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Where Does Subjective Expected Utility Fail Descriptively?

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Abstract

Subjective expected utility (SEU) rests on and implies four tenets of rational preferences; transitivity, monotonicity of consequences, independence of a common consequence, and accounting equivalences. Empirical evidence against transitivity and monotonicity is reevaluated and the opposite conclusion drawn using more recent data. The more complex accounting equivalences are descriptively implausible. The three simplest—idempotence, complementarity, and event commutativity—seem to be the only ones that may be descriptive. These, coupled with the postulate of an interval scale representation, result in a rank-dependent, weighted linear generalization of SEU. Further generalizations to nonbinary mixtures and to rank- and sign-dependent representations are also described. Problems in testing these theories are discussed.

Key words: subjective expected utility, rank-dependent linear utility, rank- and sign-dependent linear utility

The goal of any of the classical versions of the theory of subjective expected utility (SEU) has been to model normatively, and perhaps descriptively, the nature of individual choices among uncertain alternatives. All such theories have three aspects: a model for the domain of choice, i.e., a model of uncertain or risky alternatives, a set of assumptions (= axioms) that characterize preference relations over that domain, and a representation theorem into the real numbers. In this article, the alternatives, whether risky or uncertain, shall be referred to as *gambles*. The standard SEU representation involves a probability measure S over events and a utility function U over gambles with the properties that the utility function is order preserving and the utility of a gamble is given by the expectation of U evaluated over the components of the gamble relative to the measure S of the events involved in the gamble. This kind of theory originated with von Neumann and Morgenstern (1947) (see Fishburn, 1989, for an appraisal), and over the past 43

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years numerous versions of the theory have appeared. A summary of some of the later work can be found in Fishburn (1982) and Wakker (1989).

Although a fair consensus exists that SEU is normatively adequate in certain highly static situations, an at least comparable consensus holds that it fails to be descriptively adequate, even in those static situations.[1] A compilation of its failures was given over a decade ago by Kahneman and Tversky (1979) and by MacCrimmon and Larsson (1979), and the empirical situation is not greatly different today except for some subsequent studies attempting to localize the failure.

The purpose of this article is to reexamine the empirical situation, because it appears that both economists and psychologists have tended to draw incorrect inferences from the data and that, as a result, some theorists have been following a wrong path in trying to evolve a more descriptive theory. The approach here will be to return to first principles and to ask just what assumptions are being made—in particular, seeking ones that are to some degree implicit—and then to ask for each what the data suggest. As will become evident, this author disagrees with much of the conventional wisdom of the area.

Once done, the next natural question to raise is what representations are entailed by the apparently acceptable assumptions. This article shall summarize, without proof and in less than fully formal form, results from several technical papers about the resulting representations, the most general being a linear, weighted utility function with rank- and sign-dependent weights. This means that the weights depend not only on the event in question, but on the sign of the consequence associated with it and its ranked position relative to the other possible consequences of that gamble.

1. The domain

It seems that far less attention has been paid to the model of the domain than is warranted. For example, more often than not economists begin by restricting attention to monetary gambles with known probabilities, which they model as *random variables*. After all, isn't that just what a money gamble is?

One's answer depends, in this author's opinion, on just how the gamble is framed (to use a term of Tversky and Kahneman, 1981, 1986). If one considers a lottery that consists of three stages, the first yielding either nothing with some high probability and otherwise a second-stage lottery, which itself may result in a third lottery that determines the positive payoff, is the probability distribution over final outcomes actually clear to most decision makers? Or, in an example familiar to all of us, when purchasing home and auto insurance, how many of us work out the full probability distribution of the several consequences, including that of the car being in the house when the latter burns? The ability to know that distribution is exactly what one (implicitly) postulates by assuming that all monetary gambles, including compound ones, are random variables.

Put another way, one is implicitly assuming that how a gamble is framed as a random variable is immaterial to decisions. This follows by the mathematics invoked for preferences, for if two things are equal in the domain, then automatically they are assumed to be judged indifferent. Thus, by assuming that one is dealing with random variables, two different framings of the random variable *must* be evaluated by the decision maker as indifferent in preference simply because mathematically they describe the same random variable. In the language of game theory, two extensive structures with the same normal form are treated as equivalent in preference.

Once clearly stated, the implausibility of this postulate is also apparent. Hardly anyone believes that it applies to any but the most idealized decision maker. Attention and calculation are necessary in order to ascertain whether two gambling formats are identical or not, and few people devote the time or possess the skill to carry that out. Such an assumption may well be judged to be normatively compelling, provided one is willing to ignore computational capacity limitations, but surely it is not descriptive. One aspect of *bounded rationality* (Simon, 1955, 1978) is exactly this failure to analyze correctly complex presentations of random variables. Notice also that the expected utility representation not only implies such an assumption, but also is entailed by it together with certain normative assumptions (see below and Hammond, 1988). In early presentations, such as that of von Neumann and Morgenstern (1947), the indifference of the reduced version of a compound gamble to the compound gamble itself was explicitly assumed. Only later was the assumption made implicit by assuming the domain to be random variables.

It is a matter of some delicacy just how to model the domain more realistically. Not only is there a problem in dealing with compound gambles, but it is far from obvious that a Savage-like finite, state-space approach makes sense either (Savage, 1954). One probably needs a theory for comparing gambles over different chance events with different numbers of consequences.

Segal (1990) has approached the problem for money gambles with known probabilities by assuming that two-stage gambles are not automatically reduced to one-stage gambles. His development, in spirit, is somewhat like the one outlined here, but differs in detail and leads to a different representation. Roughly, the tack to be followed here is the one employed in abstract algebra, where a lot of parentheses are used to indicate the framing and one writes down as an explicit formula each alleged preference equivalence between pairs of formally equivalent framings of a gamble. Such equivalences between formally equivalent gambles shall be referred to as *accounting equivalences*.

Once one forgoes most accounting equivalences, nothing much is gained by restricting attention to monetary gambles—it is enough to speak of alternatives generated recursively from some algebra of events $\mathcal{E}$ and from some set of outcomes, money or otherwise.

