } y\uparrow 0;$$

$$\text{C or E holds} \Leftrightarrow u_\infty' \geq w_\infty' \Leftrightarrow g(y) \text{ is unbounded},$$
$$\text{D or F holds} \Leftrightarrow w_\infty' > u_\infty' \Leftrightarrow g \text{ is bounded above}, \ t = -\infty.$$

Subcase X is characterized by Axioms 1–8 and the conditions on g, or on $\{u, w\}$, in the two preceding lines that refer to X.

4. Case (II): Concave utility for losses

Assumption 0 *with* U *concave on* $\mathbb{R}^-$ *is assumed throughout this section* along with the definitions of u, w, u_0', u_∞', w_0', w_∞', and Q of Section 2. Because the segregated option for (1) [or Assumption 0(iv)] applies when $x, y > 0$ or $x, y < 0$, we focus on Q. It is convenient to split Q into its $x + y \leq 0$ and $x + y \geq 0$ regions (Thaler's regions 4 and 3, respectively) as follows:

$$Q^- = \{(x, y) \in Q: x + y \leq 0\}, \quad S^- = \{(x, y) \in Q^-: x \oplus y > x + y\},$$
$$I^- = Q^- \setminus S^-;$$
$$Q^+ = \{(x, y) \in Q: x + y \geq 0\}, \quad S^+ = \{(x, y) \in Q^+: x \oplus y > x + y\},$$
$$I^+ = Q^+ \setminus S^+.$$

The following lemmas illustrate a symmetry between Q^- and Q^+ that governs our further analysis. The definitions of $y(x)$ and $x(y)$ embedded in the lemmas apply throughout the section. See Fig. 4.

Lemma 12 (for Q^-). *For every* $x > 0$ *there is at most one* $y < 0$ *with* $x + y \leq 0$ *at which* $u(x) - w(-y) = -w(-(x + y))$. *We denote this* y *by* $y(x)$ *when it exists. When* $y(x)$ *exists, the interval* $[(x, -x), (x, y(x)))$ *is in* S^- *and* $[(x, y(x)), (x, -\infty))$ *is in* I^-.

Lemma 13 (for Q^+). *For every* $y < 0$ *there is at most one* $x > 0$ *with* $x + y \geq 0$ *at which* $u(x) - w(-y) = u(x + y)$. *We denote this* x *by* $x(y)$ *when it exists. When* $x(y)$ *exists, the interval* $[(-y, y), (x(y), y))$ *is in* S^+ *and* $[(x(y), y), (\infty, y))$ *is in* I^+.

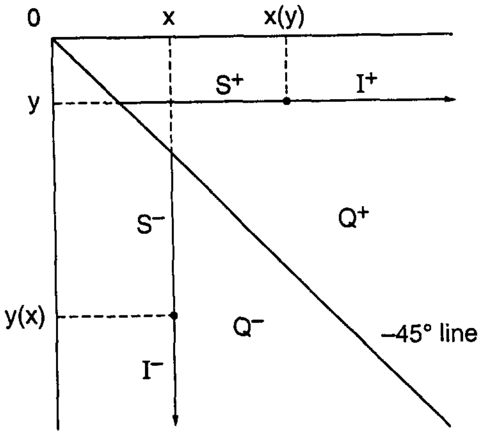

Fig. 4.

The proofs are in Appendix C.

4.1. Limiting slopes

As in Section 2, six essentially different behaviors can occur in Q according to orderings of limiting slopes. Concavity of u and convexity of w give

$$u_0' > u_\infty' \quad \text{and} \quad w_\infty' > w_0' .$$

These allow the following. The names are explained below.

$\mathrm{A^-A^+}$: $w_0' \geq u_0'$, i.e. $w_\infty' > w_0' \geq u_0' > u_\infty'$;
$\mathrm{B^-B^+}$: $u_\infty' \geq w_\infty'$, i.e. $u_0' > u_\infty' \geq w_\infty' > w_0'$;
$\mathrm{G^-G^+}$: $u_0' \geq w_\infty' > u_\infty' \geq w_0'$;
$\mathrm{G^-H^+}$: $u_0' \geq w_\infty' > w_0' > u_\infty'$;
$\mathrm{H^-G^+}$: $w_\infty' > u_0' > u_\infty' \geq w_0'$;
$\mathrm{H^-H^+}$: $w_\infty' > u_0' > w_0' > u_\infty'$.

Each of Q^- and Q^+ has four mutually exclusive alternatives. They are denoted by A, B, G and H with the corresponding superscript, and fit together in the six combinations just noted. Subcase $\mathrm{A^-A^+}$ has I throughout Q, $\mathrm{B^-B^+}$ has S throughout Q, and each of the other four has an upper left S region separated from an I region by a continuous curve in Q. Fig. 5 illustrates those four using notation of their ensuing definitions.

Characterizations of the basic behaviors for Q^- and Q^+ follow. Theorems that validate the characterizations are stated briefly and proved in Appendix C. We begin with Q^-. The $y(x)$ curve is continuous for $\mathrm{G^-}$ and $\mathrm{H^-}$.

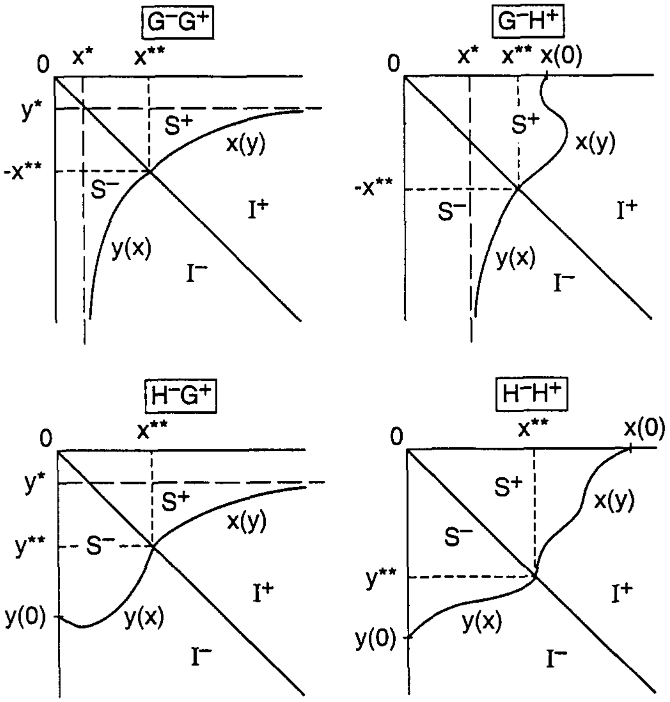

Fig. 5. Four subcases for Assumption 0(II).

A^-. $Q^- = I^-$.

B^-. $Q^- = S^-$.

G^-. There exist x^* and x^{**} with $0 \le x^* < x^{**}$ such that $y(x)$ is defined on $(x^*, x^{**}]$ with $y(x^{**}) = -x^{**}$. As x approaches x^* from the right, $y(x)$ asymptotes negatively to the vertical line through $(x^*, 0)$. S^- is the region bounded by the negative ordinate, the $-45°$ line from the origin to $(x^{**}, -x^{**})$ and the $y(x)$ curve. It excludes the left boundary and the $y(x)$ curve.

H^-. There exists $x^{**} > 0$ such that $y(x)$ is defined on $(0, x^{**}]$ with $y(x^{**}) = -x^{**}$. As $x \downarrow 0$, $y(x)$ approaches a finite negative limit denoted by $y(0)$. S^- is the region above the $y(x)$ curve, to the right of the negative ordinate, and on and below the $-45°$ line segment $((0,0), (x^{**}, -x^{**}))$.

Theorem 5. $A^- \Leftrightarrow u_0' \le w_0'$.
$B^- \Leftrightarrow w_\infty' \le u_\infty'$.
$G^- \Leftrightarrow u_\infty' < w_\infty' \le u_0'$.
$H^- \Leftrightarrow w_0' < u_0' < w_\infty'$.

The four cases for Q^+ are symmetrically similar to those for Q^-. The $x(y)$ curve is continuous for G^+ and H^+.

A$^+$. $Q^+ = I^+$.
B$^+$. $Q^+ = S^+$.
G$^+$. There exist y^* and y^{**} with $y^{**} < y^* \leq 0$ such that $x(y)$ is defined on $[y^{**}, y^*)$ with $x(y^{**}) = -y^{**}$. As y approaches y^* from below, $x(y)$ asymptotes positively to the horizontal line through $(0, y^*)$. S^+ is the region bounded by the positive abscissa, the −45° line from the origin to $(-y^{**}, y^{**})$ and the $x(y)$ curve. It excludes the upper boundary and the $x(y)$ curve.
H$^+$. There exists $y^{**} < 0$ such that $x(y)$ is defined on $[y^{**}, 0)$ with $x(y^{**}) = -y^{**}$. As $y \uparrow 0$, $x(y)$ approaches a finite positive limit denoted by $x(0)$. S^+ is the region to the left of the $x(y)$ curve, below the positive abscissa, and on and above the −45° line segment $((0,0), (-y^{**}, y^{**}))$.

Theorem 6. $\mathrm{A}^+ \Leftrightarrow u_0' \leq w_0'$.
$\mathrm{B}^+ \Leftrightarrow w_\infty' \leq u_\infty'$.
$\mathrm{G}^+ \Leftrightarrow w_0' \leq u_\infty' < w_\infty'$.
$\mathrm{H}^+ \Leftrightarrow u_\infty' < w_0' < u_0'$.

The composite behaviors implied by Theorems 5 and 6 and $\{u_0' > u_\infty', w_\infty' > w_0'\}$ are the six subcases noted earlier: $\mathrm{A}^-\mathrm{A}^+, \mathrm{B}^-\mathrm{B}^+, \ldots, \mathrm{H}^-\mathrm{H}^+$. Continuity requires $x^{**} = -y^{**}$ in the final four subcases as shown in Fig. 5.

The behaviors of $y(x)$ and $x(y)$ can be considerably more complex than shown in Fig. 5. We illustrate for H$^+$ ($u_\infty' < w_0' < u_0'$) at the left extreme of $x(y)$ defined by

$$x_0 = \inf\{x(y): y \in [y^{**}, 0)\}\ .$$

For each $x \geq 0$ let $f_x(y) = u(y) - w(x - y)$ for $y \leq x$. Then any *one or more* of four things can occur in the comparison of u and f_{x_0} on $[0, x_0]$:

1. f_{x_0} comes down to u from above u at the origin;
2. f_{x_0} is tangent to u from above at one $x_1 \in (0, x_0)$;
3. f_{x_0} is tangent to u from above at x_0;
4. f_{x_0} and u coincide on an interval $[x_1, x_2]$ in $[0, x_0]$.

Fig. 6 pictures their effects on $x(y)$ near x_0.

