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Singular points in generalized concatenation structures that otherwise are homogeneous

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A generalized concatenation structure $\mathcal{X}$ is any ordered relational structure in which at least one relation is a function of two or more variables that is monotonically increasing in each of them. A point of $\mathcal{X}$ is said to be singular if it is fixed under all automorphisms of the structure. Examples are 0 in the additive real numbers, the velocity of light in special relativity, and the status quo in some theories of utility. A translation is an automorphism with no fixed points other than the singular ones. It is assumed that $\mathcal{X}$ is order dense and finitely unique, from which it follows (Theorem 1) that at most three singular points may exist; a maximum, a minimum, and/or an interior one. Assuming also that $\mathcal{X}$ is homogeneous in translations between adjacent singularities, the singular points are further characterized (Theorem 2), with the interior one being shown to be a generalized zero (Definition 4). It is shown how to replace a generalized zero by a zero (Theorem 3) and how to partition X into homogeneous structures on either side of the interior generalized zero (Theorem 4). The major problem in representing such structures lies in understanding how the two halves relate. For structures on a continuum that are translation homogeneous between singularities, finitely unique, and solvable relative to the zero, a representation exists in terms of unit structures (Theorem 5).

Key words: Finite uniqueness; generalized concatenation structure; homogeneity; singular points.

1. Introduction

During the 1980s several authors have explored the structure of ordered relational systems that satisfy two important conditions formulated in terms of the (order-)

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automorphisms of the structure. First, these structures are assumed to be homogeneous in the sense that each point of the structure can be mapped under an automorphism of the structure into any other point. Second, they are assumed to be finitely unique in the sense that when the number of fixed points of an automorphism exceeds some fixed integer, the automorphism is forced to be the identity. These structures are highly symmetrical. Among other things, two points in the structure cannot be distinguished by differences in their properties relative to other points. The key references are Alper (1985, 1987); Luce (1986, 1987); Luce et al. (1990); Narens (1981a,b, 1985); and Roberts and Rosenbaum (1985, 1988). Application of these results to conjoint structures is found in Luce and Cohen (1983), to concatenation structures in Luce and Narens (1985), and to psychophysical laws in Luce (1990) and Roberts and Rosenbaum (1986). I assume that the reader is familiar with either Luce and Narens (1985) or Chapter 20 of Luce et al. (1990); the latter is more comprehensive.

Everyone involved in this research has been aware all along that the class of homogeneous structures fails to include a number of scientifically important examples of classical physical measurement and, quite possibly, some that are relevant to the behavioral sciences. For example, consider just concatenation structures of the form $\langle X, \succsim, \circ \rangle$ where $\succsim$ is a total order and $\circ$ is a (partial) binary operation that is monotonic. If the structure has a 'zero' element e with the property $x \circ e = x = e \circ x$, for all $x \in X$, then under the assumed monotonicity of $\circ$ it is easy to show that e is unique, and so e is unlike any other element in the structure, which implies the structure is not homogeneous. Equally, if there is an 'infinite' element e' with the property $x \circ e' = e' = e' \circ x$, for all $x \in X$, then again the structure cannot be homogeneous.

Classical length and mass measurement can be thought of as having zero elements which, in the usual axiomatic treatment, are excluded so as to be able to deal with a fully homogeneous substructure. Relativistic velocity has both a zero and an infinity (= speed of light). When both are excluded, the resulting structure is associative and homogeneous and is axiomatized in the usual way, except that instead of using an additive representation, which would map e' to $+\infty$, a non-additive one is chosen that preserves the relation that velocity = distance traversed/ time elapsed, in which case e' maps into a finite velocity. For a detailed discussion, see Krantz et al. (1971, pp. 92–96).

A superficially similar case is probability measurement in which $\circ$ is interpreted as the partial operation of the union of disjoint events. There the null event is a zero and the universal element is a maximum point (and, trivially, an infinity because it can be concatenated only with the null event). What differs significantly is that, unlike the velocity case, the structure lying between the zero and the maximal elements is not homogeneous.

The aim of the present paper is to investigate those cases for which, like velocity, the concatenation structure between the zero and the maximal element is both homogeneous and finitely unique. To develop the theory, it is sufficient that one of the defining relations of the structure be a monotonic function. As a familiar example of such a structure, consider Savage's (1954) formulation of a decision problem in terms of acts composed of pure consequences assigned to randomly determined states. Let X denote the set of consequences, where it is assumed that no two distinct consequences are indifferent; $1, \ldots, n$ denote the set of n states; and $\mathbf{x} = (x_1, \ldots, x_n)$ denote the act in which $x_i \in X$ is the consequence when state i occurs. Suppose, further, that X is sufficiently rich so that for each act $\mathbf{x}$ there is a consequence $x \in X$ with the property that the constant act $(x, \ldots, x)$ is judged indifferent to $\mathbf{x}$. Then the function F may be defined by

$$x = F(x_1, \ldots, x_n), \quad \text{iff } (x, \ldots, x) \sim \mathbf{x} = (x_1, \ldots, x_n).$$

In Savage's version of the theory, there was no interior singular point. More recent versions in which an interior singular point plays a significant role are Kahneman and Tversky (1979), Luce (1991, 1992), Luce and Fishburn (1991), and Tversky and Kahneman (1992).

The main results of this paper establish that the possible structures of this type are, once again, decidedly limited and that in the case of measurement on a continuum they can be characterized in terms of what are known as unit structures (Cohen and Narens, 1979; Luce, 1987). For the case of a binary operation, this reduces to two real functions of a single real variable and two numerical constants. The results described do not apply to structures, like probability, in which homogeneity is totally lacking nor to structures that are not generalized concatenation ones, except when these structures can be reduced to generalized concatenations, as is true for general conjoint structures.

An application of the present results to arrive at a theory of certainty equivalents for uncertain alternatives is given in Luce (1992).

2. Definitions

Let X denote a set, $\succsim$ a binary relation on X, F is a function from X^n to X, with $n \geqslant 2$ arguments, J an index set, and R_j, $j \in J$, relations of finite order on X. The first definition makes clear exactly what is meant by a generalized concatenation structure when extreme elements are admitted as a possibility.

Definition 1. A point $e \in X$ is said to be *maximal* (*minimal*) iff for all $x \in X$, $e \succsim (\precsim) x$. A point is said to be *extreme* iff it is either maximal or minimal; otherwise, it is said to be *interior*.

$\langle X, \succsim \rangle$ is said to be *order dense* iff, for each $x, y \in X$ with[1] $x \succ y$, there exists $z \in X$ with $x \succ z \succ y$.

[1] As usual, $x \succ y$ means $x \succsim y$ but not $y \succsim x$.

The relational structure $\mathscr{X}=\langle X, \succsim, F, R_j\rangle_{j\in J}$ is said to be a (closed) *generalized concatenation structure* iff the following three conditions are met:

(1) $\succsim$ is a total ordering, i.e. connected, transitive, and antisymmetric.

(2) For some $x, y \in X$, $x > y$.

(3) Monotonicity: for all $x_j, x_i' \in X$ and extreme $e \in X$,

(i) $x_i \succsim x_i'$ implies $F(x_1, \ldots, x_i, \ldots, x_n) \succsim F(x_1, \ldots, x_i', \ldots, x_n)$;

(ii) if all arguments are interior, $x_i > x_i'$ implies $F(x_1, \ldots, x_i, \ldots, x_n) > F(x_1, \ldots, x_i', \ldots, x_n)$;

(iii) $F(x_1, \ldots, x_i, x_n) > F(x_1, \ldots, x_i, \ldots, x_n)$ implies $x_i > x_i'$;

(iv) if e is extreme and x_i, $i = 1, \ldots, n$, are interior, then $F(x_1, \ldots, e, \ldots, x_n) \neq F(x_1, \ldots, x_i, \ldots, x_n)$.

This definition of a generalized concatenation structure differs in several ways from the concept of a (binary) concatenation structure formulated by Luce and Narens (1985). First, it restricts attention to closed operations; second, it admits operations more general than binary ones; and third, it admits the possibility of extreme points. The first restriction, which is automatic when homogeneity is assumed (see Definition 3 below), allows one to drop the postulate of local definability, and the third necessitates a modification of the monotonicity axiom to deal with the behavior of the extreme points. There are other possible assumptions about the behavior of the extreme points such as (iv′) if e is maximal, $F(x_1, \ldots, e, \ldots, x_n) \succsim F(x_1, \ldots, x_i, \ldots, x_n)$ and the corresponding variant for e minimal. I do not investigate these.