1.1. Binary alternatives

Initially, only the subfamily of compound gambles that are binary mixtures of consequences—gambles or pure consequences—will be discussed. That is, suppose X and Y are two possible consequences and $A \in \mathcal{E}$ is an event. Then $X \bigcirc_A Y$ denotes the gamble in which X is the consequence if A occurs and Y is the consequence if A fails to occur. The operation notation $\bigcirc_A$ is used because one draws on results from the theory of ordered systems with a binary operation.

In this notation, compound gambles may be framed explicitly. For example, $(X \bigcirc_A Y) \bigcirc_B Z$ is interpreted as follows. The underlying chance experiment is carried out, and we first focus on the event B. If B fails to occur, the decision maker receives Z and the transaction is complete; if B does occur, then the decision maker is confronted with a gamble, namely $X \bigcirc_A Y$, that is as yet unresolved. To resolve it, the experiment is run again independently of its first realization, and the focus is on the event A, with X the consequence if A occurs and Y otherwise. Note carefully that the event subscripts refer to *independent realizations* of the underlying chance experiment. Thus, somewhat contrary to usual mathematical convention, the repetition of the symbol A denotes independent realizations of the event A. It is much like using the same symbol X for the independent realizations of same random variable in a random sample.

To be formal, let $\mathcal{E}$ be a set of subsets that is closed under union and complementation, and D be a set of entities, called gambles,[2] that is endowed with closed binary operations $\bigcirc_A$, one for each $A \in \mathcal{E}$. There is assumed to be a binary relation $\gtrsim$ of weak preference over D, i.e., $\gtrsim$ is assumed to be transitive and connected. Thus, the entire structure of preferences over uncertain binary alternatives can be thought of as the algebraic system

$$\mathfrak{D} = \langle D, \gtrsim, \bigcirc_A \rangle_{A \in \mathcal{E}}.$$

We turn now to formulating normatively plausible axioms about this system and examining what is known about their empirical validity.

2. Four principles of rationality for preferences

A perusal of this literature suggests that there are just four major behavioral principles that have been invoked for normative theorizing. These shall be referred to as principles of *rationality*, although some people may be uneasy about the philosophical criteria whereby such a classification is made. This approach, in all honesty, is intuitive. All of the other assumptions involved seem to be more of a structural or technical nature. These often include some form of continuity or solvability that ensures that $\mathcal{E}$ and $\mathfrak{D}$ are sufficiently rich structures to allow certain mathematical constructions; that preference judgments can be made between any two objects of the domain; and that some form of Archimedeanness or of Dedekind completeness plus order density guarantees that we are in a situation that can be mapped into the real numbers.

The four rationality axioms, in addition to being normatively compelling, are in principle directly testable as well. This is not true for the technical and structural axioms.

The first two axioms of rationality can be derived from a single principle, to wit, when one consequence of a gamble is replaced by a more preferred consequence, then the modified gamble is preferred to the original gamble. Nevertheless, it is best to treat them separately.

2.1. Transitivity

For all $X, Y, Z \in D$,

$$\text{if } X \gtrsim Y \text{ and } Y \gtrsim Z, \text{ then } X \gtrsim Z.$$

Here, for example, one has the pair Y and Z for which Y is at least as preferred as Z, and the assertion is that if Y is replaced by X, which is assumed to be at least as preferred as Y, then in the pair X and Z, Z will never be preferred to X.

2.2. *Monotonicity*

For all $W, X, Y, Z \in D$ and $A \in \mathcal{E}$ such that A is neither null nor universal,

$$X \gtrsim Y \text{ iff } X \bigcirc_A Z \gtrsim Y \bigcirc_A Z$$
$$\text{iff } W \bigcirc_A X \gtrsim W \bigcirc_A Y.$$

Here if one begins with the mixture of Y and Z using event A and replaces Y by X, which is at least as preferred as Y, then the original mixture will never be strictly preferred to the A mixture of X and Z.

Monotonicity goes under a number of names, depending on context: independence, dominance, and substitution principle. Here the more neutral, mathematical term will be used, especially since the other concepts typically combine monotonicity with some form of accounting, to which we turn shortly.

2.3. *Independence of a common consequence*

If $A, B, C \in \mathcal{E}, A \cap C = B \cap C = \emptyset, X, Y \in D$, then

$$X \bigcirc_A Y \gtrsim X \bigcirc_B Y \text{ iff } X \bigcirc_{A \cup C} Y \gtrsim X \bigcirc_{B \cup C} Y.$$

This principle says that if two gambles have a common consequence over the same event, then the consequence can be changed to any other common consequence without affecting the ordering. Conceptually, this seems similar to monotonicity, but it is quite distinct, as we shall see.[3]

2.4. *Some accounting equivalences*

As indicated above, the general principle of accounting equivalences is that if two compound gambles—gambles with gambles as consequences—are such that each of the more primitive consequences (e.g., money) occurs under identical conditions, ignoring the order in which events are carried out, then they are judged indifferent. This, as stated, is clearly an axiom schema that embodies an infinity of more specific indifferences. This entire axiom schema shall be referred to as *universal accounting* (Luce, 1990a). The failure of such assumptions has received some attention in the more applied accounting literature under the title of *sequential aggregation* (Ronen, 1971; Snowball and Brown, 1979). Since the entire schema seems hopelessly optimistic as a descriptive principle, only some of the simplest special cases will be listed here.

Idempotence. For every $X \in D$ and $A \in \mathscr{E}$,

$$X \bigcirc_A X \sim X.$$

This simply says that the "uncertain" alternative in which X is the outcome whether or not A occurs is seen as indifferent to X itself. Violation of idempotence could arise, for example, if a decision maker simply got pleasure from or was adverse to having a chance experiment run, even though its running has no direct impact on his or her well-being.

Complementarity. For every $X, Y \in D$ and $A \in \mathscr{E}$,

$$X \bigcirc_A Y \sim Y \bigcirc_{A^c} X,$$

where A^c is the complement to A.

Here it is postulated that the decision maker recognizes that $(A^c)^c = A$, which renders the two framings identical. Although one can envisage people overlooking this fact, it does seem to be within the compass of most decision makers, even in the absence of external aids.

Event Commutativity. For every $X, Y \in D$ and $A, B \in \mathscr{E}$,

$$(X \bigcirc_A Y) \bigcirc_B Y \sim (X \bigcirc_B Y) \bigcirc_A Y.$$

This condition is slightly more complex than either of the preceding two. It simply says that the decision maker will realize that in each case the outcome X occurs when in two independent opportunities both events, A and B, occur and otherwise Y is the outcome. The two cases differ only in the temporal order of the events. This assumed irrelevance of the order of the events leads to the term *event commutativity*.