5. Axioms for case (II)

We now consider axioms for $(\mathbb{R}, \oplus, \geq)$ that are necessary and sufficient for the existence of $U: \mathbb{R} \to \mathbb{R}$ that satisfies Assumption 0 with U concave on $x \leq 0$. We say that *Assumption* 0(II) *holds for a particular set of subcases if Assumption* 0 *holds with U concave on* $\mathbb{R}^-$ *and the limiting slope conditions that define those*

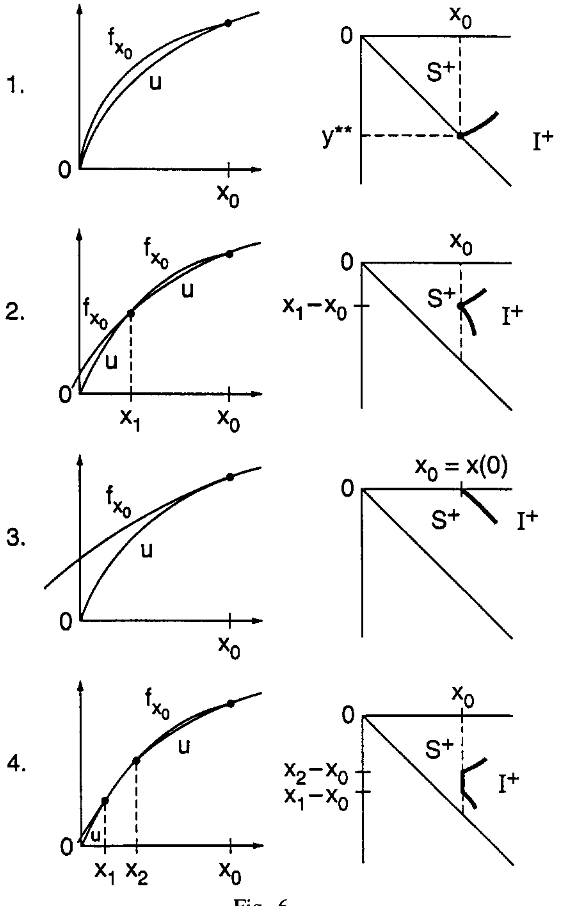

Fig. 6.

subcases are satisfied by U. The usual relationships between U, u and w are presumed.

The following axioms apply to all subcases of case (II).

Axiom 1*. Axiom 1 (see Section 3).

Axiom 2*. For all $x, y \in \mathbb{R}$,

(a) $x \oplus y \geq x + y$;

(b) $x > 0 \Rightarrow \exists \epsilon > 0$ such that $x \geq \epsilon \oplus \epsilon$;

(c) $(x, y \geq 0, x \neq y) \Rightarrow (x+y)/2 \oplus (x+y)/2 > x \oplus y$;

(d) $x < 0 \Rightarrow \exists \epsilon < 0$ such that $\epsilon \oplus \epsilon \geq x$;

(e) $(x, y \leq 0, x \neq y) \Rightarrow (x+y)/2 \oplus (x+y)/2 > x \oplus y$.

Axiom 3*. Each of $(\mathbb{R}^-, \oplus, \geq)$ and $(\mathbb{R}^+, \oplus, \geq)$ is a closed extensive structure.

As was true of case (I), Axioms 1*–3* are all necessarily true under Assumption 0(II), as is proved in Lemma 14 of Appendix D, and permit the construction of two continuous, increasing functions u and W on $\mathbb{R}^+$ with the following properties: u is concave and

$$u(x \oplus y) = u(x) + u(y) > u(x + y) ,$$

and W is convex and

$$W(-[(-x) \oplus (-y)]) = W(x) + W(y) < W(x + y) .$$

We use W instead of w in the following because Axioms 1*–3* do not imply scale alignments for u and w that are globally consistent for Assumption 0(II). We shall later set w equal to λW for some $\lambda > 0$.

Subsequent results in this section presume Axioms 1*–3* with u and W as just described. For Assumption 0(II) we take $U = u$ on $\mathbb{R}^+$ and $w = \lambda W$ on $\mathbb{R}^+$ for some $\lambda > 0$. The ensuing subcases follow the pattern in Section 3.

5.1. *Subcase* $\mathrm{A}^-\mathrm{A}^+$

We assume without loss of generality in this and later subcases of case (II) that scale units for u and W are specified, for example by $u(1) = W(1) = 1$ or in another convenient manner. Let $W_0' = \lim_{x\downarrow 0} W(x)/x$ and $W_\infty' = \lim_{x\to\infty} W(x)/x$. Then $u_0' > u_\infty'$ and $W_0' < W_\infty'$.

By Theorems 5 and 6, subcase $\mathrm{A}^-\mathrm{A}^+$ requires $u_0' \leq w_0'$. With $w = \lambda W$, there exists a $\lambda > 0$ for which $u_0' \leq w_0'$ if and only if u_0' is finite and W_0' is positive. The following axioms are necessary and sufficient for the latter properties.

Axiom A2.[5] There exist positive z and ϵ such that $z > \mu z \oplus (1 - \mu)\epsilon$ for all $0 \leq \mu < 1$.

Axiom A3. There exist negative z and ϵ such that $z > \mu z \oplus (1 - \mu)\epsilon$ for all $0 \leq \mu < 1$.

Theorem 7. *Assumption* 0(II) *holds for subcase* $\mathrm{A}^-\mathrm{A}^+$ *if and only if Axioms* 1*–3*, A2, *and* A3 *hold.*

Any λ for which $\lambda W_0' \geq u_0'$ suffices in proving this result to yield U with $U = u$ on $\mathbb{R}^+$ and $U(x) = \lambda[-W(-x)]$ on $\mathbb{R}^-$ that satisfies Assumption 0(II) for subcase $\mathrm{A}^-\mathrm{A}^+$. In succeeding cases, λ will be unique. Proofs of Theorem 7 and succeeding results in this section are found in Appendix D.

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[5] The axiom numbers continue those of Section 3 in the sense of repeating them when they are identical, and as a continuation of them when they are new.

5.2. *Subcase* $\mathrm{B^-B^+}$

Two axioms complete this subcase.

Axiom B1. $x \oplus y > x + y$ for all $(x, y) \in Q$.

Axiom B4. $\oplus$ *induces an additive conjoint structure on* $\mathbb{R} \times \mathbb{R}$.

Theorem 8. *Assumption* 0(II) *holds for subcase* $\mathrm{B^-B^+}$ *if and only if Axioms* 1*–3*, B1, *and* B4 *hold.*

5.3. *Other subcases of case* (*II*)

We consider the remaining four subcases of case (II) together and, similar to Axiom 4, approach the curves $y(x)$ and $x(y)$ directly: see Fig. 5. The four subcases in $\{\mathrm{G^-G^+}, \mathrm{G^-H^+}, \mathrm{H^-G^+}, \mathrm{H^-H^+}\}$ are differentiated by specific aspects of ψ (see below) or by limiting slope relationships, as discussed in the preceding section.

Axiom 4*. There exist $-x^{**} < y^* \le 0 \le x^* < x^{**}$, and a continuous curve ψ in Q, such that

(a) $(x^{**}, -x^{**}) \in \psi$;

(b) $x \in (x^*, x^{**}) \Rightarrow (x, y) \in \psi$ for a unique $y < -x$;

(c) $y \in (-x^{**}, y^*) \Rightarrow (x, y) \in \psi$ for a unique $x > -y$;

(d) all points in ψ are identified in (a), (b) and (c);

(e) $x^* > 0 \Rightarrow$ the y's of (b) approach $-\infty$ as $x \downarrow x^*$;

(f) $y^* < 0 \Rightarrow$ the x's of (c) approach ∞ as $y \uparrow y^*$;

(g) $x^* = 0 \Rightarrow$ as $x \downarrow 0$, the y's of (b) either approach $-\infty$ or have a unique finite limit point $y(0) < 0$;

(*h*) $y^* = 0 \Rightarrow$ as $y \uparrow 0$, the x's of (c) either approach ∞ or have a unique finite limit point $x(0) > 0$;

(i) for all $(x, y) \in Q$, $x \oplus y > x + y$ if and only if (x, y) lies in the region strictly between the origin and ψ.

In the preceding section, x of $(x, y) \in \psi$ was denoted by $x(y)$ when $x + y \ge 0$, and y of $(x, y) \in \psi$ was denoted by $y(x)$ when $x + y \le 0$. In what follows,

$$\mathscr{C} = \{\mathrm{G^-G^+}, \mathrm{G^-H^+}, \mathrm{H^-G^+}, \mathrm{H^-H^+}\}.$$

We use only one more axiom for $\mathscr{C}$. It is an inclusive conjoint structure axiom. To formulate it, let S_Q be the region in Q identified by $x \oplus y > x + y$ in Axiom 4*(i), let $S'_Q = \{(y, x) : (x, y) \in S_Q\}$ (recall that $y \oplus x = x \oplus y$), and let

$$\mathrm{c}(S) = (\mathbb{R}^+ \times \mathbb{R}^+) \cup (\mathbb{R}^- \times \mathbb{R}^-) \cup \mathrm{c}(S_Q) \cup \mathrm{c}(S'_Q).$$

Both Assumption 0(II) for a subcase in $\mathscr{C}$ and Axioms 1*–4* imply that c(S) is the closure of the region in $\mathbb{R} \times \mathbb{R}$ on which $x \oplus y > x + y$. Moreover, both also imply that there are positive-diameter disks centered at the origin which lie in c(S). Hence, even though c(S) may have an irregular shape in the mixed-sign regions, its interior is connected and has a corridor between $\mathbb{R}^+ \times \mathbb{R}^+$ and $\mathbb{R}^- \times \mathbb{R}^-$. We therefore postulate an additive conjoint structure throughout c(S).

Axiom 5*. $\oplus$ induces an additive conjoint structure over c(S).

Theorem 9. *Assumption* 0(II) *holds for a subcase in* $\mathscr{C}$ *if and only if Axioms* 1*–5* *hold.*

6. Discussion

In this paper we have examined the hedonic editing rule

$$U(x \oplus y) = \max\{U(x + y), U(x) + U(y)\}$$

for a joint-receipt operation $\oplus$ on pure sums in a riskless formulation for preference comparisons. We assumed that preference increases in amount and focused on the two main cases of (concave gains, convex losses) and (concave gains, concave losses) because of their intuitive plausibility, prevalence in the literature, and analytical tractability.

Concave gains imply the segregated option of $U(x \oplus y) = U(x) + U(y) > U(x + y)$ on the interior of the joint-gains region and provided a foothold for deriving U from axioms for $(\mathbb{R}, \oplus, \succsim)$. It is possible to have $U(x) + U(y) > U(x + y)$ throughout the joint-gains interior without U being concave everywhere on $\mathbb{R}^+$, but we did not explore this possibility.

The hedonic editing rule was investigated from two perspectives. The first focused on the rule's behavior in the mixed region $Q = \{(x, y): x > 0 \text{ and } y < 0\}$, where we discovered that its general behavior is governed by the limiting slopes of U at ± 0 and $\pm\infty$. Orderings of limiting slopes produce six different Q behaviors for each main case.