The next group of concepts are all stated in terms of automorphisms of the structure $\mathscr{X}$. These concepts have proved important and have been studied for certain classes of structures in a number of papers cited earlier. Certain modifications arise because special points, such as the extreme ones, are now being admitted.

Definition 2. Let $\mathscr{X}=\langle X, \succsim, F, R_j\rangle_{j\in J}$ be a generalized concatenation structure.

An *automorphism* of $\mathscr{X}$ is any isomorphism of $\mathscr{X}$ with itself. The group (under function composition) of automorphisms is denoted $\mathscr{A}$.

A point $e \in X$ is said to be *singular* iff e is a fixed point of every automorphism, i.e. for each $\alpha \in \mathscr{A}$, $\alpha(e) = e$.

A *translation* is either the identity or an automorphism with no fixed points aside from the singular ones. $\mathscr{T}$ denotes the set of all translations.

$\mathscr{X}$ is said to be *finitely unique* iff there is some integer N such that any automorphism with N or more fixed points is the identity. If N is the least integer for which this is true, then the structure is said to be *N-point unique*.

Luce and Cohen (1983) referred to points that are fixed under all automorphisms as ‘intrinsic zeros’. Because of the results given below, it seems better to alter the terms, as was done in Luce et al. (1990), to ‘singular point’ because some exhibit decidedly non-zero-like properties.

3. Elementary properties of singular points

The first result arrives at a number of simple properties of singular points in order-dense, finitely-unique, generalized concatenation structures. In summary, the main facts are that there is at most one of each type of singular points: minimal, maximal, and interior; that each singular point is idempotent $[F(e,\dots,e)=e]$; and that an extreme point cannot be reached by finite concatenations of interior points.

Theorem 1. *Suppose $\mathscr{X}=\langle X,\succsim,F,R_j\rangle_{j\in J}$ is an order-dense, generalized concatenation structure. Then,*

(i) *there is at most one maximal point and at most one minimal point; and*

(ii) *if $e\in X$ is an extreme point, then e is singular.*

Suppose, further, that $\mathscr{A}$ is non-trivial and $\mathscr{X}$ is finitely unique. Then, for $e,x_i\in X$,

(iii) *if e is singular, then $F(e,\dots,e)=e$;*

(iv) *if e is an extreme point and x_i, $i=1,\dots,n$, are interior, then $F(x_1,\dots,x_n)\neq e$; and*

(v) *there exists at most one interior singular point.*

Proof. (i) Trivial.

(ii) Suppose e is maximal and $\alpha\in\mathscr{A}$, then $\alpha(e)\precsim e$. If $\alpha(e)\prec e$, then applying α^{-1}, we have $e\prec\alpha^{-1}(e)$, which violates e being maximal. Thus $\alpha(e)=e$, proving e is singular. The minimal case is similar.

(iii) Suppose e is singular and interior. Let $e'=F(e,\dots,e)$. Suppose α is any automorphism, then

$$\alpha(e')=\alpha F(e,\dots,e)=F[\alpha(e),\dots,\alpha(e)]=F(e,\dots,e)=e',$$

proving that e' is also singular. By order density there exist interior x,y with $x\succ e\succ y$. By monotonicity (ii), $F(x,e,\dots,e)\succ F(e,e,\dots,e)=e'\succ F(y,e,\dots,e)$ and so e' is interior. Suppose $e'\neq e$, then by induction, $ne=F[(n-1)e,e,\dots,e]$ with $1e=e$ is either a strictly increasing or a strictly decreasing sequence of interior singular points. By finite uniqueness and the fact $\mathscr{A}$ is non-trivial, this is impossible. So $F(e,\dots,e)=e'=e$.

Next, assume e is maximal. Suppose $e'\prec e$. As above, e' is singular. By order density, there exist interior x,y with $e'\prec x\prec y\prec e$. Either e' is minimal or it is an interior singular point, so $e'\precsim F(e',\dots,e')$, whence monotonicity implies

$$e'\precsim F(e',\dots,e')\precsim F(x,\dots,x)\prec F(y,\dots,y)\precsim F(e,\dots,e)=e',$$

which is impossible. So, $F(e,\dots,e)=e$. The case e is minimal is similar.

(iv) If e is an extreme point and x_i, $i=1,\dots,n$, are interior points, then using part (iii) and monotonicity, $e=F(e,\dots,e)\succ(\prec)F(x_1,\dots,x_n)$ if e is maximal (minimal).

(v) Suppose e and e', $e\succ e'$, are both singular and interior. By monotonicity (i) and part (iii), $e=F(e,\dots,e)\succ F(e,\dots,e')\succ F(e',\dots,e')=e'$. Clearly, $e''=F(e,\dots,e')$ is interior and singular. Thus, by a finite induction, there are countably many distinct singular points, which by finite uniqueness is impossible. □

4. Further properties of singular points when translation homogeneity holds

The next definition is a generalization to structures with singular points of the concept of homogeneity that has played an essential role in our understanding of structures without such points (Alper, 1987; Luce, 1986, 1987; Luce and Narens, 1985; Narens, 1981a,b).

Definition 3. Suppose $\mathscr{X}=\langle X, \succsim, F, R_j\rangle_{j\in J}$ is a generalized concatenation structure and $\mathscr{T}$ is the set of its translations. Then, $\mathscr{T}$ is said to be *translation homogeneous between singularities* iff for each non-singular x, y not separated by a singular point, there exists $\tau\in\mathscr{T}$ such that $\tau(x)=y$.

When $\mathscr{X}$ has no singular points, this definition reduces to what Narens called homogeneity (of the translations). When invoking the concept in this paper I shall sometimes simply speak of homogeneity without making explicit that it refers just to translations and holds only between singularities.

Definition 4. Suppose that $\mathscr{X}$ is a generalized concatenation structure with a non-trivial group of automorphisms and $e\in X$ is a singular point. Then:

(1) e is said to be an *infinite point* iff for all x_i, $i=1,\dots,n$, lying between it and an adjacent singularity, if any, $F(x_1,\dots,e,\dots,x_n)=e$.

(2) e is said to be a *generalized zero* iff there exist functions θ_i: $X\to X$, $i=1,\dots,n$, such that for each interior $x\in X$ in the ith position,

$$F(e,\dots,e,x,e,\dots,e)=\theta_i(x),$$

where for some translations θ_i^j of $\mathscr{X}$, $i=1,\dots,n$, $j=+,-$,

$$\theta_i(x)=\begin{cases}\theta_i^+(x), & \text{if } x>e,\\ e, & \text{if } x=e,\\ \theta_i^-(x), & \text{if } x<e.\end{cases}$$

(3) e is said to be a *zero* iff for all interior x, $F(e,\dots,x,\dots,e)=x$, i.e. e is a generalized zero with $\theta_i=\iota$, the identity, for all positions i.

Suppose $\mathscr{X}$ is finitely unique and F is a binary operation that is associative. We show that if e is a generalized zero, then e is a zero. Using associativity, Theorem 1(iii), and the definition of a generalized zero, for all $x\in X$,

$$\theta_1(x)=F(x,e)=F[x,F(e,e)]=F[F(x,e),e]=F[\theta_1(x),e]=\theta_1^2(x).$$

By finite uniqueness, $\theta_1 = \theta_1^2$, and so θ_1 is the identity.

The following result classifies singular points in terms of the properties just defined. Observe that I impose the assumption that the translations commute, i.e. for each θ, $\tau \in \mathscr{T}$, $\theta\tau = \tau\theta$. As has been shown (Luce, 1987), this is a natural property of those general relational systems without singular points that have nice numerical representations. To the extent that structures with singular points are of this nice type between adjacent singularities, then it continues to be plausible to assume that the translations commute (e.g. this is true for the real structure of Theorem 5 below).

Theorem 2. *Let $\mathscr{X} = \langle X, \succsim, F, R_j \rangle_{j \in J}$ be a generalized concatenation structure and $\mathscr{T}$ its set of translations. Suppose that $\mathscr{X}$ is order dense and finite unique and that $\mathscr{T}$ is homogeneous between adjacent singularities and its elements commute. Then:*

(i) *A singular point cannot be both an infinite point and a generalized zero.*

(ii) *If there is an interior singular point, then it is a generalized zero.*

(iii) *If one singular point is a generalized zero, then any other singular points are infinite points.*

(iv) *If F is positive in the sense that, for all positions i and all $x_i \in X$, $F(\ldots, x_i, \ldots) > x_i$, then a minimum point is a generalized zero.*

(v) *If there is any automorphism that is not a translation, then any generalized zero is a zero.*

Proof. The proof is decomposed into several lemmas in order to make clear just what assumptions are needed to prove what, thereby giving somewhat more refined information than the theorem does itself.