There are, of course, numerous additional examples of accounting equivalences. One of the next simplest is

$$(X \bigcirc_A Y) \bigcirc_A Z \sim (X \bigcirc_A Z) \bigcirc_A (Y \bigcirc_A Z),$$

which is known as *right autodistributivity*. It is significantly more difficult to see through than event commutativity. In both cases, the same experiment is repeated and X arises if both A and A occur, Y if A and not A, and Z if not A.

For the purposes that follow, it is sufficient to focus on either universal accounting or just the first three principles above: idempotence, complementarity, and event commutativity.

It is worth observing again that limits on the wholesale use of accounting equivalences can be viewed as providing specific meaning to the somewhat vague concept of bounded rationality, which has been mentioned in some general discussions of utility (Simon, 1955, 1978).

3. Binary subjective expected utility

For the domain of binary uncertain alternatives, SEU asserts that a "utility" function U can be assigned to uncertain alternatives and pure consequences (e.g., amounts of money), and a "subjective probability" function[4] S over events such that, for all $X, Y \in D$ and $A \in \mathcal{E}$:

$$1) \; X \gtrsim Y \text{ iff } U(X) \geq U(Y),$$

$$2) \; U(X \bigcirc_A Y) = U(X)S(A) + U(Y)[1 - S(A)].$$

SEU implies all four aspects of rationality in full-blown form: transitivity, monotonicity, independence of a common consequence, and universal accounting. Conversely, in the presence of some structural assumptions about the richness of the domain of uncertain alternatives and events, these rationality assumptions imply SEU.

As noted earlier, experimentalists—both psychologists and economists—have been successful in persuading theorists that no matter how compelling SEU may seem normatively, it is not descriptive. Thus, descriptively, one or more of these tenets of rationality is regularly violated. Therefore, knowing which are the culprits matters a lot to theorists attempting to construct more accurate descriptive theories. It shall be argued here that both experimentalists and theorists have evidenced some confusion about the significance of the data and have laid the blame as to which principles are nondescriptive on the wrong doorsteps. Equally important, when the situation is understood correctly, there are promising substitutes for SEU.

Let us turn to the data.

4. Data on rationality principles

4.1. Transitivity

Evidence against transitivity has begun to be taken seriously by theorists. A recent effort is that of Fishburn and LaValle (1988), who list earlier efforts. Let us consider that evidence.

Within the context of money gambles, the major evidence[5] that has been interpreted as failure of transitivity arises from what has come to be known as the preference reversal experiment. The phenomenon, which was first reported by Lichtenstein and Slovic (1971, 1973) and which has been repeatedly replicated (Bostic, Herrnstein, and Luce, 1990; Grether and Plott, 1979; Hamm, 1980; Karni and Safra, 1987; Lindman, 1971; Mowen and Gentry, 1980; Pommerehne, Schneider, and Zweifel, 1982; Reilly, 1982; Tversky, Slovic and Kahneman, 1990; Waters and Collins, 1984), is simply this: Denote by PG a binary gamble for which there is a high probability of winning a modest amount and by \$G one for which there is a low probability of winning a large amount. Choose them to have approximately the same expected values. Let CE(PG) denote the subject's judged monetary certainty equivalent to PG, and let CE($G) denote the judged monetary certainty equivalent to $G. The method of obtaining these certainty equivalences has been varied, and has included buying prices, selling prices, and the subject's judgment as to the choice indifference point in the sense that, in a choice between the gamble and CE + ϵ, $\epsilon > 0$, the money would be preferred, whereas in a choice between the gamble and CE − δ, $\delta > 0$, the gamble would be preferred. These are referred to as *judged indifference points* (JIPs).

The oft repeated finding is that when such judgments and choices are embedded in a great complex of other, similar decisions, then between one third and one half of subjects exhibit the following behavior: when PG and $G are offered, they select PG, i.e.,

$$\text{PG} > \$\text{G},$$

but when assigning monetary equivalences, they place a higher value on $G, i.e.,

$$\text{CE(PG)} < \text{CE(\$G)}.$$

Moreover, the monetary differences are a substantial fraction of the monetary range of the gambles; the reversal is not a consequence of noisy data. Since preferences are assumed to agree with money ordering, and since judged monetary equivalences are assumed to correspond to preference indifference, this observation establishes that transitivity fails.

This conclusion is surely correct if the auxiliary assumptions are true. Since all attempts to eliminate the phenomenon experimentally have failed (see especially Grether and Plott, 1979; Hamm, 1980; Pommerehne et al., 1982; Reilly, 1982), perhaps some attention needs to be paid to these auxiliary assumptions. Pearson (1986) and Bostic et al. (1990) questioned whether JIPs actually do correspond to the choice indifference points (CIPs) that would be obtained by pitting money versus the gambles in actual choices. The practical problem faced by Bostic et al. (1990) was to avoid exasperating the subjects by requiring too many repeated comparisons of money sums with gambles. Toward that end, they borrowed from psychophysics a sequential procedure (up-down and a more refined method called PEST) of adjusting money amounts so that they converged fairly rapidly on the CIP (see von Béskéy, 1947; Dixon and Mood, 1948; Luce and Green, 1974; and Wetherill, 1966). The design involved a somewhat intricate, computer-controlled interleaving of JIPs, choices between gambles, and CIPs estimated by the *PEST* procedure. Among other considerations, both the usual JIP preference-reversal experiment and the CIP one were embedded with four gamble pairs each.

The concern of Bostic et al. (1990) that JIPs might differ from CIP was strongly confirmed, as can be seen in figure 1. For the PG gambles no difference was evident, but for the $G ones the difference is as much as $2 in a total range of outcomes from −$2 to $16.

