

A Theory of Certainty Equivalents for Uncertain Alternatives

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ABSTRACT

Certainty equivalents (CEs) of gambles are assumed to have a ratio scale representation that is strictly increasing in each consequence and the *status quo* is a singular point. A rank- and sign-dependent weighted linear representation arises as follows. Gambles with both gains and losses are reduced to the formally equivalent binary alternative with the CEs of the subgambles of gains and of losses as consequences. A plausible and partially sustained, but non-rational, assumption yields a bilinear, non-additive form. Those gambles composed entirely of gains or entirely of losses are assumed to be rationally edited by subtracting from each consequence the utility of the consequence nearest the *status quo* and adding that amount to the utility of the modified gamble. A rank-dependent, weighted average representation results in each domain separately. Because data using judged CEs of binary alternatives exhibit a pronounced non-monotonicity at the *status quo*, a version of the theory is given that takes this into account.

KEY WORDS Certainty equivalents Generalized SEU
Rank- and sign-dependent utility Cumulative prospect theory

Certainty equivalents (CEs) are frequently invoked both in gambling experiments and in decision analysis. For example, the familiar technique of folding back through a decision tree continually replaces risky sub-alternatives by certainty equivalents until the entire tree has been so reduced.

Little attempt has been made to provide a theory of CEs because, presumably, CEs did not seem problematical to most theorists. If one has a theory of choice among uncertain alternatives — and almost all theories of preferences have been concerned exclusively with choices — and if the domain includes money among the alternatives, then the CE of an uncertain alternative can be interpreted simply as that sum of money for which the decision maker exhibits choice indifference between the money and the uncertain alternative. In practice, CEs have almost never been estimated that way, but rather by eliciting them directly from subjects. However, several data sets in which judged CEs have played a crucial role have given pause to the view that judged CEs are good estimates of choice CEs. Two related examples are cited here.

Recent studies of the so-called preference reversal phenomenon, in which transitivity is violated when judged CEs and choices are intermixed, strongly suggest that such judged CEs do not agree with empirically determined choice indifference points (Bostic *et al.*, 1990; Tversky *et al.*, 1990). This is most vividly demonstrated in Bostic *et al.* (1990), who explained very explicitly to their subjects the meaning of choice indifference and asked them to estimate their choice indifferences. In addition,

they used a sequential choice procedure known as PEST to estimate actual choice indifference points. For so-called \$-gambles — those with a small probability of a moderately large payoff — these judged and choice indifferences differed widely, at least relative to the amounts of money that were involved in the gambles. For example, a difference of about \$2 was exhibited for gambles with a range of \$17.50.

Another problem with judged CEs is that in some cases they systematically violate monotonicity with changes in consequences of a gamble, as was demonstrated by Mellers *et al.* (1992). In particular, suppose a gamble over some partition into n events yields the amount $\$x_i$ if event i occurs. Let CE be its certainty equivalent. Now consider modifying the gamble as follows: keep the same event partition, increase one of the consequences and keep all remaining $n - 1$ consequences fixed. Let CE' be the certainty equivalent to the modified gamble. It certainly is more than plausible that $CE' > CE$. Mellers *et al.* (1992) found the following systematic pattern of violations of monotonicity in gambles with just two consequences, namely $(x;p;y,1 - p)$, when one of them is 0. In particular, the CE assigned when $y > 0$ should exceed that when $y = 0$, but for $p \geq 0.9$, they found the opposite ordering with consequences such as $x = \$96$ and $y = \$24$. That is, the CE assigned to $(\$96,0.9;0,0.1)$ is larger than that assigned to $(\$96,0.9;\$24,0.1)$. This phenomenon was studied carefully in a parametric design and was replicated several times; it does not appear to be a fluke and it seems to involve only the case $y = 0$. Quite possibly it depends, in part at least, on the fact that the 0 consequence makes calculating the expected value, namely px , of that gamble particularly simple. If this is in fact the reason for the non-monotonicity, then the phenomenon should not arise when $(x;p;y,q;1 - p - q)$ is compared with $(x;p;y,q;0,1 - p - q)$.