Lemma 1. *Suppose $\mathscr{X}$ is a generalized concatenation structure. Then:*

(a) *An extreme point cannot be both an infinite point and a generalized zero.*

(b) *If maximal and minimal elements exist and one is a generalized zero, then the other is an infinite point.*

Proof. (a) Suppose e is maximal, infinite, and a generalized zero, so for each position i both $F(x_1, \ldots, x_{i-1}, e, x_{i+1}, \ldots, x_n) = e$ and $F(e, \ldots, e, x, e, \ldots, e) = \theta_i(x)$, where θ_i is a translation. For $x < e$, apply monotonicity to the first equation, take note of the fact that e is maximal, and use the fact that translations are order preserving:

$$F(x, e, \ldots, e) = e > \theta_1(x) = F(x, e, \ldots, e),$$

which is a contradiction. The argument for e minimal is similar.

(b) Suppose e is maximal, e' is minimal, and e is a generalized zero, i.e. for each position i, $F(e, \ldots, e, x, e, \ldots, e) = \theta_i(x)$. In particular, $F(e, \ldots, e, e', e, \ldots, e) = \theta_i(e') = e'$. By monotonicity and the fact e' is minimal, $F(x_1, \ldots, x_{i-1}, e', x_{i+1}, \ldots, x_n) = e'$, proving that e' is an infinite point. The other case, e' a generalized zero, is similar. □

Lemma 2. *Suppose in addition that $\mathscr{X}$ is order dense. Then an infinite point is an extreme point, and so it is singular.*

Proof. Suppose e is an infinite point. If e is also interior, then by order density there exist interior x, y with $x > e > y$, and so by monotonicity (ii)

$$F(x, y, \ldots, y) > F(e, y, \ldots, y) = e = F(x, e, \ldots, e) > F(x, y, \ldots, y),$$

which is a contradiction. So e is extreme. □

Lemma 3. *Suppose that $\mathscr{X}$ is also order dense and finitely unique. Then, for $e, e' \in X$:*

(a) *If e and e' are distinct singular points and e' is interior, then e is an infinite point.*

(b) *Suppose e is a generalized zero. Every automorphism commutes with θ_i iff e is a singular point. In this case, either every automorphism is a translation or e is a zero.*

Proof. (a) Suppose e and e' are singular and e' is interior. If $e > e'$, then by monotonicity (i) and Theorem 1(iii),

$$e = F(e, \ldots, e, \ldots, e) \geqslant F(e', \ldots, e, \ldots, e') \geqslant F(e', \ldots, e', \ldots, e') = e'.$$

Observe that the middle term is singular because each of its arguments is and so if e' is interior, it follows from Theorem 1(v) that the middle term is either e or e'. The latter is, however, impossible because, by Theorem 1(v), e is maximal and, by part (iv) of monotonicity, $F(e', \ldots, e, \ldots, e') > F(e', \ldots, e', \ldots, e') = e'$. So $F(e', \ldots, e, \ldots, e') = e$. Suppose x_i lie between e and e'. Then by Theorem 1(iii) and monotonicity (i),

$$e = F(e, \ldots, e, \ldots, e) \geqslant F(x_1, \ldots, e, \ldots, x_n) \geqslant F(e', \ldots, e, \ldots, e') = e,$$

and so $F(x_1, \ldots, e, \ldots, x_n) = e$, proving that e is an infinite point. The case $e' > e$ is similar.

(b) Suppose e is a generalized zero. First, assume e is singular and α is an automorphism, then

$$\begin{aligned} \alpha\theta_i(x) &= \alpha F(e, \ldots, e, x, e, \ldots, e) \\ &= F[\alpha(e), \ldots, \alpha(e), \alpha(x), \alpha(e), \ldots, \alpha(e)] \\ &= F[e, \ldots, e, \alpha(x) e, \ldots, e] \\ &= \theta_i \alpha(x), \end{aligned}$$

establishing that α commutes with θ_i. Conversely, suppose every automorphism commutes with θ_i. If e is extreme, by Theorem 1(i) it is singular. So assume e is interior and for some automorphism α, $\alpha(e) \neq e$. Then for interior x, monotonicity and commutativity of α and θ_i imply:

$$\alpha\theta_i(x) = \alpha F(e, \ldots, x, \ldots, e) = F[\alpha(e), \ldots, \alpha(x), \ldots, \alpha(e)]$$
$$\neq F[e, \ldots, \alpha(x), \ldots, e] = \theta_i\alpha(x) = \alpha\theta_i(x),$$

which is a contradiction. So e is singular.

Assuming the above, suppose that α is such that for some $d \neq e$, $\alpha(d) = d$. Then, $\alpha\theta_i(d) = \theta_i\alpha(d) = \theta_i(d)$. By induction, $\theta_i^n(d)$ is a fixed point of α. Thus, by the fact θ_i is a translation and by finite uniqueness, $\theta_i = \iota$, thus proving that e is a zero. □

Lemma 4. *Suppose, in addition to the previous assumptions of monotonicity, order density, and finite uniqueness, that $\mathscr{T}$ is homogeneous between adjacent singularities and its elements commute, and suppose e is a singular point. For position i define*

$$\theta_i(x) = F(e, \ldots, e, x, e, \ldots, e) \quad \textit{and} \quad \eta_i(x) = F(x, \ldots, x, e, x, \ldots, x).$$

Then:

(a) *Either, for all interior x, $\theta_i(x) \equiv e$ or, between adjacent singular points, θ_i agrees with a translation.*

(b) *Either, for all interior x, $\eta_i(x) \equiv e$ or, between adjacent singular points, η_i agrees with a translation.*

(c) *An interior singular point is a generalized zero.*

(d) *If F is positive, then a minimum point is a generalized zero.*

Proof. (a) Suppose that for some interior y, $\theta_i(y) > e$. Then, by homogeneity, there is a translation τ such that $\theta_i(y) = \tau(y)$. Now, let x be any interior element $> e$, and so, again by homogeneity, there is a translation α such that $\alpha(x) = y$. Thus, using the commutativity of translations

$$\begin{aligned}
\alpha\tau(x) &= \tau\alpha(x) \\
&= \tau(y) \\
&= \theta_i(y) \\
&= \theta_i\alpha(x) \\
&= F[e, \ldots, e, \alpha(x)e, \ldots, e] \\
&= F[\alpha(e), \ldots, \alpha(e), \alpha(x), \alpha(e), \ldots, \alpha(e)] \\
&= \alpha F(e, \ldots, e, x, e, \ldots, e) \\
&= \alpha\theta_i(x).
\end{aligned}$$

Thus, taking α^{-1} shows that θ_i is identical to the translation τ, thereby proving the assertion for $x > e$. The argument for $x < e$ is similar.

(b) The argument is basically the same as for case (a).

(c) Suppose e is an interior singular point, then by monotonicity (ii) the first condition of part (a) does not obtain for any position, and so e is a generalized zero.

(d) Suppose e is minimum, then for each position i and interior x, $\theta_i(x) > x > e$, so by part (a), e is a generalized zero. □

The proof of the theorem is as follows: (i) follows from Lemmas 1(a) and 2; (ii) from Lemmas 3(a) and Lemma 4(c); (iii) from Lemmas 1(b) and Lemma 3(a); (iv) from Lemma 4(d); and (v) from Lemma 3(b). □

When an interior singular point exists, Lemma 3(a) shows that the extreme (singular) points must behave as infinities, and, in the homogeneous case with commutative translations, Lemma 4(c) shows that the interior singular point is a generalized zero. However, when the only singularities are maximal and minimal points, the situation is more complex. As was noted earlier, relativistic velocity illustrates the case where one extreme point is a generalized zero, in which case the other must be an infinite one. Lemma 4(a) and (b) makes clear that a total of four possibilities exist at each position, which reduce to three in the binary case because $\eta_1 \equiv \theta_2$ and $\eta_2 \equiv \theta_1$. The following three examples show that all three combinations actually occur. These examples are all binary operations on the extended nonnegative real numbers ordered in the usual fashion.

The first is an example in which the two extreme points are both infinite ones. Suppose $x, y \in [0, \infty]$:

$$F_1(x, y) = \begin{cases} (xy)^{1/2}, & \text{if } x < \infty,\ y < \infty, \\ \infty, & \text{if either } x \text{ or } y = \infty. \end{cases}$$

As is easily verified, under the usual ordering of the real numbers F_1 is monotonic, the translations are the similarities and so the structure is finitely unique and translation homogeneous between singular points, and 0 and ∞ are each an infinite point.