This result is consistent with a series of studies of a quite different experimental design by Tversky, Sattath, and Slovic (1988) and Tversky, Slovic and Kahneman (1990). Their interpretation was that when people are asked to judge complex stimuli along a single

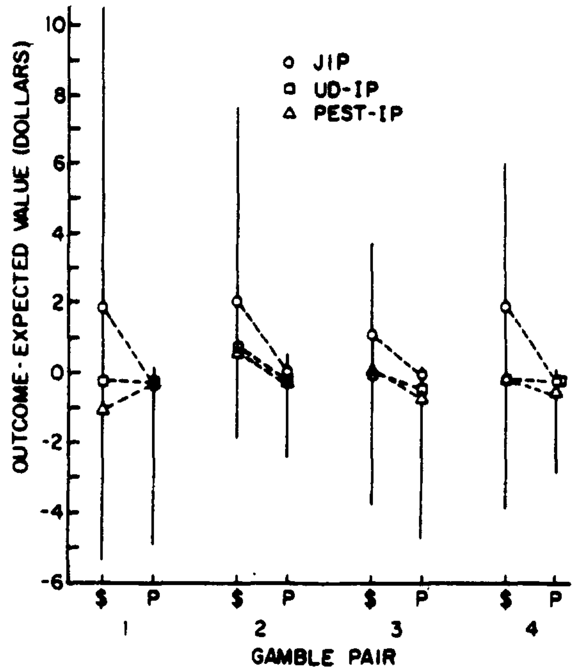

Figure 1. For each \$- and P-gamble of the four pairs studied, differences between the money assigned to the gamble and the expected value of the gamble are shown for three different procedures to estimate the indifference point. The expected value was the same for each gamble of a pair, but differed among pairs. The three procedures for assigning money values were judged indifference points (JIPs) and two forms of a sequentially determined choice indifference point (CIP). The important fact is that the JIP is substantially—between \$1 and \$2—above either CIP for \$-gambles, whereas all three are approximately the same for P-gambles. (From Bostic et al. (1990), with permission.)

dimension such as money, then that dimension becomes the most salient aspect of the complex stimuli and other aspects are less heavily involved in the judgment. Thus, the relatively larger amount of money in the \$G gamble is overweighted relative to the small probability when making money judgments as compared to what happens when making choices. Exactly why the effect fails to occur in choices when, like the judgments, they are between pure sums of money and a gamble is not entirely transparent.

Whatever the cause of the discrepancy may be, the important thing is to recognize that it exists: judgments of certainty equivalents and choices between gambles and their certainty equivalents simply do not yield the same result. Since the theory of rational choice concerns choices, not judgments, the question then is whether evidence for intransitivity disappears when CIPs are used rather than JIPs. The answer is apparently yes—at least, they are markedly reduced. Bostic et al. (1990) replicated the usual preference-reversal phenomenon when JIPs were used, but it was reduced to approximately the noise level of the data when CIPs were used.

The major conclusions from this study are two. First, maybe transitivity is not as empirically suspect as some have come to think. In this author's opinion, a theorist can, in good conscience, still postulate transitivity. Second, since a good deal of economic life does entail assigning monetary equivalences to various things, perhaps we would do well to study carefully how judged indifferences are formed and exactly how they relate to choices. Luce (in press) provides an attempt. And for psychologists this raises an added question: Does the JIP/CIP difference arise in domains other than gambles? There may be something worth studying here.

4.2. *Monotonicity*

A number of studies have been interpreted as violations of something like monotonicity, but perhaps the clearest remains the granddaddy of them all—the Allais (1952, 1953) paradox. For a comparatively recent examination of the issues surrounding this paradox, see Allais and Hagen (1979). Although this example began as a *gedanken* experiment and at first was empirically tried out very informally, many later, careful studies confirm the problem. For example, MacCrimmon and Larsson (1979) and, independently, Kahneman and Tversky (1979) report a number of sets of group data in which such violations of SEU occur. Keller (1985) has shown the degree of violation depends on exactly how the decision is represented to the subject, where a matrixlike graphic display effects the greatest reduction.

The pair of choice situations is shown graphically in figure 2. The finding is that many people choose a_1 over a_2 in the first choice and a_3 over a_4 in the second. The analysis as to why this is irrational is well known, but it will be repeated here in some detail, since it appears to have been improperly interpreted. It is laid out in figure 3. The conclusion is that the data are inconsistent with this analysis, which means that since both an accounting equivalence and monotonicity are used in the argument, at least one is in error. Because the accounting equivalence is usually simply taken for granted, the fault is traditionally attributed to monotonicity. Since, on the face of it, accounting arguments are really quite suspect descriptively, the usual inference does not seem justified. Rather, we need to study monotonicity in experimental designs that do not implicitly invoke an accounting argument. So far as this author is aware, the only published study is in Kahneman and Tversky (1979), classed under the heading *Isolation effect*. In essence, they show that when the Allais paradox is presented as a two-stage gamble, the paradox goes away, at least on average. Their data are not such that subject-by-subject comparisons are possible.

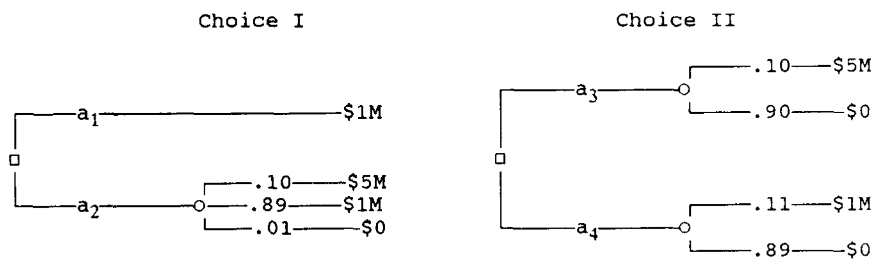

Figure 2. The basic paradigm of the Allais paradox. Many people choose a_1 over a_2 and a_3 over a_4.

Let

b_1: —— 1.00 —— \$1M

b_2: O with branches —— 10/11 —— \$5M and —— 1/11 —— \$0

By accounting,

III

$a_1 \sim \$1M \bigcirc_{.89} b_1$

$a_2 \sim \$1M \bigcirc_{.89} b_2$

By monotonicity: $a_1 \succsim a_2$ iff $b_1 \succsim b_2$.

IV

$a_3 \sim \$0 \bigcirc_{.89} b_2$

$a_4 \sim \$0 \bigcirc_{.89} b_1$

By monotonicity: $a_3 \succsim a_4$ iff $b_2 \succsim b_1$.