Some will argue that violations of monotonicity are already rampant in choices among uncertain alternatives, and so this violation is not really anything new. After all, the earliest of the paradoxes, Allais' as well as the closely related common ratio effect, are often said to evidence violations of monotonicity. Such a conclusion misinterprets the data, which actually show only that the conjunction of monotonicity together with the reduction of compound gambles fails to be confirmed. Group data in Kahneman and Tversky (1979, see the 'isolation effect') as well as a detailed empirical study of individual subjects by Brothers (1990) make clear that monotonicity, itself, probably is not the source of the trouble when choices are being made. Thus, because no reduction of compound gambles is involved, the Mellers *et al.* (1992) phenomenon is really something quite distinct from other paradoxes.

So far as we can see, we may proceed in either of two ways. One is to question the wisdom of simply asking subjects to give CEs directly, especially when we know empirically that such judgments do not agree with their choice indifference points. The other is to try to figure out what subjects are telling us when they give judged CEs. The goal of this paper is to explore each perspective. We will begin by first discussing at some length a theory of CEs that presupposes an empirical method yielding monotonic estimates. This theory arises as an application of a theory of general concatenation structures (Luce, 1992) coupled with some ideas found in Luce (1991) and Luce and Fishburn (1991). The latter two papers developed a qualitative, axiomatic theory of choices among uncertain alternatives that results in a linear, rank- and sign-dependent weighted utility representation in the following sense. First, the utility of an uncertain alternative is a weighted sum of the utilities of its component consequences. Second, the weight assigned to an event depends not only on that event, as in subjective expected utility, but also upon two aspects of its associated consequence, namely, the sign (gain or loss) of that consequence relative to the *status quo* and the rank-order position of the consequence among all of the consequences that could have arisen from the uncertain alternative in question. As an example, consider disjoint events A, B, and C and the uncertain alternative $(\$15,A; -\$10,B; -\$5,C)$, based on the event $A \cup B \cup C$, in which \$15 is the consequence if A occurs, -\$10 if B, and -\$5 if C. Then in a rank- and sign-dependent representation,

the utility of the gamble has the form $U(\$15)W(A) + U(-\$10)W(B) + U(-\$5)W(C)$, where the weights depend upon the consequences as well as the events. For example, $W(B)$ depends both on the fact that $-\$10$ is a loss and that it ranks third among the three consequences. Were the consequence associated to C changed to $-\$9$, the weight for B would remain unchanged, but were the consequence to C to be $-\$12$, then the weights assigned to both B and C would change. That to B because $-\$10$ now ranks second and that to C because it now ranks third. It follows that the notation for weights must make explicit reference both to the sign and rank order of the associated consequence.

Tversky and Kahneman (1992) have arrived at a similar representation, which they call ‘cumulative prospect theory’. The name stems from two aspects of their representation. First, it generalizes their (somewhat unrevealingly named) prospect theory (Kahneman and Tversky, 1979) to any alternatives with finitely many consequences based on uncertain events. Second, the rank-dependence of the weights arises as a difference of a function of two cumulative events (see equation (20) below). The form derived by Luce and Fishburn (1991) is somewhat more general in that the rank dependence of the weights need not have the form of a difference of a function of cumulative events. Thus, the more general representation is better characterized simply as being rank and sign dependent.

One of the most important assumptions of the theory to be developed here is monotonicity in the outcomes. Thus, it fails to describe judged CEs, at least when 0 is one of the consequences, although it may describe those that are obtained by choice procedures. In practice, the question is whether something a good deal more efficient than PEST can be devised to estimate the choice indifference. This is currently under empirical investigation by D. von Winterfeldt and the author.

Once the monotonic theory has been explained, we shall suggest how it might be modified to deal with the Mellers *et al.* (1992) data.

A THEORY OF GENERALIZED CONCATENATION STRUCTURES

This section serves to summarize some of the relevant mathematical results found in Luce (1992). Let X denote a set of entities under consideration, e.g. a set of consequences in the interpretation of this paper. Let \succeq be a binary ordering of X , e.g. the ordering of consequences by preference. Also, let $F: X^n \rightarrow X$ be a function on n arguments from X into X . For example, suppose an event is partitioned into n non-trivial sub-events, π_1, \dots, π_n , and uncertain alternatives are formed by assigning the consequences x_1, \dots, x_n , respectively to these sub-events. The CEs of this family of uncertain alternatives generate just such a function. For simplicity of statement, we shall assume that there is neither a maximum element nor a minimum element in X ; it is not difficult to treat the case where such extreme points do exist.