The second is one in which one extreme point is an infinite one and the other is neither infinite nor a generalized zero. Suppose $r > 0$ and $s > 0$ are fixed and $x, y \in [0, \infty]$:

$$F_2(x, y) = \begin{cases} [x(rx + sy)]^{1/2}, & \text{if } x < \infty,\ y < \infty, \\ \infty, & \text{if either } x \text{ or } y = \infty. \end{cases}$$

As is easily verified, F_2 meets all of the conditions of Theorem 2, ∞ is an infinite point, and 0 is infinite when on the left and a generalized zero when on the right:

$$F_2(0, y) = 0, \qquad F_2(x, 0) = r^{1/2}x.$$

The third is an example in which each extreme point is half infinite and half generalized zero. Suppose $s > r > 0$ are fixed and $x, y \in [0, \infty]$:

$$F_3(x, y) = \begin{cases} x[(rx + sy)/(x + y)], & \text{if } 0 < x < \infty,\ 0 \leqslant y \leqslant \infty, \\ \infty, & \text{if } x = \infty, \\ 0, & \text{if } x = 0. \end{cases}$$

Monotonicity for interior points follows because

$$\partial F_3(x,y)/\partial x = (rx^2+2rxy+sy^2)/(x+y)^2 > 0 \quad \text{for } x>0 \text{ or } y>0,$$

$$\partial F_3(x,y)/\partial y = x^2(s-r)/(x+y)^2 > 0 \quad \text{for } x>0 \text{ or } y>0.$$

The extreme points, 0 and ∞, behave asymmetrically, namely,

$$F_3(0,y)=0, \quad F_3(x,0)=rx; \qquad F_3(\infty,y)=\infty, \quad F_3(x,\infty)=sx,$$

from which monotonicity at the singular points follows. Multiplication by a positive constant clearly is a translation. To show that these are the only automorphisms, we use an argument given by M. Cohen and used by Narens (1985, p. 128) for positive concatenation structures. Let α be an automorphism. Taking the partial derivative with respect to y of $\alpha F_3(x,y) = F_3[\alpha(x), \alpha(y)]$, we see that

$$\alpha'[F_3(x,y)]\, \partial F_3(x,y)/\partial y = \alpha'(y)\, \partial F_3[\alpha(x), \alpha(y)]/\partial\alpha(y).$$

At $y=0$, $\partial F_3(x,y)/\partial y = s-r>0$, and so $\alpha'(0)(s-r)=\alpha'(rx)(s-r)$ which implies $\alpha'(rx)=a$ constant and $\alpha(x)=kx$, as was to be shown.

By part (iv) of the theorem we know that F_3 cannot be positive, which can be verified by choosing x sufficiently small in which case $F_3(x,y)<y$.

The subsequent discussion would be somewhat simpler were it possible to prove that any generalized zero is, in fact, a zero, i.e. each θ_i is the identity function. However, the following example with a binary operation on $[0,\infty]$ establishes that such is not the case in general. Suppose $r>0$, $s>0$ are fixed and $x, y \in [0,\infty]$:

$$F_4(x,y) = rx + sy.$$

It is easy to verify that this forms a binary concatenation structure, 0 is extreme, singular, and a generalized zero because $F_4(x,0)=rx$ and $F_4(0,y)=sy$. For $r \neq s$, 0 is not symmetric, and it is a zero iff $r=s=1$. F_4 is bisymmetric, but neither associative (unless $r=s=1$) nor commutative (unless $r=s$). And the automorphisms of the structure are multiplications by positive constants, i.e. they are translations; and multiplication by any positive constant is a translation, and so the structure is translation homogeneous between singular points and is finitely unique.

5. Replacing a generalized zero by a zero

If $\mathscr{X} = \langle X, \succsim, F, R_j \rangle_{j \in J}$ has a generalized zero, one can construct a new function G that encodes the same information and replaces the generalized zero by a zero. Explicitly:

Theorem 3. *Suppose $\mathscr{X}$ is a generalized concatenation structure that is order dense, finitely unique, translation homogeneous between adjacent singularities, and has a* *singular point e that is a generalized zero.*[2] *Let θ_i be the functions of the generalized zero (Definition 3), and define G on X by*

$$G(x_1, \ldots, x_n) = F[\theta_1^{-1}(x_1), \ldots, \theta_n^{-1}(x_n)].$$

Then:

(i) $\mathscr{Y} = \langle X, \succsim, G, R_j \rangle_{j \in J}$ *is a generalized concatenation structure.*

(ii) $\mathscr{X}$ *and* $\mathscr{Y}$ *have the same automorphism group.*

(iii) *e is a zero of* $\mathscr{Y}$.

(iv) *If e is interior,* $F(x_1, \ldots, x_n) = e$ *iff* $G[\theta_1(x_1), \ldots, \theta_n(x_n)] = e$.

(v) *Suppose that under the one-to-one mapping ϕ from X onto R, a subset of the real numbers,* $\mathscr{Y}$, *is isomorphic under ϕ to* $\mathscr{Y}' = \langle R, \geqslant, G', R_j' \rangle_{j \in J}$, *then, under* ϕ, $\mathscr{X}$ *is isomorphic to* $\mathscr{X}' = \langle R, \geqslant, F', R_j' \rangle_{j \in J}$, *where*

$$F'(\ldots, x_i, \ldots) = G'[\ldots, \phi\theta_i\phi^{-1}(x_1), \ldots].$$

Proof. (i) Trivial because θ_i are order preserving.

(ii) By Lemma 3(b), θ_i must commute with every automorphism α, so

$$\begin{aligned}\alpha G(x_1, \ldots, x_n) &= \alpha F[\theta_1^{-1}(x_1), \ldots, \theta_n^{-1}(x_n)] \\ &= F[\alpha\theta_1^{-1}(x_1), \ldots, \alpha\theta_n^{-1}(x_n)] \\ &= F[\theta_1^{-1}\alpha(x_1), \ldots, \theta_n^{-1}\alpha(x_n)] \\ &= G[\alpha(x_1), \ldots, \alpha(x_n)].\end{aligned}$$

That is to say, G is invariant under the automorphism group of $\mathscr{X}$. Moreover, the argument goes the other way, and so the two operations have the same automorphism group.

(iii) Observe that e is a zero of $\mathscr{Y}$ because

$$\begin{aligned}G(e, \ldots, e, x, e, \ldots, e) &= F[\theta_1^{-1}(e), \ldots, \theta_{i-1}^{-1}(e), \theta_i^{-1}(x), \theta_{i+1}^{-1}(e), \ldots, \theta_n^{-1}(e)] \\ &= F[e, \ldots, e, \theta_i^{-1}(x), e, \ldots, e] \\ &= \theta_i\theta_i^{-1}(x) \\ &= x.\end{aligned}$$

(iv) Suppose e is interior, then

$$\begin{aligned}G[\theta_1(x_1), \ldots, \theta_n(x_n)] &= F[\theta_1\theta_1^{-1}(x_1), \ldots, \theta_n\theta_n^{-1}(x_n)]. \\ &= F(x_1, \ldots, x_n) \\ &= e.\end{aligned}$$

(v) Define $F'(\ldots, x_i, \ldots) = G'[\ldots, \phi(x_i), \ldots]$. Then ϕ is an isomorphism of $\mathscr{X}$ and $\mathscr{X}'$ because

[2] By Lemmas 1(b) and 3(a) there is at most one.

$$\phi F(\ldots, x_i, \ldots) = \phi G[\ldots, \theta_i(x_i), \ldots] \quad \text{(definition of } G)$$
$$= G'[\ldots, \phi\theta_i(x_i), \ldots] \quad \text{(isomorphism of } \mathscr{Y} \text{ and } \mathscr{Y}')$$
$$= F'[\ldots, \phi(x_i), \ldots] \quad \text{(definition of } F'). \quad \square$$

It is important to recognize that the two structures $\mathscr{X}$ and $\mathscr{Y}$ of Theorem 3, although very closely related, are not isomorphic. In particular, nice properties in one of them need not be mapped into very nice ones in the other.

6. Partitioning into substructures

The next result concerns the substructures that lie between adjacent singular points.

Suppose $\mathscr{X} = \langle X, \succsim, F, R_j \rangle_{j \in J}$ is a generalized concatenation structure and $e \in X$ is an interior singular point. Let X^* denote the set of interior points, $X^+ = \{x: x \in X^* \ \& \ x > e\}$ and $X^- = \{x: x \in X^* \ \& \ x < e\}$, and denote by $\mathscr{X}^i$ the restriction of $\mathscr{X}$ to X^i, $i = +, -$. If there is no interior singularity, let $\succsim^*$ be the restriction of $\succsim$ to X^*, F^* and R_j^* the restrictions of F and R_j to X^*, and $\mathscr{X}^* = \langle X^*, \succsim^*, F^*, R_j^* \rangle_{j \in J}$.