Figure 3. The standard rational analysis of the Allais paradox. The a_i are written as a probability mixture of a fixed sum and the appropriate b_i. This rewriting is the assertion of an accounting equivalence. Specifically, one uses the relatively innocent properties of idempotence and monotonicity to rewrite

$$a_1 = \$1M \sim \$1M \bigcirc_{.89} \$1M \sim \$1M \bigcirc_{.89} b_1.$$

The rewriting of the three-outcome gamble a_2 as a two-stage gamble:

$$a_2 = (\$5M, .10; \$1M, .89; \$0, .01) \sim \$1M \bigcirc_{.89} (\$5M \bigcirc_{10/11} \$0)$$

rests on a rather stronger accounting property that identifies the three-outcome gamble with the two-stage binary gambles that have the same probability distribution. Then monotonicity is invoked to show that the observed behavior is contrary to the rational analysis. The only conclusion that can be drawn is that either monotonicity or the accounting equivalence or both fail to be true for a substantial fraction of subjects.

A detailed, as yet unpublished study of 30 subjects has been carried out by Brothers (1990), and the unambiguous upshot is that, although the common ratio paradox is well evidenced when gambles are presented in normal form, it is substantially reduced in extensive form—monotonicity is sustained.

A recent related study by Mellers, Weiss, and Birnbaum (1992) exhibits a pronounced violation of monotonicity in binary gambles when one of the consequences is 0, but not otherwise. However, it is important to realize that this is obtained by a procedure in which subjects provide what amount to certainty equivalents to gambles. One does not expect to see $\$96 \bigcirc_{0.95} 0$ preferred to $\$96 \bigcirc_{0.95} \$24$ in a choice situation, even though the certainty equivalents are ordered in this way.

In wholly different domains of choice where probabilistic models have been developed, a close analogue of monotonicity is simple scalability, which is equivalent to an observable property called *order independence*. This property has repeatedly been shown to be violated (for a summary, see Suppes et al., 1989, pp. 410–413). This fact, while not directly relevant to the present case, is some source for concern.

So given the current choice data concerning monotonicity, which is not really sufficiently robust yet to justify any degree of strong conviction about its correctness, it is at least plausible for a theorist to postulate monotonicity provided no strong accounting equivalences are invoked.

4.3. Independence of a common consequence

The major evidence against this principle is the famous paradox of Ellsberg (1961) and the subsequent experimental verifications (see, e.g., MacCrimmon and Larsson, 1979). It is robustly violated, and unlike violations of transitivity and monotonicity, which subjects find disconcerting, these violations are often stoutly defended. Furthermore, the relevant experiments do not invoke any accounting equivalence. Therefore, it seems unwise to invoke this principle in descriptive theories. The major impact in dropping it is that the weights can no longer be shown to be finitely additive.

4.4. Accounting equivalences

To this author's knowledge, no systematic empirical data have been published about any of the accounting equivalences, although in commenting on the isolation effect mentioned under the discussion of monotonicity, Kahneman and Tversky (1979, p. 272) remark " . . . [It] is particularly significant because it violates the basic supposition of a decision-theoretical analysis, that choices between prospects are determined solely by the probabilities of final states." In particular, no published evidence about idempotence, complementarity, or event commutativity seems to exist, aside from the studies in Brothers's (1990) dissertation. He did not attempt to study either idempotence or complementarity. Idempotence seems difficult to realize in any way that does not seem stupid to the subjects, but complementarity can be studied indirectly by finding choice certainty equivalents using something like the PEST procedure mentioned earlier.

For event commutativity, Brothers (1990) found that only about a quarter of his subjects—young undergraduates—saw the equivalence when the two gambles were presented simultaneously. Considerably better support was found when each gamble was evaluated separately in terms of certainty equivalents. Of those who perceived the equivalence in event commutativity, only a small subset also recognized the equivalence in right autodistributivity, and none of the other subjects did. Thus, from a descriptive point of view, what evidence there is strongly suggests that theorists should be most restrained in invoking accounting equivalences; certainly their wholesale inclusion, as when the domain is treated as random variables, is totally unjustified.

The consequences of not having universal accounting are considerable. One simply can no longer simplify the theory to the study of very large first-order gambles over a fixed state space, as has been traditionally done in modeling SEU. One has to deal explicitly both with compound gambles and with gambles based on different event partitions.

5. Binary rank-dependent linear utility

In attempting to devise a more descriptive theory, we must abandon both the independence of a common consequence and universal accounting. It seems reasonable, at present, to retain transitivity and monotonicity. As to special accounting equivalences, no clear evidence against idempotence and complementarity appears to exist, probably because they are fairly transparent, so it seems reasonable to accept them tentatively. Less certain is event commutativity, but for present purposes, let us assume it. The question is: Where do these lead?

Probably it is simplest first to state the representation that will arise.

Definition. *Rank-Dependent Linear Utility (RDLU)* obtains (in the two-outcome or binary case) provided there is a utility function U over the set of uncertain alternatives D and a pair of weighting functions,[6] $S_>$ and $S_<$, over the set of events E and with range contained in [0, 1] such that for all $X, Y \in D$ and $A \in \mathcal{E}$:

1) $X \gtrsim Y$ iff $U(X) \geq U(Y)$.

$$2)\ U(X \bigcirc_A Y) = \begin{cases} U(X)S_>(A) + U(Y)[1 - S_>(A)], & \text{if } X \gtrsim Y, \\ U(X)S_<(A) + U(Y)[1 - S_<(A)], & \text{if } Y \gtrsim X. \end{cases}$$

This binary RDLU representation can also be written:

$$U(X \bigcirc_A Y) = U(X)S(A) + U(Y)[1 - S(A)] + R(A)|U(X) - U(Y)|,$$

where

$$S(A) = \tfrac{1}{2}[S_>(A) + S_>(A)],$$

$$R(A) = \tfrac{1}{2}[S_>(A) - S_<(A)].$$

Since the standard deviation of the gamble measured in units of U and computed relative to S is proportional to $|U(X) - U(Y)|$, the representation may be viewed as saying that the utility of the gamble is a mean plus a weighting of the standard deviation. Thus, except for the fact S need not be finitely additive, the representation is that of binary SEU plus a correction term that depends on the standard deviation.