The three major structural assumptions of a *generalized concatenation structure* are:

- (1) \succeq is a total order, i.e. it is transitive, connected, and antisymmetric.¹
- (2) The structure is non-trivial in the sense that there exist x, y in X such that $x > y$.
- (3) The function F is monotonic in the usual sense that for each position i and each $x_1, \dots, x_i, y_i, \dots, x_n$,

$$x_i \succeq y_i \text{ if and only if } F(x_1, \dots, x_i, \dots, x_n) \succeq F(x_1, \dots, y_i, \dots, x_n)$$

Such a structure is said to be *order dense* if and only if for every x, y in X for which $x > y$, there exists z in X such that $x > z > y$.

Until the mid-1980s, the typical pattern of research in measurement theory was to impose additional structural assumptions and then show the existence of a numerical representation with some degree of uniqueness, such as uniqueness up to similarity transformations (called a ratio scale) or up to positive affine transformations (called an interval scale). Beginning with Narens (1981a,b) and extended

by Alper (1985, 1987) and Luce (1986, 1987, 1992), and Luce and Narens (1985) a different strategy has been pursued that seems quite effective. (For a general survey of these ideas until 1989, see Luce *et al.*, 1990.) Basically, the idea is to formulate, at the qualitative level, conditions about the scale type and see what this entails when coupled with some weak structural assumptions.

At first, this approach may seem rather abstract and indirect, but the results are so powerful that it is well worth while. One happy feature of this new strategy is that a great deal of the mathematical complexity usually found in axiomatic treatments is absorbed within the general theorems that have been proved, and the additional specialization to a particular theory is not mathematically very hard to follow. In a sense, one may view the present paper as illustrating this comment.

The abstract notion of scale type is captured through the concept of the automorphism group of the structure, as is made clear below. As is standard, an ‘automorphism’ is an isomorphism of the structure onto itself; physicists use instead the term a ‘symmetry’ of the structure. To be explicit, a mapping α from X onto itself is an *automorphism* if and only if it is 1:1 and it preserves the structure in the following sense:

- (1) $x \succeq y$ if and only if $\alpha(x) \succeq \alpha(y)$; and
- (2) For all values of the arguments

$$F[\alpha(x_1), \dots, \alpha(x_i), \dots, \alpha(x_n)] = \alpha[F(x_1, \dots, x_i, \dots, x_n)]$$

Under function composition, the set of all automorphisms forms a mathematical group because composition is associative; the identity transformation, which is an automorphism, serves as the group identity; and each automorphism, being order preserving and 1:1, has an inverse function that is easily shown to be an automorphism.

Now we explore the sense in which studying the automorphism group is the same as studying the scale type of a family of representations. Suppose, for example, that the structure has an isomorphic representation onto the positive real numbers that forms a ratio scale in the sense that φ and ψ are representations if and only if for some positive constant r , $\psi = r\varphi$. Now, consider the mapping $\alpha_r = \varphi^{-1}r\varphi$ within the original structure. It is not difficult to show that α_r is an automorphism. Moreover, if α is an automorphism of the structure, then one can show that there is a positive constant r_α such that $r_\alpha\varphi = \varphi\alpha$. Therefore, the automorphism group captures within the structure the admissible transformations of the representation and vice versa.

A point x of the structure is said to be *fixed* under an automorphism α if and only if $\alpha(x) = x$. A point e in X is said to be *singular* — i.e. structurally different from all other points — if and only if it is fixed under every automorphism. Had we admitted extreme points to the structure, they would necessarily have been singular because an automorphism, which is order preserving and 1:1, must map an extreme point onto itself. Moreover, they are obviously structurally unlike all other points, none of which are extreme. We shall be interested in an interior singular point that serves, in our application, the role of distinguishing between gains and losses — the *status quo* or, perhaps, an aspiration level.

A structure is said to be *finitely unique* if and only if there is a fixed integer N such that whenever any automorphism has N or more fixed points, then necessarily it is the identity map. Note that a structure with a ratio scale representation (i.e. those for which multiplication by positive constants are its automorphisms) and no singular points is a case that is finitely unique with $N = 1$. An interval scale is one with $N = 2$.