Theorem 4. *Suppose $\mathscr{X}$ is an order-dense, finitely-unique, generalized concatenation structure with a non-trivial automorphism group. Then:*

(i) *The induced structures of $\mathscr{X}$, either $\mathscr{X}^*$ or both $\mathscr{X}^+$ and $\mathscr{X}^-$ are generalized concatenation structures with no singular points.*

(ii) *The restriction of an automorphism (translation) of $\mathscr{X}$ to X^i, $i = *, +, -$, is an automorphism (translation) of $\mathscr{X}^i$.*

Proof. (i) We first show that the induced F^* is a function. By Theorem 1(iv) it lies between the extreme points, and if e is an interior singular point and $x_i > e$, then by monotonicity $F(x_1, \ldots, x_n) > F(e, \ldots, e) = e$. It is easy to see that the restriction of any automorphism of $\mathscr{X}$ to an induced substructure is an automorphism of it. Thus, were the substructure to have a singularity, that would mean $\mathscr{X}$ has an additional singularity, which by Theorem 1 and the construction of the substructure(s) is impossible. Therefore, monotonicity (i), (ii), and (iii) hold in the substructure(s), proving they are closed concatenation structures.

(ii) As was mentioned in (i), this is trivial. $\square$

Corollary. *Under the assumptions of the theorem:*

(i) *if the translations commute, then their restriction is* 1*-point unique;*

(ii) *if $\mathscr{X}$ is translation homogeneous between singularities, then the restriction is either* 1*-point unique or it is idempotent and either* 1*- or* 2*-point unique.*

Proof. (i) This follows from the theorem and the Corollary to Theorem 20.4 of Luce et al. (1990).

(ii) The following is a slight generalization of Theorem 2.2 of Luce and Narens (1985). Let x and y be interior points. By homogeneity, $F(x,\ldots,x)\geq x$ iff $F(y,\ldots,y)\geq y$. Suppose, first, $F(x,\ldots,x)>x$ and suppose α is an automorphism with $\alpha(x)=x$. Observe that $F(x,\ldots,x)$ is also a fixed point, and so by induction there are countably many. Thus, by finite uniqueness, α is the identity and the restricted structure is 1-point unique. The case $F(x,\ldots,x)<x$ is similar. So, suppose $F(x,\ldots,x)=x$. Suppose $x<y$ are both fixed points of an automorphism α. By monotonicity, $x<F(x,y,\ldots,y)<y$, and $F(x,y,\ldots,y)$ is a fixed point of α. By a finite induction, there are countably many fixed points, and so α is the identity, proving that the restricted structure is either 1- or 2-point unique. □

7. Structures on a continuum

Suppose that a generalized concatenation structure on a continuum is finitely unique and its translations are homogeneous between singularities. Theorem 4 shows that the extreme points, if any, can always be dropped, and matters are thereby reduced either to the already developed theory of the homogeneous case (Alper, 1987; Luce, 1987) or to the case with no extreme points but with an interior singularity, which by Theorem 2 must be a generalized zero. It is clear that the structures $\mathscr{X}^i$, $i=+,-$, on either side of this interior singularity are concatenation structures with no singularities, and so from previous results (Cohen and Narens, 1979; Luce and Narens, 1985; Luce, 1987) we know that their forms are unit structures. Thus, the entire problem reduces to gaining an understanding of how the two halves are linked together. To this end, we need a concept of solvability.

Definition 5. Suppose $\mathscr{X}$ is a generalized concatenation structure and e is an interior point. It is *solvable relative to e* if and only if for each interior x there exists for each position i an element $s_i(x)$ such that $F(x,\ldots,x,s_i(x),x,\ldots,x)=e$.

In working with structures on a continuum without extreme points, no loss of generality results from selecting it to be Re. Moreover, if $\mathscr{R}'=\langle Re,\geq,F',R_j'\rangle_{j\in J}$ is a generalized concatenation structure with e an interior generalized zero, then it can be replaced by $\mathscr{R}=\langle Re,\geq,F,R_j\rangle_{j\in J}$, where, for all $x_i\in Re$,

$$F(x_1,\ldots,x_n)=F'(x_1+e,\ldots,x_n+e)-e,$$

and R_j is similarly defined. We show that 0 is a generalized zero of $\mathscr{R}$. Let θ_i' be the translations of the generalized zero e. Define $\theta(x)=\theta_i'(x+e)-e$. Then

$$\begin{aligned}F(0,\ldots,0,x,0,\ldots,0)&=F'(e,\ldots,e,x+e,e,\ldots,e)-e\\&=\theta_i'(x+e)-e\\&=\theta_i(x).\end{aligned}$$

We must show that θ_i is a translation. If it has a fixed point x, then $x+e$ is a fixed point of θ_i', which is impossible. And it meets the invariance condition because, for example,

$$\begin{aligned}F[\theta_i(x_1),\dots,\theta_i(x_n)] &= F'[\theta_i(x_1)+e,\dots,\theta_i(x_n)+e]-e\\ &= F'[\theta_i'(x_1+e),\dots,\theta_i'(x_n+e)]-e\\ &= \theta_i'F'(x_1+e,\dots,x_n+e)-e\\ &= \theta_i'[F(x_1,\dots,x_n)+e]-e\\ &= \theta_i F(x_1,\dots,x_n).\end{aligned}$$

Furthermore, $\phi(x)=x+e$ is an isomorphism from $\mathscr{R}'$ onto $\mathscr{R}$.

For a translation-homogeneous structure that is onto the real numbers and has an interior singularity, part (ii) of the following theorem establishes that there are no automorphisms other than the translations. In particular, no interval scale cases exist.

Theorem 5. *Suppose $\mathscr{R}=\langle Re, \geqslant, F, R_j\rangle_{j\in J}$ is a generalized concatenation structure with no extreme points and* 0 *is an interior generalized zero. Suppose further that $\mathscr{R}$ is solvable relative to* 0, *finitely unique, and translation homogeneous between singularities. Then $\mathscr{R}$ is isomorphic to $\mathscr{R}'=\langle Re, \geqslant, F', R_j'\rangle_{j\in J}$ that has the following properties*:

(i) *The translations of $\mathscr{R}'$ are multiplications by positive constants.*

(ii) *Every automorphism of $\mathscr{R}$ is a translation.*

(iii) *For all $x\in Re$, the solutions to $F'(x,\dots,x,S_i(x)x,\dots,x)=0$ are for some constants $A_i>0$ and $B_i<0$*:

$$S_i(x)=-x\begin{cases}A_i, & x>0,\\ 0, & x=0,\\ B_i, & x<0.\end{cases}$$

Proof. By Theorem 4 and the assumption of homogeneity, it follows from results of Alper (1987) that $\mathscr{R}^i$, $i=+,-$, are isomorphic under ϕ^i, $i=+,-$, to unit structures, whose translations are multiplication by a positive constants. Define the function ϕ on Re to agree with ϕ^+ on X^+, with ϕ^{-1} on X^{-1}, and 0 at 0. Let F'' be defined by:

$$F''(\dots,x_i,\dots)=\phi^{-1}F[\dots,\phi(x_i),\dots].$$

Define R_j'' similarly. Clearly, $\mathscr{R}''=\langle Re, \geqslant, F'', R_j''\rangle_{j\in J}$ is isomorphic to $\mathscr{R}$. Because each component is a unit representation in which the translations are multiplications by positive constants, we need to understand how the two constants defining a translation of the entire structure relate to one another. Three lemmas are required, which will be stated generally although they are used only with $\mathscr{R}''$.