Our other perspective asked what must be true of $(\mathbb{R}, \oplus, \succsim)$ for U to be derived so as to satisfy the conditions of various subcases of the main cases. Our axioms made extensive use of closed extensive structures and additive conjoint structures from the theory of measurement. Other axioms were used to induce limiting slope inequalities, continuity, and concavity or convexity, and to describe the general features of the boundary in Q between its segregated and integrated regions.

Acknowledgments

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Appendix A: Proof of Theorem 1 on region Q for case (I)

Our proof of Theorem 1 derives the inverse mapping h of g, with

$$h(x) = y \Leftrightarrow g(y) = x ,$$

on an interval $K \subseteq \mathbb{R}^+$ for subcases C–F. As in the theorem and Fig. 3, K is open on the right and closed on the left with $h(\min K) = 0$. We obtain h as continuous and strictly decreasing over K with S below and I above the h curve in Q.

As an aid in the proof, we define a family of functions $\{f_x \text{ on } (-\infty, x]\colon x \geq 0\}$ by

$$f_x(y) = u(x) - w(x - y) = U(x) + U(y - x) \quad \text{for all } y \leq x .$$

f_0 is identical to U on losses. When $x > 0$, f_x is the translation of f_0 that results when f_0's right terminus $(0, 0)$ is moved to $(x, u(x)) = (x, f_x(x))$ on the u curve, with no rotation. We are interested in whether, and where, f_x intersects U to the left of $(x, u(x))$. Indeed, if $f_x(z) = U(z)$, then $U(z) = U(x) + U(z - x)$ or, with $y = z - x$, $U(x) + U(y) = U(x + y)$, so that z effectively determines the transition point from I to S as we move downward through Q on the vertical that begins at $(x, 0)$.

An informal description of what we shall show in the next few pages goes as follows. As x increases and the right terminus of f_0 translated to f_x slides up along the u curve:

- if f_x always lies below U except at $(x, u(x))$, then $U(x) + U(y - x) < U(y)$ for all $y < x$, and A obtains;
- if f_x always lies above U except at $(x, u(x))$, then $U(x) + U(y - x) > U(y)$ for all $y < x$, and B obtains;
- if f_x sometimes intersects U, then the x's for intersection form an interval K and, as x increases over this interval, the intersection points define h according to

  $$h(x) = z - x \text{ when } f_x(z) = U(z) .$$

Moreover, h is continuous and strictly decreasing, and the particular form of K, or of the nonintersection cases, is determined by the limiting slope orderings that characterize subcases A through F.

To be more rigorous, we begin by defining four cases of nonintersection and intersection of f_x and U for $x>0$.

Case 1: $f_x(y)>U(y)$ for all $y<x$.

Case 2: $f_x(z)=u(z)$ for some $0\leq z<x$.

Case 3: $f_x(z)=f_0(z)$ for some $z<0$.

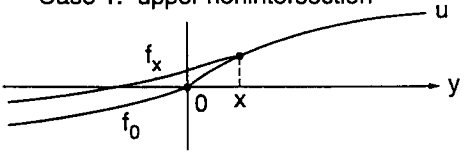
Case 4: $f_x(y)<U(y)$ for all $y<x$.

Fig. 7 pictures these cases. The vertical distance from $u(x)$ to the point on the ordinate at which f_x crosses the ordinate is $w(x)$, i.e. $f_x(0)=u(x)-w(x)$ by definition, so $w(x)=u(x)-f_x(0)$. Case 1 requires $u(x)>w(x)$, or $f_x(0)>0$, case 2 requires $u(x)\geq w(x)$, and cases 3 and 4 require $w(x)>u(x)$, or $f_x(0)<0$.

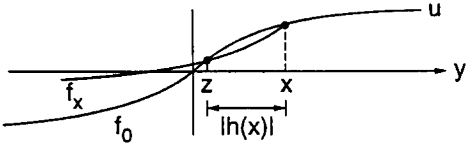
The following result is an easy consequence of monotonicity and the curvature properties. We assume $x>0$.

Lemma 3. *If* $u(x)\geq w(x)$ *then* f_x *intersects* u *at most once to the left of* x, *and* f_x

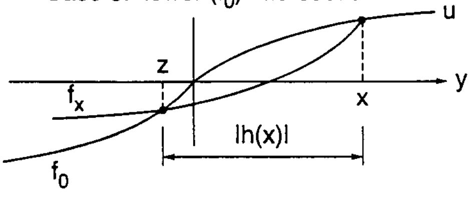
Case 1: upper nonintersection



Case 2: upper (u) intersection



Case 3: lower (f_0) intersection



Case 4: lower nonintersection

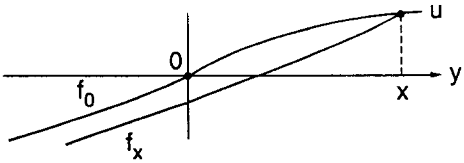


Fig. 7. Translations of f_0.

never intersects f_0 *to the left of* 0 (*cases* 1 *and* 2). *If* $w(x) > u(x)$ *then* f_x *never intersects* u, *and* f_x *intersects* f_0 *at most once* (*cases* 3 *and* 4).

Suppose f_x does intersect U at $z < x$. If $z_1 < z < z_2 < x$ then

$$f_x(z_1) > U(z_1)\,, \quad \text{or } U(x) + U(z_1 - x) > U(z_1)\,, \qquad [S]$$

$$f_x(z) = U(z)\,, \quad \text{or } U(x) + U(z - x) = U(z)\,, \qquad [I]$$

$$f_x(z_2) < U(z_2)\,, \quad \text{or } U(x) + U(z_2 - x) < U(z_2)\,. \qquad [I]$$

We then define $h(x)$ as $z - x < 0$ so that, on the vertical line down through Q from $(x, 0)$, I obtains for all (x, y) for which $h(x) \leq y \leq 0$, and S obtains for all (x, y) for which $y < h(x)$. When f_x never intersects U to the left of x, $h(x)$ is undefined.

Lemma 3 implies that h, where defined, is single-valued. Our next lemma begins an analysis of slopes. For $0 < x < \infty$, $u'_l(x)$ and $u'_r(x)$ denote the left and right derivatives respectively of u at x. Their existence is guaranteed by u being increasing and concave. The following applies to each $x > 0$.

Lemma 4. Case 1 *holds* $\Leftrightarrow w'_0 \leq u'_l(x)$
Case 2 *holds* $\Leftrightarrow u'_l(x) < w'_0$ *and* $u(x) \geq w(x)$
Case 3 *holds* $\Leftrightarrow w(x) > u(x)$ *and* $w'_\infty < u(x)/x$
Case 4 *holds* $\Leftrightarrow w(x) > u(x)$ *and* $w'_\infty \geq u(x)/x$.

Proof. It is easily seen that the curve of u lies above the curve of f_x immediately to the left of $(x, u(x))$ if and only if the left derivative of u at x is strictly less than w'_0, the initial slope of f_x at its right end. (See Fig. 8(a) for case 2 where $u(x) \geq w(x)$ then implies that f_x crosses u at the origin or to the right of the origin.) This and the first part of Lemma 3 give the results for cases 1 and 2 in Lemma 4. As remarked above, cases 3 and 4 require $w(x) > u(x)$. Then case 4 holds if and only if $f_0(y) > f_x(y)$ for all $y < 0$, which is tantamount to $f_0(y) > u(x) + f_0(y - x)$ for all $y < 0$, or to

$$w(x + z) - w(z) > u(x) \quad \text{for all } z > 0\,.$$

Divide both sides of this inequality by x. By Lemma 2, the limit of the left side as $z \to \infty$ is w'_∞ and, since that side decreases in z by the concavity of w, case 4 holds if and only if $w'_\infty \geq u(x)/x$. Hence, given $w(x) > u(x)$, case 3 holds if and only if $w'_\infty < u(x)/x$. □

Corollary 1. *Suppose* $0 < x_1 < x_2$. *If case* 1 *holds at* x_2, *it holds also at* x_1. *If case* 4 *holds at* x_1, *it holds also at* x_2.

Proof. Given $0 < x_1 < x_2$, suppose case 1 holds at x_2. By Lemma 4, $w'_0 \leq u'_l(x_2)$. Since u'_l decreases as x increases by Lemma 1(ii), it follows that $w'_0 < u'_l(x_1)$,

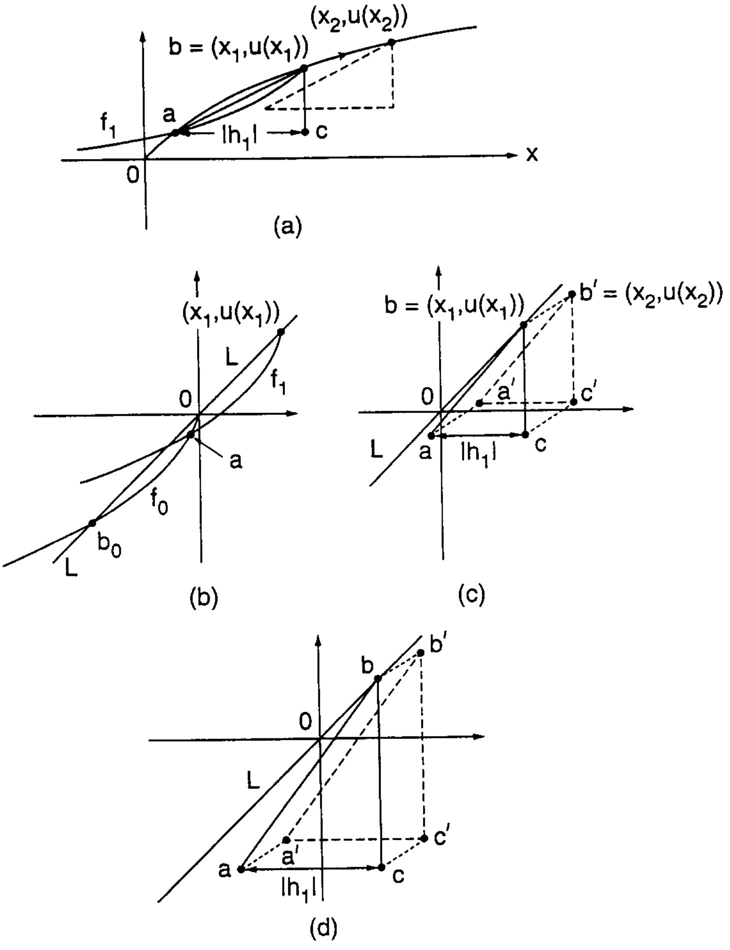


Fig. 8.

hence that case 1 holds at x_1. Suppose case 4 holds at x_1. By Lemma 4, $w'_\infty \geq u(x_1)/x_1$ and, since $u(x)/x$ by Lemma 1(iii) decreases in x, $w'_\infty > u(x_2)/x_2$. Because $w(x)/x > w'_\infty$ for all $x > 0$, we also have $w(x_2) > u(x_2)$, and conclude that case 4 holds at x_2. □

The next corollary is an easy consequence of Corollary 1 and Lemma 4. We elaborate its proof to observe the emergence of subcases A through F.