It is not difficult to show that a generalized concatenation structure that is order dense and finitely unique has at most one interior singular point (see Luce, 1992).

Among all automorphisms, it is useful to single out for special study those that will turn out to correspond to ratio scale transformations in the representation. These are called translations because, with positive structures, the ratio transformation $x \rightarrow rx$ converts under a logarithmic

transformation to $\log x \rightarrow \log x + \log r$, which is what one normally means by a translation. The trick is to capture abstractly what is involved. Narens (1981b) suggested that an automorphism be called a translation if and only if it is either the identity map or it has no fixed point. In this more general context of structures with singular points, it is appropriate to define a *translation* to be any automorphism that is either the identity or has no fixed points other than the singular ones.

A finitely unique, unbounded structure is said to be *translation homogeneous* if and only if for each pair of points x, y both lying on the same side of the interior singular point, if any, there exists a translation τ such that $\tau(x) = y$. (When there is no singular point, this property is met both in ratio and interval scale structures.) Now, under these assumptions plus the property of order density (defined above), Luce (1992) has shown that the interior singular point, e , if one exists, has the following very simple property: for each position i there exists a mapping θ_i such that:

- (1) θ_i agrees with a translation in the region above e and with another translation in the region below e ; and
- (2) $F(e, \dots, e, x_i, e, \dots, e) = \theta_i(x_i)$.

Because of the latter property, the singular point e is referred to as a *generalized zero*; it is simply a *zero* when each θ_i is the identity map, i.e. for each i , $F(e, \dots, e, x_i, e, \dots, e) = x_i$.

The structure is said to be *solvable relative* to e if and only if for each position i and for each element x , there is an element $s_i(x)$ such that

$$F[x, \dots, x, s_i(x), x, \dots, x] = e$$

Finally, we say $\langle X, \succeq \rangle$ forms a continuum if and only if it is order isomorphic to $\langle Re, \geq \rangle$, where Re denotes the real numbers.

The major result of Luce (1992, Theorem 5), the proof of which depends critically upon Alper's (1987) theorem for the homogeneous, finitely unique, non-singular case, is as follows. Suppose $\mathcal{X} = \langle X, \succeq, F \rangle$ is a generalized concatenation structure for which $\langle X, \succeq \rangle$ forms a continuum, is unbounded, and has an interior singularity, e . If \mathcal{X} is also finitely unique, translation homogeneous, and solvable relative to e , then \mathcal{X} is isomorphic to a real relational structure $\mathcal{R} = \langle Re, \geq, G \rangle$ whose automorphisms are all multiplications by positive constants (ratio scale) and whose unique singular point is 0. Thus, every automorphism of such a structure is a translation.

Because the singular point is a generalized zero, it follows immediately that there are constants $W_{s,i} > 0$, $s = +, -$, such that for u_i in Re ,

$$G(0, \dots, 0, u_i, 0, \dots, 0) = W_{s,i} u_i \quad s = \text{sgn } u_i$$

We will take considerable advantage of this last observation.

CERTAINTY EQUIVALENTS AS A GENERALIZED CONCATENATION STRUCTURE

Suppose $\pi = \{\pi_1, \dots, \pi_n\}$ partitions an event Π into n non-trivial sub-events, and let f denote the gamble that associates the money amount x_i to the event π_i . Denote by $F_\pi(x_1, \dots, x_n) = \text{CE}(f)$ the amount of money that is the certainty equivalent to f . This notation treats the partition as a parameter and focuses primarily on the gamble as a function over consequences. As was emphasized above, considerable delicacy is required in interpreting what certainty equivalent means and how best to estimate it. We assume it is done in such a way that, as a function of the consequences, it is monotonic.

Obviously, in contrast to the general theory, here we have a collection of functions, one for each

possible partition of an event. This is, of course, true for any of the classical subjective-expected utility theories as well as of the revisionist, weighted-utility theories in which the weights depend upon such aspects of the consequences as their sign, or their rank order relative to other consequences in the gamble, or both. We shall suppose, as is true for these theories, that all of the CE functions have the same automorphism group. This is, in fact, a very strong assumption. It formalizes the intuition that there is a single coherent theory for all uncertain alternatives, an assumption over which some commentators have expressed doubts (e.g. Krantz, 1990).