Lemma 5. *Suppose $\mathscr{X}$ is an order-dense generalized concatenation structure that has no extreme points, e is an interior generalized zero, and $\mathscr{X}$ is solvable relative to e. Then for any $x>e$, there exist $y<e$ and $z<e$ such that, for each position i, $F(x,\ldots,x,y,x,\ldots,x)>e>F(x,\ldots,x,z,x,\ldots,x)$, and a similar assertion holds for $x<e$.*

Proof. Let $y'=\max_i[s_i(x)]$ and $z'=\min_i[s_i(x)]$. By monotonicity both $y'<e$ and $z'<e$. By order density, there exist y with $e>y>y'$ and by the fact that there is no extreme point there is a z such that $e>z'>z$. Monotonicity yields the assertion. □

Lemma 6. *On the same assumptions as in Lemma 5, if α,β are two automorphisms that agree on one of $\mathscr{X}^+$ and $\mathscr{X}^-$, then they agree everywhere.*

Proof. With no loss of generality, suppose they agree on $\mathscr{X}^+$. By Lemma 5, for any $x<e$, select $y>e$ such that $F(x,y,\ldots,y)>e$. By the fact α,β are automorphisms and agree on $\mathscr{X}^+$:

$$\begin{aligned} F[\alpha(x),\beta(y),\ldots,\beta(y)] &= F[\alpha(x),\alpha(y),\ldots,\alpha(y)] \\ &= \alpha F(x,y,\ldots,y) \\ &= \beta F(x,y,\ldots,y) \\ &= F[\beta(x),\beta(y),\ldots,\beta(y)], \end{aligned}$$

and so by monotonicity, $\alpha(x)=\beta(x)$, proving $\alpha\equiv\beta$. □

Lemma 7. *Suppose $\mathscr{R}=\langle Re,\geqslant,F,R_j\rangle_{j\in J}$ is a real concatenation structure, 0 is an interior generalized zero, $\mathscr{R}$ is solvable relative to 0, $\mathscr{R}$ is translation homogeneous between singularities, and for each translation τ, there exist positive constants u and t such that $\tau(x)=ux$ if $x>0$, $\tau(x)=tx$ if $x<0$, and $\tau(0)=0$. Then there exists a fixed constant $\varrho>0$ such that for each $u>0$, $t=u^\varrho$.*

Proof. By Lemma 6, $t=T(u)$ is a function of u. We show that T is a strictly increasing function: suppose $u<u'$ and $x>0$, then by solvability $F[s(x)x,\ldots,x]=0$, where $s(x)<0$. Applying the two translations $\langle u,T(u)\rangle$ and $\langle u',T(u')\rangle$, we see

$$F[us(x),\ldots,T(u)x]=0=F[u's(x),\ldots,T(u')x]. \tag{1}$$

Then,

$$\begin{aligned} u<u' &\text{ iff } us(x)>u's(x) && [\text{because } s(x)<0] \\ &\text{ iff } T(u)x<T(u')x && [\text{monotonicity on equation (1)}] \\ &\text{ iff } T(u)<T(u') && (x>0). \end{aligned}$$

So T is a strictly increasing function of t.

By Lemma 5, for $x<0$ select $y>0$ such that $F(x,\dots,x,y)<0$, and apply the translation $\langle u, T(u)\rangle$ to it:

$$uF(x,\dots,x,y)=F[ux,\dots,ux,T(u)y]. \tag{2}$$

Select $u=1/x$, and we see from equation (2):

$$F(x,\dots,x,y)=xF[1,\dots,1,T(1/x)y]. \tag{3}$$

Substituting (3) into both sides of (2):

$$uxF[1,\dots,1,T(1/x)y]=uxF[1,\dots,1,T(1/ux)T(u)y].$$

Cancel the common factor ux and then by monotonicity:

$$T(1/x)=T(1/ux)T(u).$$

Setting $x=1$ yields $T(u)=T(1)/T(1/u)$, but by (2) and monotonicity we see $T(1)=1$. Thus,

$$T(ux)=T(u)T(x), \tag{4}$$

and it is well known that the only strictly increasing functions satisfying (4) are $T(u)=u^{\varrho}$, for some $\varrho>0$. □

Let ϱ be the constant defined in Lemma 7 when it is applied to $\mathscr{R}''$, and define $\lambda: Re\to$ (onto)Re by

$$\lambda(x)=\begin{cases} x, & \text{if } x>0,\\ 0, & \text{if } x=0,\\ -(-x)^{1/\varrho}, & \text{if } x<0.\end{cases}$$

And define F' (and similarly R_j') on Re by

$$F'(\dots,x_i,\dots)=\lambda^{-1}F''[\dots,\lambda(x_i),\dots].$$

Because, as is easily seen, λ is strictly monotonic increasing, $\mathscr{R}''$ and $\mathscr{R}'=\langle Re,\geqslant,F',R_j'\rangle_{j\in J}$ are isomorphic under λ, and so $\mathscr{R}$ and $\mathscr{R}'$ are isomorphic under $\lambda\phi$.

(i) For $t>0$, consider the mapping $\tau(x)=tx$ in $\mathscr{R}'$. Obviously, τ is order preserving. We show $\tau F'(\dots,x_i,\dots)=F'[\dots,\tau(x_i),\dots]$, which establishes that it is an automorphism and so a translation because $tx=x$ only for $t=1$ or $x=0$. Observe the following properties of λ:

$$\lambda[\tau(x)]=\lambda(tx)=\begin{cases} tx,\\ t^{1/\varrho}x,\end{cases}\qquad \lambda^{-1}(x)=\begin{cases} x, & x\geqslant 0,\\ -(-x)^{\varrho}, & x<0.\end{cases}$$

We now demonstrate that multiplication by a constant in $\mathscr{R}'$ is a translation because it corresponds to one in $\mathscr{R}''$:

$$\lambda^{-1}(tx)=\begin{cases} tx, & x\geqslant 0,\\ -(-tx)^{\varrho}, & x<0,\end{cases}$$

$$=\begin{cases} t\lambda^{-1}(x), & x\geqslant 0,\\ t^{\varrho}\lambda^{-1}(x), & x<0.\end{cases}$$

(ii) By Alper (1987), we know that any automorphisms aside from the translations in $\mathscr{R}''$ are of the form:

$$\alpha(x)=\begin{cases} ax^{b}, & x>0,\\ -c(-x)^{d}, & x<0.\end{cases}$$

By the fact that these are unit structures we have:

$$F''(x,y,\ldots,y)=yF''(x/y,1,\ldots,1).$$

Applying α to $F''(u,1,\ldots,1)$ with $u>0$ yields:

$$aF''(u,1,\ldots,1)^{b}=F''(au^{b},a1^{b},\ldots,a1^{b})=aF''(u^{b},1,\ldots,1).$$

By the usual arguments, this holds for all rational b and so with the monotonicity of F'', the only solution is $F''(x,1,\ldots,1)=x^{\sigma}$, for some positive σ. Thus,

$$F''(x,y,\ldots,y)=y(x/y)^{\sigma}=x^{\sigma}/y^{\sigma-1}.$$

If $\sigma\neq 1$, then this approaches ∞ as y approaches 0, which is impossible because by Lemma 3, 0 is a zero and so $F''(x,0,\ldots,0)=x$. So $\sigma=1$, in which case $F''(x,y,\ldots,y)=x$, which violates the monotonicity in y. This contradiction means that the supposition of automorphism other than translations is false.

(iii) To find the solutions in F', we first explore the intermediate case F''.

Lemma 8. *Under the conditions of Lemma 7, there exist constants $a_i>0$, $b_i>0$ such that the solutions relative to 0 (Definition 5) are given by*

$$s_i(x)=\begin{cases} -a_i x^{\varrho}, & \text{if } x>0,\\ 0, & \text{if } x=0,\\ b_i(-x)^{1/\varrho}, & \text{if } x<0,\end{cases}$$

where ϱ is the constant asserted in Lemma 7.

Proof. Suppose $x>0$. Applying the translation $\langle u,u^{\varrho}\rangle$ to $0=F[x,\ldots,x,s_i(x),x,\ldots,x]$, we see

$$0=F[ux,\ldots,ux,u^{\varrho}s_i(x),ux,\ldots,ux]=F[ux,\ldots,ux,s_i(ux),ux,\ldots,ux],$$

whence by monotonicity

$$s_i(ux)=u^{\varrho}s_i(x).$$

Setting $x=1$ and $a_i=-s_i(1)$, this reduces to $s_i(u)=-a_i u^{\varrho}$. For $x<0$ a similar argument shows $s_i(u^{\varrho}x)=us_i(x)$, and setting $x=-1$ and $b_i=s_i(-1)>0$, yields $s_i(y)=b_i(-y)^{1/\varrho}$ for $y<0$. □

To complete the proof of the theorem, let $S_i(x)$ be the solution in F'. Thus,

$$\lambda^{-1}S_i(x)=s_i[\lambda^{-1}(x)]=\begin{cases}-a_i\lambda^{-1}(x)^{\varrho} \\ b_i[-\lambda^{-1}(x)^{1/\varrho}]\end{cases}=\begin{cases}-a_i x^{\varrho}, & x\geqslant 0, \\ -b_i x, & x<0.\end{cases}$$