Corollary 2. *The subset of* $\mathbb{R}^+$ *on which* h *is defined with* $h(x) < 0$ *is either empty or a nondegenerate open interval.*

Remark. It is easily verified that, when h is defined on a nondegenerate interval, it is continuous on the interval. Moreover, as x decreases toward the infimum of the interval of definition, $h(x)$ approaches 0. We therefore extend h by including the infimum in its domain, with $h = 0$ at that point.

Proof of Corollary 2. By Corollary 1, case 4 either holds for no $x > 0$, for a left-closed interval $K_4 = [k, \infty)$ with $k > 0$, or for all $x > 0$. The last possibility obtains if and only if $w'_\infty \geq u'_0$, which is the defining inequality of subcase A, and for this subcase I holds throughout Q. When the K_4 alternative obtains, left closure follows from continuity of $u(x)/x$.

By Corollary 1, case 1 either holds for no $x > 0$, for a bounded nondegenerate right-closed interval $K_1 = (0, k']$, or for all $x > 0$. The last possibility holds by Lemma 4 if and only if $w'_0 \leq u'_\infty$, which is the defining inequality of subcase B. Because f_x lies above U for case 1, S holds throughout Q when B obtains. In the K_1 alternative, right closure of K_1 follows from the fact that u'_l is continuous from the left.

Suppose neither A nor B obtains. There are four combinations of the other than 'all $x > 0$' alternatives in the preceding paragraphs:

K_1	K_4	inequalities	h's domain	subcase
absent	absent	$w'_0 \geq u'_0,\ u'_\infty \geq w'_\infty$	$(0, \infty)$	C
absent	present	$w'_0 \geq u'_0,\ w'_\infty > u'_\infty$	$(0, k)$	D
present	absent	$u'_0 > w'_0,\ u'_\infty \geq w'_\infty$	(k', ∞)	E
present	present	$u'_0 > w'_0,\ w'_\infty > u'_\infty$	(k', k)	F

Because exclusion of A and B entails $u'_0 > w'_\infty$ and $w'_0 > u'_\infty$, the subcases identified here are those defined earlier. Their domains for h, the complements in $(0, \infty)$ of the unions of K_1 and K_4, are equivalent to the K intervals for C through F of Theorem 1 when we extend h to the infimum of its original domain as noted in the remark that follows Corollary 2. □

The proof of Theorem 1 is essentially completed by showing that h strictly decreases on the open interval where $h(x) < 0$ when neither A nor B obtains. Let (s_1, s_2) denote the interval as identified in the penultimate column of the above display. When s_2 is finite (D and F), monotonicity and openness imply that $h(x) \downarrow -\infty$ as $x \to s_2$. This corresponds to $t = -\infty$ in lines D and F of Theorem 1. Hence all parts of Theorem 1 are accounted for as soon as we prove the following lemma.

Lemma 5. *Suppose* $h(x) < 0$ *for all* $x \in (s_1, s_2)$*. Then* h *decreases over* (s_1, s_2).

Proof. Given the hypotheses, we are to show that $h(x_1) > h(x_2)$, or equivalently that $|h(x_1)| < |h(x_2)|$, whenever $s_1 < x_1 < x_2 < s_2$.

Assume that $s_1 < x_1 < x_2 < s_2$. For convenience let $h_i = h(x_i)$ and $f_i = f_{x_i}$.

Suppose f_1 crosses u (case 2): see Fig. 8(a). As the triangle abc is translated without rotation up and to the right so that b moves from $(x_1, u(x_1))$ to $(x_2, u(x_2))$, vertex a will fall beneath the u curve. Therefore $|h_1| < |h_2|$.

Assume henceforth that f_1 crosses f_0 below 0 (case 3). Let a be the intersection point: see Fig. 8(b). Also let L denote the line through $(x_1, u(x_1))$ and the origin. If f_0 never crosses L, a is to the right of L. If f_0 crosses L (see b_0), then f_1 will cross L above b_0 and, since f_0 is to the right of L between $(0, 0)$ and b_0, a is to the right of L.

Given a to the right of L, translate the triangle abc as before: see Fig. 8(c). Curve f_2 passes through a', which is also to the right of L. Hence if a' is to the right of the vertical line through $(0, 0)$ below the horizontal axis, or if a' is on or above the horizontal axis (as pictured), then $|h_1| < |h_2|$. The only possible way to have $|h_1| \geq |h_2|$ is for a' to be below $(0, 0)$ to the left of the ordinate.

Suppose this is so: see Fig. 8(d). We again translate abc to $a'b'c'$ with $b' = (x_2, u(x_2))$ so that f_2 passes through a'. Observe that a' is right of the line through a and b. If $|h_1| \geq |h_2|$, then f_0 must cut f_2 at or above right of a' (and must intersect line $a'c'$ at or to the right of a'). But then f_0 would also intersect f_1 above a', thus contradicting a as the sole intersection point of f_1 and f_0. Hence $|h_1| \geq |h_2|$ is impossible.

It follows that $|h_1| < |h_2|$ in all situations. □

Appendix B: Proofs of Theorems 2–4 on axioms for case (I)

Lemma 6. *If Assumption* 0(I) *holds, then Axioms* 1, 2 *and* 3 *hold.*

Proof. See the paragraph that follows Assumption 0 in Section 1 with regard to Axiom 1. Axiom 2(a) is implied by increasing u and the joint-receipt property (iv) of Assumption 0, and 2(b) follows from continuity. Axiom 2(c) follows from the fact that, by the concavity of U and Lemma 1(i),

$$U(x+y) < U(x) + U(y) < 2U\left(\frac{x+y}{2}\right),$$

and so

$$\begin{aligned} U\left(\frac{x+y}{2} \oplus \frac{x+y}{2}\right) &= \max\left[U(x+y), 2U\left(\frac{x+y}{2}\right)\right] \\ &= 2U\left(\frac{x+y}{2}\right) \\ &> U(x) + U(y) \end{aligned}$$

$$\begin{aligned}&= \max[U(x+y), U(x)+U(y)]\\ &= U(x \oplus y)\,.\end{aligned}$$

Axiom 2(d) follows from convexity of U on $\mathbb{R}^-$. Axiom 3 follows from the properties of U on $\mathbb{R}^+$ in Assumption 0, and from the implications for $\oplus$ and $\geq$ in Axioms 1 and 2. We refer to Krantz et al. (1971, p. 73) for further details. □

We begin our derivation of U with the following result on gains.

Lemma 7. *Axioms* 1–3 *imply the existence of a continuous, increasing and concave* $u\colon \mathbb{R}^+ \to \mathbb{R}^+$ *with* $u(0)=0$ *such that, for all* $x, y > 0$, $u(x \oplus 0) = u(x)$ *and*

$$u(x \oplus y) = u(x) + u(y) > u(x+y)\,,$$

and u thus specified is unique up to multiplication by a positive constant.

Proof. Let Axioms 1–3 hold. Theorem 3.1 in Krantz et al. (1971) implies that there exists a strictly increasing $u\colon \mathbb{R}^+ \to \mathbb{R}^+$ with $u(0)=0$ such that, for all $x, y > 0$, $u(x \oplus 0) = u(x)$ and

$$u(x \oplus y) = u(x) + u(y)\,,$$

with u unique up to multiplication by a positive constant. By Axiom 2(b), every $x > 0$ has an $\epsilon > 0$ for which $x \gtrsim \epsilon \oplus \epsilon$, i.e. $u(x)/2 \geq u(\epsilon) > 0$, and it follows that u is continuous at 0. Given $x, y \geq 0$ and $x \neq y$, Axiom 2(c) yields

$$u\left(\frac{x+y}{2}\right) > \tfrac{1}{2}u(x) + \tfrac{1}{2}u(y)\,.$$

It is well known that this mid-point concavity condition with u increasing implies that u is continuous and concave. Then Lemma 1 implies that $u(x) + u(y) > u(x + y)$ for all $x, y > 0$. □

The following lemma and subsequent results in this appendix presume Axioms 1–3 *with u on* $\mathbb{R}^+$ *as in Lemma* 7.

Lemma 8. $u_0' < \infty$ *if and only if Axiom* A2 *holds.*

Proof. Suppose Axiom A2 holds with z and ϵ as specified. For every $x \in [0, z)$ define $p(x)$ as the unique gain for which $x \oplus p(x) = z$, as implied by Lemma 7. Let $\lambda = x/z$. Then, by A2,

$$x \oplus p(x) = z > \lambda z \oplus (1-\lambda)\epsilon = x \oplus (1 - x/z)\epsilon\;;$$

hence, by monotonicity, $p(x) > (1 - x/z)\epsilon$, or equivalently $(z-x)/p(x) < z/\epsilon$. Because $u(z) = u(x) + u(p(x))$, we have, with the help of Lemma 1(iv),

$$\frac{u(p(x))}{p(x)} = \frac{u(z)-u(x)}{p(x)} = \left[\frac{u(z)-u(x)}{z-x}\right]\left[\frac{z-x}{p(x)}\right] < \left[\frac{u(z)}{z}\right]\left[\frac{z}{\epsilon}\right] = \frac{u(z)}{\epsilon} < \infty .$$

Since $p(x)\downarrow 0$ as $x\uparrow z$, it follows that

$$u_0' = \lim_{x\uparrow z} u(p(x))/p(x) \le u(z)/\epsilon ,$$

and we conclude that $u_0' < \infty$.

Conversely, suppose $u_0' < \infty$. Given arbitrary $z > 0$ and $0 \le \lambda < 1$, let $x = \lambda z$ and $\epsilon = zu_l'(z)/u_0'$. Then

$$\begin{aligned} u((1-\lambda)\epsilon) &= u((1-x/z)\epsilon) < u_0' \cdot (1-x/z)\epsilon = (z-x)u_l'(z) \\ &< (z-x)\frac{u(z)-u(x)}{z-x} = u(z)-u(x) , \end{aligned}$$

and therefore $u(\lambda z) + u((1-\lambda)\epsilon) < u(z)$. Lemma 7 says that this is the same as $u(\lambda z \oplus (1-\lambda)\epsilon) < u(z)$, so $\lambda z \oplus (1-\lambda)\epsilon < z$. Our definition of ϵ did not depend on λ, so we conclude that for every $z > 0$ there is a positive ϵ such that $z > \lambda z \oplus (1-\lambda)\epsilon$ for all $0 \le \lambda < 1$. Any one $z > 0$ shows that Axiom A2 holds. □

Necessity Proof of Theorem 2. The necessity of Axiom A2 follows from Lemma 8 and subcase A's defining characteristic of $w_\infty' \ge u_0'$. The necessity of A1 follows from Corollary 2 and its proof: see also the final paragraph of the next proof.