Binary RDLU does not necessarily make the several predictions that ordinary SEU does and that are found empirically to be false. For example, Kahneman and Tversky (1979), in addition to illustrating what they called the substitution and isolation effects that SEU clearly handles, noted several others which in the presence of SEU imposed constraints on the form of U. They were called the reflection effect, probabilistic insurance, and a second isolation effect having to do with changes in total assets. Of these, the reflection effect—changing the signs of the outcomes reverses the preference order—has no necessary implications about the form of U in RDLU because it could arise from shifting from $S_>$ to $S_<$. We cannot at this point analyze the case of probabilistic insurance because it involves comparing a binary gamble with a trinary one. (Probabilistic insurance poses no problem for the more general theory outlined below because no strong accounting equivalences are involved.) And the second isolation effect receives exactly the same analysis as in SEU which, in effect, shows that U cannot be linear with money. Kahneman and Tversky's (1979) prospect theory, which is both rank- and sign-dependent, was designed to accommodate these findings, which was achieved in part by placing special restrictions on U. Aside from the nonlinearity of U, the data do not require special restrictions on U in binary RDLU.

There are some things to note about the binary RDLU representation. Proofs can be found in Luce and Narens (1985).

First, it reduces to binary SEU if and only if $S_> \equiv S_< \equiv$ subjective probability. Without stating the result precisely, the next level of complexity in accounting equivalences beyond the three assumed to get RDLU forces this equality to hold. For example, right autodistributivity, $(X \bigcirc_A Y) \bigcirc_A Z \sim (X \bigcirc_A Z) \bigcirc_A (Y \bigcirc_A Z)$, forces $S_> \equiv S_<$. In order for the common function to be a finitely additive measure, as in probability, the principle of the independence of a common consequence must also be imposed. We do not do this, so the S_i must be viewed not as probabilities, but only as weights.

Second, like SEU, binary RDLU means that utility is unique only up to positive linear transformations, i.e., it is measured on an interval scale.

Third, binary RDLU implies transitivity, monotonicity, idempotence, and event commutativity.

Fourth, assuming binary RDLU, complementarity obtains if and only if for each $A \in \mathcal{E}$, $S_>(A) + S_<(A^c) = 1$. Thus, if complementarity holds, this binary model has exactly the same number of parameters as SEU, but they are organized differently—neither $S_>$ nor $S_<$ need itself be additive over complementary events.

Fifth, as in the SEU case, there is a partial converse in the form of a representation theorem: in the presence of a sufficiently rich set of alternatives, and limiting attention to interval scale representations, then transitivity, monotonicity, idempotence, and event commutativity imply the RDLU representation. One can replace the assumption of an interval scale representation by qualitative axioms on preference, but that theorem is too intricate to be worth stating here (see Luce, 1986, 1988).

It may not be amiss to outline one aspect of the proof, because it makes very clear the role of event commutativity. Under the conditions of the theorem, but omitting event commutativity, if one looks at the substructure $\mathfrak{D}_A$ of $\mathfrak{D}$ that is generated recursively from all the pure outcomes and one repeated event A, then an RDLU representation arises, but with U subscripted by A. That is, the representation of the substructure based on B may have U_B different from U_A. Adding event commutativity allows one to prove them identical. Thus, clearly, if event commutativity is not descriptive, as may well be the case, then the representation becomes exceedingly complex and it becomes unclear how to proceed.

Accepting event commutativity, one may reasonably ask: How limiting is it to attend only to interval scale representations? The answer probably is "Not very," for the following reason. An ordered relational structure on the real numbers is said to be *finitely unique* if there is some integer N such that whenever two admissible transformations of the representation agree at N points, then they are identical. It is said to be *homogeneous* if for every pair of alternatives an admissible transformation of the representation can be found that takes one alternative into the other. (Both concepts can be stated qualitatively entirely in terms of the structure.) For structures on the reals that are both homogeneous and finitely unique, then there are only three scale types: ratio ($x \to rx, \forall r > 0$), interval ($x \to rx + s, \forall r > 0, \forall$ real s), and a case intermediate between ratio and interval of the form $x \to rx + s$, where r lies in a subgroup of the multiplicative real numbers and s is real. When there is suitable continuity around, as with a monotonic operation, the only intermediate case that can arise, called *discrete interval*, has r of the form k^n, where n ranges over the integers. This deep result was partially discovered by Narens (1981a, 1981b) and was completed by Alper (1987); it is summarized in Luce et al. (1990).

The discrete interval case does not correspond to any measurement structure of which this author is aware. Thus attention must be focused on the ratio and interval cases. M. Cohen[7] has proved under somewhat plausible assumptions that for these domains of gambles the ratio case degenerates into the interval one. In all likelihood, we should consider carefully what happens when his assumptions do not hold, because the above results make it clear that we do not have much room for maneuver in the presence of transitivity, monotonicity, idempotence, and event commutativity.

6. Generalizations to more complex gambles

The discussion so far has been restricted to recursive compounding of binary gambles. This is a serious restriction because, without universal accounting equivalences, we have no way of analyzing any single-stage gamble that involves three or more distinct outcomes. There is, however, a problem in effecting a generalization. For gambles with more than two outcomes, the form of the representation is not uniquely determined by the conditions so far mentioned and we do not yet have a full understanding of the possibilities.

6.1. Moments

One suggestion made early on by Allais (1979, 1984) and later pursued by Múnera (1985, 1986) and Múnera and Neufville (1983) involves weighted linear sums of terms of the form of the nth roots of the absolute value of the nth moments about the mean. This is a natural generalization of the second form of the binary RDLU model, mentioned above. And such a proposal certainly is consistent with an interval scale representation of utility. But such sums do not, in general, exhibit monotonicity (these authors do not address the question). Actually, it is quite unclear under what conditions monotonicity holds; moments and monotonicity simply do not seem to be very comfortable companions. One can see the difficulty by computing the partial derivative of such a term with respect to any of the variables, and it soon is evident that it is exceedingly difficult to determine whether the slope is positive or negative.