Applying λ to this expression,

$$S_i=\begin{cases}-(a_i x^{\varrho})^{1/\varrho} \\ b_i(-x)\end{cases}=-x\begin{cases}A_i, & x\geqslant 0, \\ B_i, & x<0,\end{cases}$$

where $A_i=a_i^{1/\varrho}$ and $B_i=b_i$. □

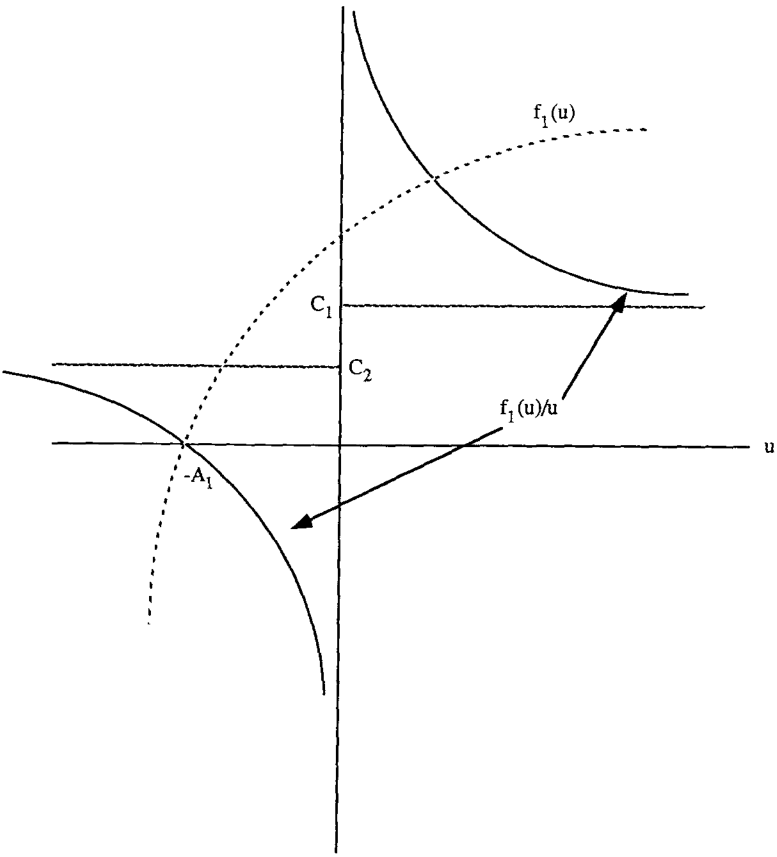

Fig. 1. An example of the typical behavior of $f_1(u)$ and $f_1(u)/u$ versus u. The behavior of f_2 is similar except that the roles of C_1 and C_2 are reversed and each branch of $f_2(u)/u$ ultimately falls below its appropriate C_i.

Corollary. *Under the conditions of the theorem, suppose that F is binary. Then, there exist positive constants C_1, C_2, and functions $f_i : Re \to$ (onto)Re, $i=1,2$, such that (see Fig. 1):*

(i) *For $i=1,2$, f_i is strictly increasing.*

(ii) *For $i=1,2$, f_i/ι, where ι = identity, is strictly decreasing except at 0.*

(iii) *For all $x, y \in Re$,*

$$F'(x,y)=\begin{cases} yf_1(x/y), & \text{if } y>0,\\ C_1x, & \text{if } y=0,\ x\geqslant 0,\\ C_2x, & \text{if } y=0,\ x<0,\\ yf_2(x/y), & \text{if } y<0.\end{cases}$$

(iv) *$f_1(u)/u>C_i$, where $i=1$ for $u>0$ and $i=2$ for $u<0$, $f_2(u)/u<C_i$, where $i=1$ for $u<0$ and $i=2$ for $u>0$.*

(v) *$f_i(u)/u$ approaches $-\infty$ as u approaches 0 from below and $+\infty$ as u approaches 0 from above.*

(vi) *The constants A_i and B_i of part* (iii) *of the theorem satisfy*:

$$A_1=1/B_2, \qquad A_2=1/B_1, \qquad f_1(-A_1)=f_2(B_1)=0.$$

(vii) *$f_1 \equiv f_2$ iff, for all $x, y \in Re$, $F'(-x,-y)=-F'(x,y)$.*

Proof. Let $f_1(z)=F'(z,1)$, $f_2(z)=-F'(-z,-1)$, $C_1=F'(1,0)$, and $C_2=-F'(-1,0)$.

(i) The monotonicity of F' in the first argument clearly implies f_i is strictly increasing.

(iii) Using the unit structure aspect, we have for $y>0$,

$$F'(x,y)=F'[y(x/y), y1]=yF'(x/y,1)=yf_1(x/y),$$

and for $y<0$,

$$F'(x,y)=F'[(-y)(-x/y),(-y)(-1)]=-yF'(-x/y,-1)=yf_2(x/y).$$

For $y=0$, because 0 is a generalized zero, there are translations C_i, $i=1,2$, of the two halves such that

$$F'(x,0)=\begin{cases} C_1x, & x>0,\\ 0, & x=0,\\ C_2x, & x<0.\end{cases}$$

(ii) Suppose $x>0$, then $z=x/y<x/y'=z'$ iff $y>y'$ iff $F'(x,y)>F'(x,y')$. Using the monotonicity of F' in the second component and the fact that $x>0$ yields:

$$F'(x,y)/x>F'(x,y')/x \quad \text{iff } f_i(z)/z=f_i(x/y)/(x/y)>f_i(x/y')/(x/y')=f_i(z')/z',$$

where $i=1$ if $y>0$ and $i=2$ if $y<0$. A similar argument, using $x<0$, establishes the decreasing character of the f_i/ι for the entire domain except, of course, at 0.

(iv) For $y>0$, monotonicity yields $F'(x,y)>F'(x,0)$, and so by part (iii),

$$yf_1(x/y)>C_i x, \quad \text{where } i=1 \text{ if } x\geqslant 0 \text{ and } i=2 \text{ if } x<0.$$

Set $u=x/y$, then

$$f_1(u)/u>C_i, \quad \text{where } i=1 \text{ if } u\geqslant 0 \text{ and } i=2 \text{ if } u<0.$$

The argument for f_2 is similar.

(v) For $u>0$, $f_i(u)>f_i(0)$, and so $\lim f_i(u)/u \geqslant f_i(0)/u$, which approaches $+\infty$ as u approaches 0. The argument for $u<0$ is similar.

(vi) Substituting the expression from part (iii),

$$0=F'[S_1x),x]=\begin{cases} F'(-xA_1,x) \\ F'(-xB_1,x) \end{cases}=\begin{cases} xf_1(-A_1), & x>0, \\ xf_2(B_1), & x<0. \end{cases}$$

(vii) Suppose $x>0$, $y>0$, then because $-x<0$, $-y<0$,

$$F'(-x,-y)=-yf_2(x/y) \quad \text{and} \quad -F'(x,y)=-yf_1(x/y),$$

so for $x/y>0$, the assertion holds. Next, suppose $x>0$, $y<0$, then

$$F'(-x,-y)=-yf_1(x/y) \quad \text{and} \quad -F'(x,y)=-yf_2(x/y),$$

and so it also holds for $x/y<0$. □

8. Generalized associativity

It is well known (see for example Aczél, 1966, p. 254) that an associative binary concatenation structure that satisfies the assumptions of Theorem 5 has a ratio scale representation with $F'(x,y)=x+y$. The question to be addressed is the degree to which this can be generalized to operations with n arguments. The answer is 'completely', as formulated in the following theorem.

Theorem 6. *Suppose $\mathscr{R}=\langle Re, \geqslant, F, R_j\rangle_{j\in J}$ is a generalized concatenation structure that fulfills the assumptions of Theorem 5 and in addition:*

(i) *is associative in the sense that if F is of order n and $x_1, x_2, \ldots, x_{2n-1}$ are any $2n-1$ arguments, then any grouping of n successive ones such as*

$$F[x_1,\ldots,x_i,F(x_{i+1},\ldots,x_{i+n}),x_{i+n+1},\ldots,x_{2n-1}]$$

is equivalent to any other such grouping; and

(ii) *has inverses in the sense that for each real x, there exists real x^{-1} such that*

$$F(x^{-1},x,0,\ldots,0)=0.$$

Then there exists a real ratio-scale representation F' such that for all $x_i\in Re$, $i=1,2,\ldots,n$,

$$F'(x_1,\ldots,x_n)=\sum x_i.$$

Proof. Once again, it is easiest to break the proof into a series of lemmas. Throughout, we work with the ratio scale representation of Theorem 5, but for simplicity of notation, I omit ′.