Sufficiency Proof of Theorem 2. Suppose Axioms A1 and A2 hold. Because u is concave and $u_0' < \infty$, there exists concave w on $\mathbb{R}^+$ with $w(0) = 0$ and $w_\infty' \ge u_0'$. Any such increasing w suffices for the representation of Asssumption 0(I) when we take $U(-x) = -w(x)$ for all $x \in \mathbb{R}^+$. In particular, U is then strictly convex on $\mathbb{R}^-$, and for negative x and y,

$$\max\{U(x+y), U(x)+U(y)\} = -w(-(x+y))$$

in accord with Axiom 2(d).

Axiom A1 says that $x \oplus y = x + y$ throughout Q, so to be consistent with part (iv) of Assumption 0 we require

$$\max\{U(x+y), U(x)+U(y)\} = U(x+y) \text{ in } Q .$$

Assume that $(x, y) \in Q$. If $x + y \ge 0$, we require $u(x+y) \ge u(x) - w(-y)$ or, with $z = x + y$, $w(x-z) \ge u(x) - u(z)$. This is true because concavity of w implies by Lemma 1 that

$$w(x-z) = w(x-z) - w(0) > w(x) - w(z) ,$$

and $w_\infty' \ge u_0'$ implies $w(x) - w(z) > u(x) - u(z)$. If $x + y \le 0$, we require $-w(-(x + y)) \ge u(x) - w(-y)$ or, with $z = -(x+y)$, $w(z+x) - w(z) \ge u(x)$. This also is true because, in view of Lemma 2 applied to w,

$$[w(z+x)-w(z)]/x > w'_\infty \geq u'_0 > u(x)/x\,. \quad \square$$

Necessity Proof of Theorem 3. Suppose Assumption 0(I) holds for subcase B. We have strictly increasing and concave u and w on $\mathbb{R}^+$ with $u(0)=w(0)=0$, $u'_\infty \geq w'_0$, $S_Q = Q$ (Theorem 1), and

$$U(x\oplus y) = U(x)+U(y) = u(x)+u(y) \qquad \text{for } x, y \geq 0\,,$$
$$U(x\oplus y) = U(x)+U(y) = u(x)-w(-y) \quad \text{for } x>0,\, y<0\,.$$

Axiom B1 is immediate from $S_Q = Q$, Axiom B2 follows directly from the preceding expressions for $U(x\oplus y)$, and Axiom B3 is implied by the latter expression and w's concavity as follows. The hypotheses of B3 give $u(x)=w(-y)$, $u(x')=w(-y')$, and $u(x^*)=w(-(y+y')/2)$. Because w is concave, we have $2w(-(y+y')/2) > w(-y)+w(-y')$. Hence $2u(x^*) > u(x)+u(x')$ which, by Lemma 7, implies $x^*\oplus x^* > x\oplus x'$. $\square$

Sufficiency Proof of Theorem 3. Assume that Axioms B1, B2 and B3 hold. By B2, there are U_1 on $\mathbb{R}^+$ and U_2 on $\mathbb{R}$ such that U_1+U_2 is an additive representation on $\mathbb{R}^+\times\mathbb{R}$ (see (4) of Section 3). Admissible transformations allow $U_1(0)=U_2(0)=0$, so assume this henceforth. It follows from $\mathbb{R}^+\times\mathbb{R}^+\subset\mathbb{R}^+\times\mathbb{R}$, Lemma 7, and scale alignment that we can presume

$$U_1 = U_2 = u \text{ on } \mathbb{R}^+\,,$$

where u has the properties specified in Lemma 7, including $u(x\oplus y)=u(x)+u(y)>u(x+y)$ for all $x, y>0$. We then define w by

$$w(-y) = -U_2(y) \text{ on } \mathbb{R}^-\,,$$

with $w(0)=0$ and w increasing on $\mathbb{R}^+$ because

$$\begin{aligned} 0\geq y>z &\Rightarrow -y\oplus y > -y\oplus z && \text{(by Axiom 1)}\\ &\Rightarrow u(-y)-w(-y) > u(-y)-w(-z) && \text{(by (4))}\\ &\Rightarrow w(-z)>w(-y)\,. \end{aligned}$$

Given u, w is uniquely defined, for if $y<0$ then $-y\oplus y>0$ by Axiom B1, so $u(-y)-w(-y)>0$ by (4), i.e. $u(-y)>w(-y)>0$. Because u is continuous, there exists x with $0<x<-y$ for which $u(x)=w(-y)$, and therefore w is unique. Since $u(x)-w(-y)=0=u(0)-w(0)$ for such an x, we have $x\oplus y = 0\oplus 0 = 0$.

Suppose $x>\epsilon>0$, so $(x,-\epsilon)\in Q$. By Axiom 2(a), $x\oplus(-\epsilon)\gtrsim x-\epsilon>0=0\oplus 0$, so (4) implies $u(x)-w(\epsilon)>0$, or $u(x)>w(\epsilon)>0$. Since u is continuous at 0, it follows that w is continuous at 0. Moreover, Axiom B3, (4), and the concavity of u, which gives $2u(x^*)>u(x)+u(x')$ via $x^*\oplus x^* > x\oplus x'$ in B3 under its hypotheses, imply that

$$w\left(\frac{-y-y'}{2}\right) > \tfrac{1}{2}w(-y)+\tfrac{1}{2}w(-y') \quad \text{for all } y, y'\leq 0,\, y\neq y'\,.$$

Hence, by analogy with the proof of Lemma 7, w is continuous and concave.

It remains to verify the hedonic rule (i.e. (1), or (iv) of Assumption 0(I)) on Q. If $(x, y) \in Q$ and $x + y \geq 0$, we have $x \oplus y > x + y = (x + y) \oplus 0$ by Axioms B1 and A1, so, by (4), $u(x) - w(-y) > u(x + y)$. If $(x, y) \in Q$ and $x + y < 0$, we have $x \oplus y > x + y = 0 \oplus (x + y)$, so (4) gives $u(x) - w(-y) > -w(-(x + y))$. Hence $U(x \oplus y) > U(x + y)$ throughout Q. □

Lemma 9. *If Assumption* 0(I) *holds for a subcase in* {C, D, E, F} *then Axiom* 4 *holds.*

Proof. The axiom is an easy consequence of Theorem 1. □

Lemma 10. *If Assumption* 0(I) *holds for a subcase in* {C, D, E, F}, *then axioms* 5, 6 *and* 7 *hold. Given Axioms* 1–7 *and* u *as in Lemma* 7, *there is a unique increasing, continuous and concave* $w: \mathbb{R}^+ \to \mathbb{R}^+$ *with* $w(0) = 0$ *such that, for all* $(x, y), (x', y') \in \mathrm{c}(S_Q)$,

$$x \oplus y > x' \oplus y' \Leftrightarrow u(x) - w(-y) > u(x') - w(-y') . \tag{5}$$

Proof. Suppose Assumption 0(I) holds for a subcase in {C, D, E, F}. We have strictly increasing continuous and concave u and w on $\mathbb{R}^+$ with $u(0) = w(0) = 0$, S_Q as characterized by Axiom 4, $U = u$ on $\mathbb{R}^+$, $U(y) = -w(-y)$ on $\mathbb{R}^-$, $U(x \oplus y) > U(x' \oplus y') \Leftrightarrow x \oplus y > x' \oplus y'$ throughout $\mathbb{R}^2$, and

$$U(x \oplus y) = u(x \oplus y) = u(x) + u(y) \quad \text{for all } (x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ ,$$
$$U(x \oplus y) = u(x) - w(-y) \quad \text{for all } (x, y) \in \mathrm{c}(S_Q) .$$

Axiom 5 follows immediately from the preceding line and order preservation for U for $\oplus$. The hypotheses of Axiom 6 give

$$u(x_1) - w(-y_1) = u(x_2) - w(-y_2) ,$$
$$u(x_2) - w(-y_1) = u(x_3) - w(-y_2) ,$$

so $u(x_1) + u(x_3) = 2u(x_2)$, i.e. $x_1 \oplus x_3 = x_2 \oplus x_2$. The hypotheses of Axiom 7 imply that there is a number α such that

$$u(x) - w(-y) = \alpha ,$$
$$u(x') - w(-y') = \alpha ,$$
$$u(x^*) - w\left(-\frac{y + y'}{2}\right) = \alpha .$$

Therefore $[2u(x^*) - u(x) - u(x')] - [2w(-(y + y')/2) - w(-y) - w(-y')] = 2\alpha - \alpha - \alpha = 0$. Concavity of w implies $2u(x^*) > u(x) + u(x')$, hence $x^* \oplus x^* > x \oplus x'$ as in the conclusion of Axiom 7.

Assume henceforth that Axioms 1–7 hold with u as in Lemma 7 and S_Q as characterized by Axiom 4. Let $R_1 = \mathrm{c}(S_Q)_1$. It follows from Axiom 5 and admissible transformations that there exist continuous real-valued functions U_1 on R_1 and U_2 on $\mathbb{R}^-$ which satisfy (4) on $\mathrm{c}(S_Q)$ with $U_1(0) = U_2(0) = 0$. U_1 increases on R_1, for if $x > x' \geq 0$ with $x \in R_1$, then $(x, y) \in \mathrm{c}(S_Q)$ for some $y < 0$, and Axiom 1 gives $x \oplus y > x' \oplus y$ so, by (4), $U_1(x) > U_1(x')$. U_2 also increases since if $0 \geq y > z$, then $(0, y)$, $(0, z) \in \mathrm{c}(S_Q)$, so $U_2(y) > U_2(z)$ by (4) on $\mathrm{c}(S_Q)$.

Suppose $x_3 > 0$, $x_3 \in R_1$. For large negative y_2 with $(x_3, y_2) \in S_Q$, continuity, monotonicity and (4) ensure the existence of $(x_2, y_1) \in S_Q$ with $0 < x_2 < x_3$ and $0 > y_1 > y_2$ such that $x_2 \oplus y_1 = x_3 \oplus y_2$, i.e.

$$U_1(x_3) - U_1(x_2) = U_2(y_1) - U_2(y_2).$$

If the difference here is suitably small, we also have $0 < x_1 < x_2$ such that $x_1 \oplus y_1 = x_2 \oplus y_2$ with $(x_1, y_1) \in S_Q$, so

$$U_1(x_2) - U_1(x_1) = U_2(y_1) - U_2(y_2).$$

The preceding equations imply

$$U_1(x_2) = \tfrac{1}{2}U_1(x_1) + \tfrac{1}{2}U_1(x_3),$$

and Axiom 6 yields $x_1 \oplus x_3 = x_2 \oplus x_2$, so by Lemma 7 we have

$$u(x_2) = \tfrac{1}{2}u(x_1) + \tfrac{1}{2}u(x_3).$$

By repeating this process within $(0, x_3]$ and letting $U_2(y_1) - U_2(y_2)$ be arbitrarily small, we saturate the interval $(0, x_3]$ with triples $x_1' < x_2' < x_3'$ for which

$$U_1(x_2') = \tfrac{1}{2}U_1(x_1') + \tfrac{1}{2}U_1(x_3') \quad \text{and} \quad u(x_2') = \tfrac{1}{2}u(x_1') + \tfrac{1}{2}u(x_3'),$$

and conclude that there is a $\lambda > 0$ such that

$$U_1(x) = \lambda u(x) \quad \text{for all } x \in [0, x_3].$$

We then increase $x_3 \in R_1$ to conclude that $U_1 = \lambda u$ on R_1.