6.2. Rank-dependent linear utility (RDLU)

Another natural generalization extends the first form of the representation so that the utility of a gamble is a weighted average of the utility of its components, with the weights depending upon two things: the event, as in SEU, and the ranking of the payoffs. Mathematically, we may formulate this representation as follows:

Let $\pi = \{\pi_i\}$, $i = 1, 2, \ldots, n$, be an event partition into n disjoint and exhaustive subevents, and let X_i be n outcomes. Then $G_\pi(X_1, X_2, \ldots, X_n) = (X_1, \pi_1; X_2, \pi_2; \ldots; X_n, \pi_n)$ is the gamble in which X_i is the payoff when event π_i occurs. Let ρ be the permutation of $\{1, 2, \ldots, n\}$ for which the outcomes are ranked from worst to best; then there are weights $S(\pi_i, \rho)$ and a utility function U such that:

1. U preserves preference ordering;
2. For each ρ, the weights sum to 1;
3. $U(G_\pi) = \Sigma_i S(\pi_i, \rho) U(X_i)$ is the utility of the gamble.

RDLU reduces to SEU if the weights are probabilities that are independent of the ranking of the outcomes. It is sufficiently like SEU that it may appeal to economists who wish to use a theory of individual decision making as the basis of other derivations.

As stated, this representation need not be monotonic. Luce (1988) showed that monotonicity is equivalent to a simple condition on the weights: with the event partition fixed, whenever two rankings of the payoffs are identical except that the order of two adjacent payoffs is interchanged, then the weights are identical for all of the other events. This greatly reduces the number of free parameters—from $(n - 1)n!$ to $2^n - 2$ independent weights, as was proved by K. L. Manders in an appendix to Luce (1988). For $n = 5$, the reduction is from 480 to 30.

In many ways, this representation is closely related to the ones found by Gilboa (1987), Quiggin (1982), Segal (1987a, 1987b, 1989), and Yaari (1987), but there are major differences. For the most part, those authors restrict attention to random variables, and they weaken significantly the monotonicity postulate in such a way as to restrict indirectly the accounting equivalences used. In a number of cases, it is shown that the weights associated with the event in rank i are of the form:

$$S(\pi_i, \rho) = \phi\left[\sum_{\rho(j) \le i} p_j\right] - \phi\left[\sum_{\rho(j) < i} p_j\right],$$

where ϕ is strictly increasing and $p_i = \Pr(\pi_i)$. It is easy to verify that because permutations within each sum do not alter the sum, these weights satisfy the above criterion for monotonicity. However, not every RDLU representation has this Quiggin-Yaari cumulative form (see Luce, 1988). For this reason the term *cumulative*, which Tversky and Kahneman (1990) have proposed and which Peter Wakker has accepted (rather than his earlier *Choquet-expected utility*), seems a less appropriate designation of the general rank-dependent form.

The obvious question is what axioms on preferences yield this representation. So far, two kinds of axiomatization have been suggested.

Recursive reduction. The following assumptions are sufficient for the rank-dependent representation:

1. The assumptions for the binary case, previously discussed;

2. Transitivity of preference;
3. Monotonicity of preference;
4. The assumption that gambles on n events can be partitioned into a binary gamble involving the most preferred payoff as one payoff and the other a gamble involving the other $n - 1$ payoffs.

The only empirically new thing here is the last *decomposition* assumption. It is sufficiently important that it will be stated quite precisely. For each integer $n > 2$ and each partition π into n events, there exists a partition σ into $n - 1$ events and an event A such that for all outcomes X_i with $X_1 < X_2 < \ldots < X_n$:

$$G_\pi(X_1, X_2, \ldots, X_n) \sim G_\sigma(X_1, X_2, \ldots, X_{n-1}) \bigcirc_A X_n.$$

In a theory of rational preferences where the event probabilities P are known, we would expect σ and A to be chosen so that

$$P(A) = 1 - P(\pi_n) \text{ and } P(\sigma_i) = P(\pi_i)/P(A), i = 1, \ldots, n - 1.$$

Comonotonic tradeoff consistency. If a structure has a rank-dependent representation, then so long as the rank order among the consequences is unchanged, it is indistinguishable from an ordinary weighted-utility model. Moreover, if one takes a probability mixture of two gambles that have the same ordering of consequences, the mixture exhibits the same ordering. Wakker (1989) pointed out that this suggests defining a subset of gambles over a fixed state space—as in the Savage (1954) axiomatization of SEU—to be *comonotonic* if and only if the consequences rank order the states in the same way, and then to assume that an analogue of independence (= monotonicity of mixtures) holds in every comonotonic subspace. He calls the property *comonotonic tradeoff consistency.* From that, together with some other rationality postulates and some structural assumptions, the rank-dependent representation follows.

Although this does indeed yield the representation, it is not scientifically very compelling. It builds the rank dependence explicitly into the axioms, rather than letting it arise naturally from processes of analyzing the gambles.

6.3. *Rank- and sign-dependent linear utility (RSDLU)*

In this model there is a concept of a status quo, from which any other consequence is viewed as being either a gain if it is preferred to the status quo or a loss otherwise. The status quo is singular in the sense of being unlike any other consequences. The representation is similar to the rank-dependent one, except that the weights also depend upon the sign of the consequence. In fact, if the gamble is divided into gains and losses, one can think of the representation as consisting of a linear weighted sum of a rank-dependent expression for the gains, conditional on a gain occurring, and a similar expression for the losses. The weight applied to the gain term plus the weight applied to the loss term need not add to one. Because of the singularity of the status quo, the representation is of ratio, not interval, scale type.

Several axiomatizations of this representation have been given (see Luce, in press, 1991c; Luce and Fishburn, 1991; Tversky and Kahneman, 1990; and Wakker, in a talk). The latter two involved a fixed state-space approach and variants on the above approach of Walker which, in this context, is called *sign-comonotonicity tradeoff consistency*. The earlier objection to this approach is unchanged. All of the Luce articles involve an additional primitive concept, namely, the joint receipt of two things. Luce (1991) deals with recursive, binary uncertain outcomes; Luce (in press), which rests on results in Luce (1990b), gives a model for certainty equivalents of gambles having money outcomes; and Luce and Fishburn (1991) deal with arbitrary, finite, first-order gambles.