Lemma 9. *$C_i = 1$, $i = 1, \ldots, n$, where these are the constants of the generalized zero.*

Proof. We first show that $C_1 = 1$:

$$\begin{aligned} F(x_1, \ldots, x_{n-1}, C_1 x_n) &= F[x_1, \ldots, x_{n-1}, F(x_n, 0, \ldots, 0)] && \text{(property of } C_1\text{)} \\ &= F[F(x_1, \ldots, x_{n-1}, x_n), 0, \ldots, 0] && \text{(associativity)} \\ &= C_1 F(x_1, \ldots, x_{n-1}, x_n) && \text{(property of } C_1\text{)} \\ &= F(C_1 x_1, \ldots, C_1 x_{n-1}, C_1 x_n) && \text{(ratio scale property).} \end{aligned}$$

By monotonicity, the result follows. A similar proof shows that $C_n = 1$. Next we show that $C_2 C_{i-1} = C_i$:

$$\begin{aligned} C_2 C_{i-1} x &= C_{i-1} F(0, x, 0, \ldots, 0) && \text{(property of } C_2\text{)} \\ &= F(0, C_{i-1} x, 0, \ldots, 0) && \text{(ratio scale)} \\ &= F[0, F(0, \ldots, 0_{i-2}, x, 0, \ldots, 0), 0, \ldots, 0) && \text{(property of } C_{i-1}\text{)} \\ &= F[F(0, \ldots, 0_{i-1}, x, 0, \ldots, 0), 0, \ldots, 0) && \text{(associativity)} \\ &= F(0, \ldots, 0_{i-1}, x, 0, \ldots, 0) && (C_1 = 1) \\ &= C_i x. && \text{(property of } C_i\text{).} \end{aligned}$$

By induction, for $i = 3, \ldots, n$, $C_i = C_2^{i-1}$. Thus, $1 = C_n = C_2^{n-1}$, so $C_i = 1$. □

Lemma 10. *For each position i,*

$$F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) = F(x_1, \ldots, x_{i-2}, 0, x_{i-1}, \ldots, x_n).$$

Proof.

$$\begin{aligned} &F(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n) \\ &\quad = F[x_1, \ldots, x_{i-2}, F(0, x_{i-1}, 0, \ldots, 0), 0, x_{i+1}, \ldots, x_n] && (C_2 = 1) \\ &\quad = F[x_1, \ldots, x_{i-2}, 0, F(x_{i-1}, 0, \ldots, 0), x_{i+1}, \ldots, x_n] && \text{(associativity)} \\ &\quad = F[x_1, \ldots, x_{i-2}, 0, x_{i-1}, x_{i+1}, \ldots, x_n] && (C_1 = 1). \quad \square \end{aligned}$$

So, by induction, one can move a 0 anywhere in the sequence.

Define $h(x, y) = F(x, y, 0, \ldots, 0)$.

Lemma 11. *h is associative.*

Proof.

$$\begin{aligned} h[h(x,y),z] &= F[F(x,y,0,\dots,0),z,0,\dots,0] && \text{(definition of } h) \\ &= F[x,F(y,0,\dots,0,z),0,\dots,0] && \text{(associativity)} \\ &= F[x,F(y,z,0,\dots,0),0,\dots,0] && \text{(Lemma 10)} \\ &= h[x,h(y,z)] && \text{(definition of } h). \quad \square \end{aligned}$$

Lemma 12. *There exists an isomorphism ϕ such that $\phi h(x,y)=\phi(x)+\phi(y)$.*

Proof. By Lemma 11, h is associative; by assumption, for each x there exists x^{-1} such that $h(x^{-1},x)=0$; and by Lemma 9, $h(x,0)=h(0,x)=x$. According to Aczél (1966), such a structure is isomorphic to addition, which is a ratio scale representation. □

Lemma 13. $F(x_1,x_2,\dots,x_{n-1},x_n)=h(x_1,h(x_2,\dots,h(x_{n-1},x_n)\dots))$.

Proof.

$$\begin{aligned} &F(x_1,x_2,\dots,x_{n-1},x_n) \\ &\quad = F[x_1,x_2,\dots,x_{n-2},F(0,x_{n-1},0,\dots,0),x_n] && (C_2=1) \\ &\quad = F[x_1,x_2,\dots,x_{n-2},0,F(x_{n-1},0,\dots,0,x_n)] && \text{(associativity)} \\ &\quad = F[0,x_1,x_2,\dots,x_{n-2},F(x_{n-1},x_n,0,\dots,0)] && \text{(Lemma 10)} \\ &\quad = F[0,x_1,x_2,\dots,x_{n-2},h(x_{n-1},x_n)] && \text{(definition of } h). \end{aligned}$$

The proof is completed inductively. □

Define

$$F'(x_1,x_2,\dots,x_n)=\phi F[\phi^{-1}(x_1),\phi^{-1}(x_2),\dots,\phi^{-1}x_n)],$$

then the theorem follows immediately from Lemmas 12 and 13. □

9. Representation of continuous conjoint structures

The terminology of this section follows that of Luce and Cohen (1983) or, equally well, Section 19.6 of Luce et al. (1990). Most of the definitions will not be repeated here. I simply outline how for a binary conjoint structure $\mathscr{C}=\langle A\times P, \succsim\rangle$ that is solvable relative to a singular point a_0p_0 (one that is invariant under the factorizable automorphisms of $\mathscr{C}$) the results of the corollary to Theorem 5 can be used to arrive at a representation of $\mathscr{C}$.

Recall that with the usual definitions of π, ξ, and $\circ$, namely $a_0\pi(a)\sim ap_0$, $\xi(a,p)\pi_0\sim ap$, and $a\circ b=\xi[a,\pi(b)]$, then $\mathscr{I}_A=\langle A, \succsim_A, \circ\rangle$ is a total concatenation structure. Because a_0p_0 is singular, Theorem 19.12 of Luce et al. (1990) established that a one-to-one function from A onto A is the A-component of a factorizable automorphism iff it is an automorphism of $\mathcal{I}_A$. Thus a_0 is a singular point of $\mathcal{I}_A$.

To invoke the corollary to Theorem 5, $\mathcal{I}_A$ must meet the following conditions: solvability relative to a_0, finite uniqueness, translation homogeneity between singularities, and be isomorphic to a real structure. We examine each of these.

To achieve solvability, we need to add to the solvability conditions of $\mathscr{C}$ that for each $a \in A$, there exists an element $\sigma(a) \in P$ such that $a, \sigma(a) \sim a_0 p_0$, and for each $p \in P$ there exists $\eta(p) \in A$ such that $\eta(p), p \sim a_0 p_0$. It is easy to verify that $s_2(a) = \pi^{-1}\sigma(a)$ and $s_1(a) = \eta\pi(a)$ are the solutions of a relative to 0.

The appropriate definitions of uniqueness and homogeneity are given in Definition 9 of Chapter 20 of Luce et al. (1990).

The mapping onto the reals can be assured in various ways, perhaps the simplest being to suppose that it is true for $\mathscr{C}$.

That done, then one has a representation ϕ onto a numerical concatenation structure meeting the conditions of the corollary to Theorem 5. From that, one induces a representation ψ of $\mathscr{C}$ onto the same structure simply by defining $\psi(a,p) = \phi\xi(a,p)$. This can, of course, be worked out in detail in special cases.

All of the above generalizes in an obvious way to general conjoint structures by using the results of Theorem 5, rather than the corollary.

10. Concluding remarks

Concatenation and conjoint structures are the mainstays of applied measurement models, and for a number of attributes it is reasonable to assume that the strong assumption of homogeneity is satisfied. However, there exist sufficiently many familiar examples of non-homogeneous structures that we must continue to study them. The strategy of this paper has been to focus on structures with singular points, i.e. points fixed under every automorphism, and to weaken 1-point homogeneity only to the extent of saying that it continues to hold between adjacent singular points.

The case worked out in detail in the paper involves a monotonicity condition that precludes structures, like probability, in which two non-extreme points can concatenate to equal an extreme one. The present theory treats only structures, like relativistic velocity, for which the extreme points (zero velocity and the velocity of light) can be dropped or, like prospect theory, in which they do not exist, and the remaining structure continues to be a concatenation structure. It may have an interior singularity, as in prospect theory, which under the assumption of finite uniqueness was shown to be unique and to behave as a generalized zero.

For the continuous case, the possible forms for the representation were worked out, and certainty equivalents of money gambles are shown in Luce (1992) to be one possible interpretation of generalized concatenation structures. In addition, an outline was provided of how to get representations of continuous conjoint structures from these results on generalized concatenation structures.

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