Given $U_1 = \lambda u$ on R_1, define w on $\mathbb{R}^+$ by

$$w(-y) = -U_2(y)/\lambda \quad \text{for all } y \leq 0.$$

Then, by (4), for all (x, y), $(x', y') \in \mathrm{c}(S_Q)$,

$$x \oplus y > x' \oplus y' \Leftrightarrow u(x) - w(-y) > u(x') - w(-y'). \tag{5}$$

Given u, w is uniquely defined, and it is increasing and continuous with $w(0) = 0$.

It remains only to prove that w is concave. Axiom 7 and (5) on $\mathrm{c}(S_Q)$ imply

$$w\left(-\frac{y + y'}{2}\right) > \tfrac{1}{2}w(-y) + \tfrac{1}{2}w(-y')$$

whenever the hypotheses of Axiom 7 hold as stated. Moreover, if this w inequality holds on the adjacent triples in the series $r_1 > r_2 > r_3 > r_4 > r_5 > 0$ of evenly spaced points, i.e. if

$$w(r_2) > \tfrac{1}{2}w(r_1) + \tfrac{1}{2}w(r_3)\,, \quad r_2 = (r_1 + r_3)/2\,,$$
$$w(r_3) > \tfrac{1}{2}w(r_2) + \tfrac{1}{2}w(r_4)\,, \quad r_3 = (r_2 + r_4)/2\,,$$
$$w(r_4) > \tfrac{1}{2}w(r_3) + \tfrac{1}{2}w(r_5)\,, \quad r_4 = (r_3 + r_5)/2\,,$$

then it holds also for r_1, r_3 and r_5 by substitution of the first and third inequalities on the right hand side of the second inequality. For any given $y_0 < 0$ there is an interval $[y_0, z]$ with $y_0 < z \leq 0$ such that the hypotheses of Axiom 7 hold for every y and y' for which $y_0 \leq y < y' \leq z$. By using the same small $x_3 > 0$ throughout, a finite series of such steps (the first of which replaces y_0 by z when $z < 0$) allows the ultimate z to be arbitrarily close to 0 (as $x_3 \to 0$). It then follows that the original inequality in this paragraph holds for all $y < y' < 0$ and, as in the penultimate paragraph of the sufficiency proof of Theorem 3, w on $\mathbb{R}^+$ is concave. □

Lemma 11. *If Assumption* 0(I) *holds for a subcase in* {C, D, E, F} *then Axiom* 8 *holds. Given Axioms* 1–8 *and u and w as in the latter part of Lemma* 10, *for all* $y \in (t, 0)$

$$g(y) + y \leq 0 \Rightarrow u(g(y)) - w(-y) = -w(-(g(y) + y))\,, \tag{6a}$$
$$g(y) + y > 0 \Rightarrow u(g(y)) - w(-y) = u(g(y) + y)\,. \tag{6b}$$

Proof. Suppose Assumption 0(I) holds for a subcase in {C, D, E, F}, and $t < y < 0$, $g(y) + y > 0$ and $x \oplus y = 0$. Part (iv) of Assumption 0 implies $u(z) - w(-y) > u(z + y)$ when $-y \leq z < g(y)$, and $u(z + y) \geq u(z) - w(-y)$ when $z \geq g(y)$, so continuity of u gives

$$u(g(y) + y) = u(g(y)) - w(-y)\,.$$

By our hypotheses and (iv), $g(y) + y > 0 = x \oplus y \geq x + y$, so $g(y) > x$. Therefore $(x, y) \in S_Q$, so $x \oplus y = 0 > x + y$ and $U(x \oplus y) = 0 = u(x) - w(-y) > U(x + y)$. Then, since $w(-y) = u(x)$,

$$u(g(y) + y) + u(x) = u(g(y))\,,$$

and therefore Lemmas 6 and 7 yield $(g(y) + y) \oplus x = g(y) \oplus 0 = g(y)$, the conclusion of Axiom 8.

Suppose the hypotheses of the latter part of Lemma 11 hold. To verify (6a), suppose $g(y) + y \leq 0$. By Axioms 2(a) and 4, $g(y) \oplus y = g(y) + y = 0 \oplus (g(y) + y)$. Since $(g(y), y)$, $(0, g(y) + y) \in c(S_Q)$, (5) implies (6a). Suppose $t < y < 0$ and $g(y) + y > 0$ as in (6b). Continuity and (5) imply that there is an x with $0 < x < g(y)$ such that $x \oplus y = 0$, i.e. $u(x) - w(-y) = 0$. Hence, by Axiom 8, $(g(y) + y) \oplus x = g(y) = g(y) \oplus 0$, which yields $u(g(y) + y) + u(x) = u(g(y))$ by Lemma 7. Therefore $u(g(y) + y) = u(g(y)) - w(-y)$, which is (6b). □

Proof of Theorem 4. The necessity of Axioms 1–8 was verified by Lemmas 6 and

9–11. Lemma 7 and the latter part of Lemma 10 imply all aspects of Assumption 0(I) for a subcase in {C, D, E, F} except for part (iv) in Q. Given $(x, y) \in Q$, we partition that part into the following four pieces:

p1. $[x+y \leq 0, (x,y) \in S_Q] \Rightarrow u(x) - w(-y) > -w(-(x+y))$,
p2. $[x+y \leq 0, (x,y) \not\in S_Q] \Rightarrow -w(-(x+y)) \geq u(x) - w(-y)$,
p3. $[x+y > 0, (x,y) \in S_Q] \Rightarrow u(x) - w(-y) > u(x+y)$,
p4. $[x+y > 0, (x,y) \not\in S_Q] \Rightarrow u(x+y) \geq u(x) - w(-y)$.

These implications are established as follows.

p1. The hypotheses and Axiom 4 give $x \oplus y > x+y = 0 \oplus (x+y)$ with (x,y), $(0, x+y) \in \mathrm{c}(S_Q)$. Hence $u(x) - w(-y) > -w(-(x+y))$ by (5).

p2. Since $(x,y) \not\in S_Q$, $x \geq g(y)$. Let $x = g(y) + \Delta$. Since $x+y \leq 0$, $g(y)+y \leq 0$, so (6a) of Lemma 11 implies $u(g(y)) = w(-y) - w(-(g(y)+y))$. By Lemma 1(ii),

$$\frac{u(g(y)+\Delta)}{g(y)+\Delta} \leq \frac{u(g(y))}{g(y)} = \frac{w(-y) - w(-(g(y)+y))}{g(y)}$$
$$\leq \frac{w(-y) - w(-(g(y)+\Delta+y))}{g(y)+\Delta}.$$

Therefore $u(x) \leq w(-y) - w(-(x+y))$.

p3. Suppose $x+y > 0$ and $(x,y) \in S_Q$. Then $x \oplus y > x+y$ by Axiom 4. If $(x+y, 0) \in \mathrm{c}(S_Q)$, (5) implies $u(x) - w(-y) > u(x+y)$. Suppose $(x+y, 0) \not\in \mathrm{c}(S_Q)$. Then $g(v) = x$ for some $y < v < 0$. By (6b), $u(g(v)) - w(-v) = u(g(v)+v)$, so concavity yields

$$\frac{w(-y)}{-y} < \frac{w(-v)}{-v} = \frac{u(x) - u(x+v)}{-v} < \frac{u(x) - u(x+y)}{-y}.$$

Therefore $u(x) - u(x+y) > w(-y)$.

p4. Suppose $x+y > 0$ and $(x,y) \not\in S_Q$. Let $x = g(y) + \Delta$, $\Delta \geq 0$. If $g(y)+y \leq 0$, (6a) gives $u(g(y)) - w(-y) = -w(-(g(y)+y))$ so, by concavity

$$\frac{u(g(y)+\Delta) - u(g(y)+\Delta+y)}{-y} \leq \frac{u(g(y))}{g(y)} = \frac{w(-y) - w(-(g(y)+y))}{g(y)}$$
$$\leq \frac{w(-y)}{-y}.$$

Hence $u(x) - u(x+y) \leq w(-y)$. If $g(y)+y > 0$, (6b) and concavity of u give the same conclusion. □

Appendix C: Proofs of Theorems 5 and 6 on region Q for case (II)

Proof of Lemma 12. Given $x > 0$, suppose $y < 0$ satisfies $x+y \leq 0$ and $-w(-(x+y)) = u(x) - w(-y)$. This equation is identical to $f_0(x+y) = f_x(x+y)$, so that $y(x)$ exists if and only if f_x intersects f_0 at or to the left of the origin. Concavity of f_0

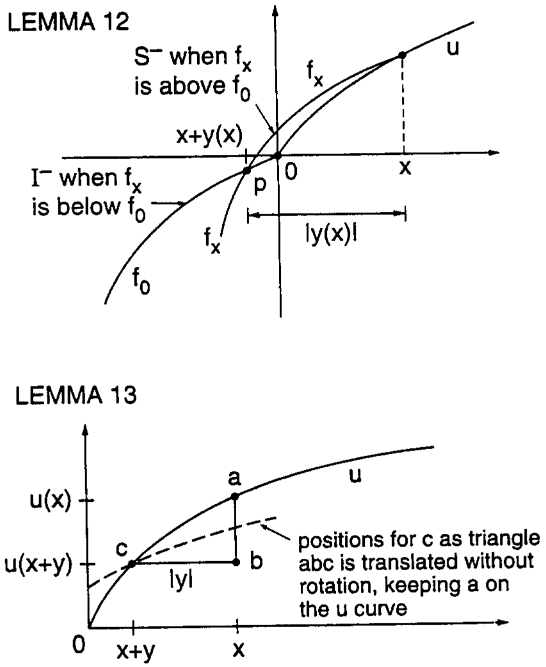


Fig. 9.

guarantees no more than one such intersection for a given $x > 0$: see p on the upper part of Fig. 9. The assertions for S^- and I^- follow easily. □

Proof of Lemma 13. Given $y < 0$, suppose $x > 0$ satisfies $x + y \geq 0$ and $u(x + y) = u(x) - w(-y)$, i.e. $u(x+y) = f_x(x+y)$. Hence $x(y)$ is defined if and only if f_x crosses or touches (by tangency) u at $x + y$. This happens at most once for a given y, for consider the triangle abc at the bottom of Fig. 9 with $|y|$ the distance between b and c, and $c = (x + y, u(x + y))$. Suppose $u = f_x$ at c. Translate the triangle without rotation by sliding point a along the u curve. Then c will move along the dashed curve, by concavity, and $u(z+y) \neq f_z(z+y)$ for all $z \neq x$, $z + y \geq 0$. In particular,

$$-y \leq z < x \Rightarrow f_z(z+y) > u(z+y) \Rightarrow (z, y) \in S^+ ,$$

$$x \leq z \Rightarrow u(z+y) \geq f_z(z+y) \Rightarrow (z, y) \in I^+ ,$$

with strict inequality throughout the latter when $x \neq z$. □

Lemma 13 can also be proved from Lemma 12 and the symmetry induced by the concave–concave structure. We explain and exploit this symmetry later in the following proof.