The joint-receipt operation plays two distinct roles. First, it is used to formulate the idea that when a gamble has only gains or losses, one can think of it as being indifferent to the joint receipt of the smallest gain or loss together with the gamble that results by having all consequences reduced by that amount. Although this seems eminently rational, it in fact provides a recursive reduction scheme that is the ultimate source of the rank dependence in the model. Second, it is used to formulate the basic nonrationality of the theory, which is here referred to as a *decomposition* assumption: a mixed gamble of gains and losses is viewed as equivalent to the joint receipt of two gambles, one being the gains pitted against the status quo and the other the losses pitted against the status quo. The latter has been called a *duplex gamble*. Although this is not an accounting equivalence, apparently people often treat it as such (see Slovic and Lichtenstein, 1968, for the only directly relevant data known of by this author).

In this approach to RSDLU, the weighted linear form across gains and losses arises primarily from the nonrational decomposition assumptions, and many theorists doubt that it will be sustained empirically. However, within the general framework of ratio scale representations, much more general nonlinear possibilities exist that take the general form $U^+g(U^-/U^+)$, where U^+ refers to the utility of the gains and U^- to that of the losses and g is an increasing function with the property that $g(u)/u$ is deceasing. A similar generalization exists in another approach of Chew Soo-Hong and Amos Tversky (personal communication). These possibilities need much more attention.

7. Testing RDLU and RSDLU

What should experimentalists interested in preferences among gambles focus on at this point? Three lines of attack come to mind.

First, although RDLU and RSDLU do not necessarily make the predictions of SEU that we know to be wrong empirically, it does not follow automatically that either can actually fit the existing data. In principle, one could try to estimate the parameters of the model and attempt to evaluate the fit to data. Indeed, Tversky and Kahneman (1990) used sequential methods to estimate certainty equivalents and then estimated the weight and utility parameters for the RSDLU model for individual subjects. They did this by assuming a three-parameter power function for U and a particular one-parameter family for the positive and negative weights. These parameters were estimated for each subject separately. These estimated functions exhibited the sorts of regularities expected. The authors did not actually test the goodness of fit of the model to the data. Since, for the most part, global approaches were not especially useful in evaluating SEU, it does not seem likely this is the best way to test these models.

Second, one can seek unusual experimental predictions that may prove these models wrong. This is not necessarily easy, because they have a great deal of flexibility in the many weights and free utility function. But then, originally SEU did not appear to be easy to challenge, and yet ultimately a number of predictions were derived and tested.

Third, we can focus on the straightforward necessary properties of these models. They are as follows.

Transitivity. We certainly need additional assurance beyond that of Bostic et al. (1990) that transitivity of *choices* is valid. Moreover, we clearly need to understand much more fully the discrepancy between judged and choice indifference points.

Monotonicity. The issue is, of course, monotonicity in isolation from accounting equivalences. The evidence that we have from Kahneman and Tversky (1979) and Brothers (1990) supports it. There is reason for concern in the failure of simple scalability to hold in probabilistic choice. Furthermore, Mellers et al. (1992) show that certainty equivalents for binary gambles violate monotonicity when one consequence is the status quo. Because monotonicity is a keystone to all existing theories of choices among uncertain alternatives, it is essential that we decide whether or not it is generally applicable. If not, it's back to the drawing boards.

Accounting. Much has been made here of the failure to pay close attention to the strong property of universal accounting in SEU. Of course, RDLU manages to get by with only two of the weakest accounting equivalences: idempotence and commutativity of events. Only the latter has been studied empirically, with at best weak support; clearly, it needs to be studied further. For RSDLU, the accounting properties are the same as those of RDLU for gains and losses separately, but for mixed gambles matters are more complex.

Nonrational decomposition. If one accepts the idea of joint receipt, then the distinctive property of RSDLU that needs to be studied is the nonrational analogue of an accounting equivalence that was mentioned earlier. Since the one test that exists (Slovic and Lichtenstein, 1968) tends to support it, it is certainly worth additional study.

8. Conclusions

The major thrust of this article is that in testing the descriptive adequacy of axioms of rationality, it is essential that the accounting equivalences be kept distinct from other properties, such as transitivity and monotonicity, and that each be tested in isolation. There is little reason to expect universal accounting to be descriptive, and imposing only some of the accounting equivalences is one way to make concrete what is meant by bounded rationality.

A second conclusion is that further, careful empirical testing is needed of both transitivity and monotonicity. In particular, transitivity should be tested using only choice procedures, not a mixture of choices and judgments, and monotonicity should be tested without invoking accounting principles extraneous to it. Clearly, since choice and judged indifference points do not agree, a comprehensive theory of both choices and judgments is needed. This could be an important theoretical development, both for psychology and economics.

Third, at the present time, RSDLU (also called cumulative prospect theory) is the best alternative to SEU that is both known not to be contradicted by data and to have a sufficiently simple structure that economists may be able to use it in applications. Further options for generalization may exist along at least two lines. First, the lines of argument that lead to a weighted linear expression across gains and losses are somewhat strained, and more general nonlinear representations are available and need to be studied (see, e.g., Luce, in press). This will probably indirectly impact the rank-dependent representation of gains and losses separately. Second, some question exists whether monotonicity breaks down when one of the outcomes is the status quo. Little is known about such models.

Notes

1. The question of recurrent choices is not considered here; see, for example, Herrnstein and Prelec (1991) and Rachlin (1990).
2. A number of other terms are used, including *uncertain alternative*, *act* (Savage, 1954), and *prospect* (Kahneman and Tversky, 1979; Tversky and Kahneman, 1990).
3. For example, the main purpose of the independence-of-a-common-consequence principle is to establish the finite additivity of the weighting functions on events.
4. This is a function from $\mathcal{E}$ into [0, 1] that is finitely additive in the sense that $S(\emptyset) = 0$ and for $A, B \in \mathcal{E}$, if $A \cap B = \emptyset$, then $S(A \cup B) = S(A) + S(B)$.
5. For preferences among complex stimuli with many dimensions, such as automobiles, intransitivities appear to arise for other reasons probably having to do with context-dependent, differential weighting of the dimensions. For many situations, this may be important, but for the typical gambling studies that have been performed it does not seem to be a major factor.
6. Luce and Narens (1985) and Luce (1988) used the notation S^+ for $S_>$ and S^- for $S_<$, but subsequently, with the development of a sign-dependent as well as rank-dependent models, it becomes apparent that the notation had to be changed, so that + and − referred to the sign of the consequence, not its rank order.
7. Reported in Luce and Narens (1985).

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