Proof of Theorems 5 and 6. We prove only Theorem 5 in detail because Theorem 6 follows from Theorem 5 by symmetry around the $-45°$ line through the origin that is induced by U being concave on both $\mathbb{R}^+$ and $\mathbb{R}^-$. If we look at U from a point down on the $-45°$ line in Q, the left-to-right and right-to-left picture scans are similar. The only discordant note in this viewing occurs when u is bounded above since U on $\mathbb{R}^-$, or f_0, cannot be bounded on the left. This does not affect the general behaviors based on orderings of limiting slopes although it can affect fine details not covered by the theorems.

The algebraic transformations between the Q^- focus of Theorem 5 and the Q^+ focus of Theorem 6 can be described as follows. The defining conditions for $(x, y) \in S^-$ are

$$x > 0,\ y < 0,\ x + y \le 0 \quad \text{and} \quad U(x) + U(y) > U(x + y)\,.$$

If we let $t = U(x)$, then the pair $(x, U(x))$ is $(U^{-1}(t), t)$. The symmetry described in the preceding paragraph maps this into $(-t, -U^{-1}(t))$ because axes are interchanged along with the $+$, $-$ orientations. The $x > 0 \ldots$ line displayed above then maps into

$$-t > 0,\ -s < 0,\ -t - s \le 0 \quad \text{and} \quad -U^{-1}(t) - U^{-1}(s) > -U^{-1}(t + s)\,.$$

Let $V = -U^{-1}$. The preceding line is

$$s > 0,\ t < 0,\ t + s \ge 0 \quad \text{and} \quad V(t) + V(s) > V(t + s)\,,$$

with V concave on $t \ge 0$ and on $t \le 0$. Moreover, V is increasing and continuous with $V(0) = 0$. Hence V has the same behavior as U, assuming that U is not bounded above, and if we now replace (s, t, V) in the preceding display by (x, y, U) we obtain

$$x > 0,\ y < 0,\ x + y \ge 0 \quad \text{and} \quad U(x) + U(y) > U(x + y)\,,$$

which gives the defining conditions for $(x, y) \in S^+$.

As suggested earlier, the symmetry between Q^- and Q^+ is directly reflected in the defining characterizations of $\{A^-, B^-, G^-, H^-\}$ and $\{A^+, B^+, G^+, H^+\}$. The net effect on limiting slopes of the transformation from Q^- to Q^+ replaces (u, w) by $(-w, -u)$. In particular,

$$(u_0' \le w_0') \text{ goes into } (u_0' \le w_0')\,,$$
$$(w_\infty' \le u_\infty') \text{ goes into } (w_\infty' \le u_\infty')\,,$$
$$(u_\infty' < w_\infty' \le u_0') \text{ goes into } (w_0' \le u_\infty' < w_\infty')$$
$$(w_0' < u_0' < w_\infty') \text{ goes into } (u_\infty' < w_0' < u_0')\,.$$

These four lines begin with the slope characterizations in Theorem 5 and conclude with those in Theorem 6. Therefore Theorem 6 is a direct consequence of Theorem 5.

We now prove Theorem 5. By Lemma 12 and its proof, $y(x)$ fails to exist at $x > 0$ if and only if either

(a) $u(x) < w(x)$ (or $f_x(0) < 0$), in which case I obtains in Q^- on the vertical line down from $(x, -x)$, or

(b) $w(x - y) - w(-y) < u(x)$ for all $y \leq 0$ [or $f_0(y) < f_x(y)$ for all $y \leq 0$], in which case S obtains in Q^- on the vertical down from $(x, -x)$.

Convexity of w (the analog of Lemma 1(ii) for convex functions) and Lemma 2 imply

$$\left[\frac{w(x-y) - w(-y)}{x} < \frac{u(x)}{x} \text{ for all } y \leq 0\right] \Leftrightarrow [w'_\infty \leq u(x)/x] ,$$

so (b) is the same as $w'_\infty \leq u(x)/x$. We record the following implication of (a) and (b) for use later:

(c) $y(x)$ exists for $x > 0 \Leftrightarrow [w(x)/x \leq u(x)/x < w'_\infty]$.

Part A$^-$ of Theorem 5 says that $I^- = Q^- \Leftrightarrow u'_0 \leq w'_0$. By (a), (b) and Lemma 12, $I^- = Q^- \Rightarrow [u(x) \leq w(x)$ for all $x > 0] \Rightarrow u'_0 \leq w'_0$. Conversely, if $u'_0 \leq w'_0$, we have $u(x)/x < u'_0 \leq w'_0 < w(x)/x$ by the curvatures (i.e. concavity of u and convexity of w), hence we have $u(x) < w(x)$ for all $x > 0$, so $I^- = Q^-$ by (a).

Part B$^-$ of Theorem 5 says that $S^- = Q^- \Leftrightarrow w'_\infty \leq u'_\infty$. By (a), (b) and Lemma 12, $S^- = Q^- \Rightarrow [f_0(y) < f_x(y)$ for all $y \leq 0$ and all $x > 0] \Rightarrow [w'_\infty \leq u(x)/x$ for all $x > 0] \Rightarrow w'_\infty \leq u'_\infty$. If $w'_\infty \leq u'_\infty$ then curvatures and Lemma 2 yield

$$\frac{w(x-y) - w(-y)}{x} < w'_\infty \leq u'_\infty < \frac{u(x)}{x} \text{ for all } x > 0, \quad \text{all } y \leq 0 ,$$

and therefore $S^- = Q^-$ by (b).

We assume henceforth in the proof that neither A$^-$ nor B$^-$ holds, so

$$\{w'_0, u'_\infty\} < \{u'_0, w'_\infty\} .$$

Then lines three and four of Theorem 5 say that $\text{G}^- \Leftrightarrow w'_\infty \leq u'_0$ and $\text{H}^- \Leftrightarrow u'_0 < w'_\infty$. We show next that, regardless of whether $w'_\infty \leq u'_0$ or $u'_0 < w'_\infty$,

(d) $y(x)$ is defined on a finite interval $(x^*, x^{**}]$ with $0 \leq x^* < x^{**}$, it is continuous on this interval, (b) holds for all $0 < x \leq x^*$, (a) holds for all $x > x^{**}$, and $y(x^{**}) = -x^{**}$.

The claims about S^- made in the defining characterizations of G$^-$ and H$^-$ follow directly from Lemma 12, (a), (b) and (d). The G$^-$ versus H$^-$ distinction is noted below.

Proof of (d). Because w is convex, u is concave, $w'_0 < u'_0$ and $u'_\infty < w'_\infty$, there is a unique $z = x^{**} > 0$ at which $u(z) = w(z)$, so for $x > 0$

$$w(x) \leq u(x) \Leftrightarrow x \leq x^{**}, \quad \text{with } w(x) = u(x) \text{ at } x = x^{**} .$$

Because $u'_\infty < w'_\infty$ and $u(z)/z$ decreases in z, there is a unique $z = x^* \geq 0$ such that for $x > 0$

$$u(x)/x < w'_\infty \Leftrightarrow x > x^* .$$

Suppose $0 < x^{**} \leq x^*$. Then we have $w(x^{**})/x^{**} = u(x^{**})/x^{**} \geq u(x^*)/x^* = w'_\infty$, contrary to convexity of w. Therefore

$$x^* < x^{**}$$

and, by (c), $y(x)$ is defined on $(x^*, x^{**}]$. The defining equation for $y(x)$ in Lemma 12, i.e. $u(x) - w(-y(x)) = -w(-(x + y(x)))$, ensures that $y(x)$ is continuous.

If $0 < x \leq x^*$, then $w'_\infty \leq u(x)/x$ and (b) obtains. If $x > x^{**}$ then $u(x) < w(x)$ and (a) obtains. Since $w(x^{**}) = u(x^{**})$, hence $u(x^{**}) - w(-x^{**}) = -w(-(x^{**} - x^{**})) = 0$, we have $y(x^{**}) = -x^{**}$. □

We conclude by proving

(e_1) $w'_\infty < u'_0 \Rightarrow x^* > 0$, and $y(x)\downarrow -\infty$ as $x \downarrow x^*$;

(e_2) $w'_\infty = u'_0 \Rightarrow x^* = 0$, and $y(x)\downarrow -\infty$ as $x \downarrow 0$;

(e_3) $u'_0 < w'_\infty \Rightarrow x^* = 0, \lim_{x\downarrow 0} y(x) = y(0) > -\infty$, where $(y(0), f_0(y(0)))$ is the point on f_0 at which a line with slope u'_0 touches f_0 but does not otherwise intersect f_0.

Because the hypotheses of (e_1)–(e_3) are mutually exclusive and exhaustive, and their conclusions differ, the implications go right to left also and complete the proof of Theorem 5.

(e_1) Suppose $w'_\infty < u'_0$. Then (b) holds for small $x > 0$, so $x^* > 0$ and $w'_\infty = u(x^*)/x^*$. By the defining equation for $y(x)$ and u's concavity, we have

$$\frac{w(-y(x)) - w(-(x + y(x)))}{x} = \frac{u(x)}{x} < \frac{u(x^*)}{x^*} = w'_\infty , \quad x \in (x^*, x^{**}] .$$

Since $u(x)/x \uparrow u(x^*)/x^*$ as $x \downarrow x^*$,

$$\frac{w(-y(x)) - w(-(x + y(x)))}{x} \uparrow w'_\infty \text{ as } x \downarrow x^* .$$

Lemma 2 says that the limit of the ratio is indeed w'_∞ if $-y(x) \rightarrow \infty$ as $x \downarrow x^*$. Moreover, if $y(x)$ has a finite limit point z as $x \downarrow x^*$, we contradict the given convergence. Hence $y(x)\downarrow -\infty$ as $x \downarrow x^*$.

(e_2) If $u'_0 = w'_\infty$, then $u(x)/x < w'_\infty$ for all $x > 0$, so $x^* = 0$. The convergence proof for this case is similar to that for (e_1).

(e_3). Suppose $u'_0 < w'_\infty$. As in (e_2), $x^* = 0$. For small $x > 0$ we have

$$\frac{w(-y(x)) - w(-(x + y(x)))}{x} = \frac{u(x)}{x} \downarrow u'_0 < w'_\infty \text{ as } x \downarrow 0 .$$

It follows that there is a unique point $(-y(0), w(-y(0)))$ on the w curve such that the line with slope u'_0 through this point intersects the w curve nowhere else.

Moreover, by considering left and right derivatives of w near this point, it is apparent that $y(x)$ converges to $y(0)$ as $x \to 0$. □

Appendix D: Proofs of Theorems 7–9 on axioms for case (II)

Lemma 14. *If Assumption* 0(II) *holds then Axioms* 1*, 2* *and* 3* *hold.*

Proof. The necessity of the axioms for Assumption 0(II) follows from Lemma 6 except for Axioms 2*(d), 2*(e) and the $\mathbb{R}^-$ part of Axiom 3*. The latter follows from $U(x) + U(y) > U(x + y)$ on $x, y < 0$ by concavity of U on $\mathbb{R}^-$ and Lemma 1, via (iv) of Assumption 0(II) which therefore requires $U(x \oplus y) = U(x) + U(y)$ for $x, y \leq 0$. When $x < 0$, increasing U and continuity of U at 0 imply $U(\epsilon \oplus \epsilon) = 2U(\epsilon) \geq U(x)$ for some $x < \epsilon < 0$, hence $\epsilon \oplus \epsilon \geq x$ for some $\epsilon < 0$: Axiom 2*(d) holds. By concavity of U,

$$(x, y \leq 0, x \neq y) \Rightarrow \left[U\left(\frac{x+y}{2} \oplus \frac{x+y}{2}\right) = 2U\left(\frac{x+y}{2}\right) > U(x) + U(y) \right.$$
$$\left. = U(x \oplus y) \right],$$

so Axiom 2*(e) holds. □

Lemma 15. *Axioms* 1*–3* *imply the existence of a continuous, increasing and concave* $u: \mathbb{R}^+ \to \mathbb{R}^+$ *and a continuous, increasing and convex* $W: \mathbb{R}^+ \to \mathbb{R}^+$ *with* $u(0) = W(0) = 0$, $u(x \oplus 0) = u(x)$ *and* $W(x \oplus 0) = W(x)$ *for all* $x > 0$, *and, for all* $x, y > 0$,

$$u(x \oplus y) = u(x) + u(y) > u(x + y)\,,$$
$$W(-[(-x) \oplus (-y)]) = W(x) + W(y) < W(x + y)\,.$$

Each of u and W thus specified is unique up to multiplication by a positive constant.

Proof. See Lemma 7 for the u part of the theorem. By analogy, Axioms 1*, 2*(a,d,e) and the $\mathbb{R}^-$ part of Axiom 3* imply the existence of a continuous, increasing and concave $V: \mathbb{R}^- \to \mathbb{R}^-$ with $V(0) = 0$, $V(x \oplus 0) = V(x)$ for all $x < 0$, and $V(x \oplus y) = V(x) + V(y) > V(x + y)$ for all $x, y < 0$, with V unique up to multiplication by a positive constant. This translates into the W assertions of Lemma 15 when we define W by

$$W(-x) = -V(x) \quad \text{for all } x \leq 0\,.$$

Note in particular that for $x, y > 0$, $-V(-x) - V(-y) = W(x) + W(y)$ and $-V((-x) \oplus (-y)) = W(-[(-x) \oplus (-y)])$. □

Proof of Theorem 7. Lemma 8 and its proof, which involves only gains, show that Axiom A2 is necessary and sufficient for $u_0' < \infty$.

To consider Axiom A3 for losses, let V on $\mathbb{R}^-$ be as specified in the proof of Lemma 15: $V(x) = -W(-x)$ for all $x \le 0$. Because $V_0' = \lim_{x\uparrow 0} V(x)/x = W_0'$, we show that A3 is necessary and sufficient for $V_0' > 0$.

Suppose Axiom A3 holds for z and ϵ as specified, and for every $x \in (z, 0]$ let $p(x)$ be the unique loss for which $x \oplus p(x) = z$. For $\mu = x/z$, Axiom A3 gives

$$x \oplus p(x) = z > \mu z \oplus (1-\mu)\epsilon = x \oplus (1 - x/z)\epsilon\ ,$$

so $p(x) > (1 - x/z)\epsilon$ by monotonicity. Since $p < 0$, this is the same as $(z-x)/p(x) > z/\epsilon$. Because $V(z) = V(x) + V(p(x))$ by Lemma 15, and $[V(z) - V(x)]/(z-x) > V(z)/z$ by concavity in losses,

$$\frac{V(p(x))}{p(x)} = \frac{V(z) - V(x)}{p(x)} = \left[\frac{V(z) - V(x)}{z - x}\right]\left[\frac{z - x}{p(x)}\right] > \left[\frac{V(z)}{z}\right]\left[\frac{z}{\epsilon}\right] = \frac{V(z)}{\epsilon} > 0\,.$$

Since $p(x)\uparrow 0$ as $x \downarrow z$, $V_0' > 0$.

Suppose $V_0' > 0$. Given $z < 0$ and $0 \le \mu < 1$, let $x = \mu z$ and $\epsilon = zV_r'(z)/V_0'$, where V_r' denotes right derivative. Then

$$\begin{aligned} V((1-\mu)\epsilon) &< V_0'((1 - x/z)\epsilon) = (z-x)V_r'(z) < (z-x)[V(z) - V(x)]/(z-x) \\ &= V(z) - V(x)\,. \end{aligned}$$

Therefore $V(\mu z \oplus (1-\mu)\epsilon) = V(x) + V((1-\mu)\epsilon) < V(z)$, so $\mu z \oplus (1-\mu)\epsilon < z$. This is true for every $\mu \in [0, 1)$, so Axiom A3 holds.

Assume that Axioms A2 and A3 hold. We have $W_0' > 0$ and $\mu_0' < \infty$. Let $\lambda > 0$ be such that, with $w = \lambda W$, $w_0' = \lambda W_0' \ge \mu_0'$. Subcase $\mathrm{A}^-\mathrm{A}^+$ requires I throughout Q, so we want

$$U(x+y) \ge U(x) + U(y) \quad \text{for all } x > 0 \text{ and } y < 0\,.$$

If $x + y \le 0$, we want $-w(-(x+y)) \ge u(x) - w(-y)$, and this is true because

$$\frac{w(-y) - w(-(x+y))}{x} > w_0' \ge u_0' > \frac{u(x)}{x}\,.$$

If $x + y \ge 0$, we want $u(x+y) \ge u(x) - w(-y)$, and this follows from $w(-y)/(-y) > w_0' \ge u_0' > (u(x) - u(x+y))/(-y)$. Hence part (iv) of Assumption 0(II) holds on Q and hence everywhere as required for subcase $\mathrm{A}^-\mathrm{A}^+$. □

Proof of Theorem 8. The necessity of Axiom B1 follows from Theorems 5 and 6 for $\mathrm{B}^-\mathrm{B}^+$, and Axiom B4 is immediate from the fact that $\mathrm{B}^-\mathrm{B}^+$ requires $U(x \oplus y) = U(x) + U(y)$ throughout $\mathbb{R} \times \mathbb{R}$.

Suppose Axioms B1 and B4 hold. It follows from Axiom B4, (4) and Lemma 15 that $U_1 = U_2$ on $\mathbb{R}$ with $U_1 = \lambda^+ u$ on $\mathbb{R}^+$ and $U_1 = \lambda^- V$ on $\mathbb{R}^-$, where $V(x) = -W(-x)$. Taking $\lambda^+ = 1$ without loss of generality, λ^- is then uniquely determined. Let $U = U_1$, so $U(x \oplus y) = U(x) + U(y)$ everywhere by (4) and Axiom B4. When $(x, y) \in Q$, Axiom B1 gives $x \oplus y > x + y = (x + y) \oplus 0$, so $U(x) + U(y) = U(x \oplus y) > U(x + y) + U(0) = U(x + y)$. Hence (iv) holds throughout $\mathbb{R} \times \mathbb{R}$ in the manner required for subcase $\mathrm{B}^-\mathrm{B}^+$. □

Proof of Theorem 9. The necessity of Axiom 4* follows directly from Theorems 5 and 6. The necessity of Axiom 5* follows from the fact that when Assumption 0(II) holds for a subcase in $\mathscr{C}$, $U(x \oplus y) = U(x) + U(y) \geq U(x + y)$ for all $(x, y) \in \mathrm{c}(S)$, and therefore $x \oplus y \geq x' \oplus y' \Leftrightarrow U(x) + U(y) \geq U(x') + U(y')$ for all (x, y), $(x', y') \in \mathrm{c}(S)$.

Conversely, suppose that Axioms 1*–5* hold. Lemma 15, the inclusive additive conjoint structure of Axiom 5*, and the nature of S_Q by Axiom 4*, which gives unique determination of $\lambda > 0$ for $w = \lambda W$, routinely imply all aspects of Assumption 0(II) for a subcase in $\mathscr{C}$ excepting (iv) on Q. We verify (iv), using U, u and w in the usual manner.

Suppose $(x, y) \in S_Q$. By Axioms 4* and 1*, $x \oplus y > x + y = (x + y) \oplus 0$. Because (x, y), $(x + y, 0) \in \mathrm{c}(S)$, we have $U(x) + U(y) = U(x \oplus y) > U(x + y) + U(0) = U(x + y)$, hence $U(x) + U(y) > U(x + y)$ on S_Q.

Suppose $(x, y) \in \psi$ on the boundary of S_Q. By Axiom 2*(a), $x \oplus y \gtrsim x + y$. By Axiom 4*(i), $x \oplus y \lesssim x + y$. Hence $x \oplus y \sim x + y$, and because (x, y), $(x + y, 0) \in \mathrm{c}(S)$ we get $U(x + y) = U(x \oplus y) = U(x) + U(y)$. Therefore

$$(x, y) \in \psi \Rightarrow U(x) + U(y) = U(x + y) .$$

It remains to prove that when $(x, y) \in Q \backslash \mathrm{c}(S_Q)$, i.e. on the opposite side of ψ from the origin, we have $U(x) + U(y) \leq U(x + y)$. We show that

$$[(x, y) \in Q \backslash \mathrm{c}(S_Q), x + y \geq 0] \Rightarrow u(x) - w(-y) < u(x + y) .$$

The proof for $x + y \leq 0$ is similar.

Given $(x, y) \in Q \backslash \mathrm{c}(S_Q)$ and $(x + y) \geq 0$, suppose first that $-x^{**} = y^{**} \leq y < y^*$ so that (x, y) is to the right of ψ in the Q^+ region. Let z satisfy $(z, y) \in \psi$ with $0 < z < x$. Then, by concavity of u and the preceding result for ψ,

$$u(x) - u(x + y) < u(z) - u(z + y) = w(-y)$$

and therefore $u(x) - w(-y) < u(x + y)$.

Suppose next that $(x, y) = (t, -t)$ with $-t < y^{**}$, i.e. on the $-45°$ line down from the origin past $(x^{**}, -x^{**})$. The curvature properties give

$$\frac{u(t)}{t} < \frac{u(x^{**})}{x^{**}} = \frac{w(x^{**})}{x^{**}} < \frac{w(t)}{-t}$$

so $u(t) - w(t) < u(t - t) = 0$.

Finally, if $(x, y) = (t, s)$ with $t > x^{**}$, we have $u(t) - u(t + s) < u(-s)$ by concavity, and $u(-s) < w(-s)$ by the preceding paragraph, so $u(t) - w(-s) < u(t + s)$. □

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