Because money is almost always available as a possible consequence, it probably is not too much of an idealization to suppose that the domain of gambles, $\langle X, \geq \rangle$, forms a continuum. Moreover, it is reasonable to suppose that there is a consequence e which can be interpreted as 'no change from the *status quo*' and that it is singular.

If we now suppose that the structure is finitely unique and translation homogeneous, which, as we have remarked, is true for every utility theory that has been proposed, then by the theorem quoted (Luce, 1991b, Theorem 5) we know that there is an isomorphism U such that the following are true:

$$U(e) = 0 \quad (1)$$

For any real u_i , the function corresponding to F_π under the isomorphism U , namely,

$$G_\pi(u_1, \dots, u_n) = UF_\pi[U^{-1}(u_1), \dots, U^{-1}(u_n)] \quad (2)$$

is invariant under multiplication by any positive constant, i.e. for $r > 0$

$$G_\pi(ru_1, \dots, ru_n) = rG_\pi(u_1, \dots, u_n) \quad (3)$$

In particular, for $\pi = \{A, B\}$, it is shown in the Appendix that

$$G_{\{A,B\}}(u, 0) = W_{s,2}(A, B)u, \quad G_{\{A,B\}}(0, u) = W_{s,2}'(A, B)u \quad (4)$$

where

$$W_{s,2}(A, B) = sG_{\{A,B\}}(s1, 0) \text{ and } W_{s,2}'(A, B) = sG_{\{A,B\}}(0, s1), \quad s = +, -$$

The reason for the subscript 2 is to distinguish the weights in the binary case from those arising with more complex gambles.

Note that if we make the usual assumption that the order in which the event partition is written is immaterial, i.e.

$$G_{\{A,B\}}(u, v) = G_{\{B,A\}}(v, u) \quad (5)$$

then

$$W_{s,2}'(A, B) = W_{s,2}(B, A) \quad (6)$$

Superficially, it would appear that there are many equations like equation (4), but based on non-binary partitions. However, since all of the consequences save one are 0, it is plausible to assume that they all reduce to the simple binary case. An important consequence of this is developed below.

PARTITIONING INTO CEs OF GAINS AND LOSSES

Our next assumption concerns gambles with both gains and losses. It says that the CE can be computed as follows. Determine the CE of the gains, conditional on a gain occurring, and call it CE^+ ; determine

the CE of the losses, conditional on a loss occurring, and call it CE^- ; and then determine the CE of the binary gamble of CE^+ pitted against CE^- . The assumption is that this number is identical to the CE of the gamble. Formally, suppose $x_i > e$ for $i = 1, \dots, k$, $x_i \leq e$ for $i = k + 1, \dots, n$. Set $A = \pi_1 \cup \dots \cup \pi_k$ and $B = \pi_{k+1} \cup \dots \cup \pi_n$. Then,

$$F_\pi(x_1, \dots, x_n) = F_{\{A,B\}}[F_{\pi|A}(x_1, \dots, x_k), F_{\pi|B}(x_{k+1}, \dots, x_n)] \tag{7}$$

where $\pi|A$ denotes the restriction of the partition π to the event A. This assumption is testable directly in terms of CEs, although, to the author's knowledge, it has not been.

Introducing the definition of G_π (equation (2)), it follows immediately from equation (7) that

$$G_\pi(u_1, \dots, u_n) = G_{\{A,B\}}[G_{\pi|A}(u_1, \dots, u_k), G_{\pi|B}(u_{k+1}, \dots, u_n)] \tag{8}$$

CONDITIONAL FORM FOR WEIGHTS

If we add to equation (7) the assumption that two events that have the same consequence can be collapsed in the partition, in particular that

$$F_{\{A,B,C\}}(x, e, e) = F_{\{A,B \cup C\}}(x, e) \tag{9}$$

then, as is shown in the Appendix, the weights have the form

$$W_{s,2}(A, B) = W_s(A) / W_s(A \cup B) \tag{10}$$

where $W_s(A) = W_{s,2}(A, A^c)$ and A^c is the complement of A relative to some universal event Ω . In general, $A \cup B$ is a proper subset of Ω . Thus, on the assumption of equation (10) the weight takes the form of a conditional probability of A given $A \cup B$, except, of course, the weights $W_s(A)$ are not probabilities because they need not exhibit finite additivity (see, however, below).

DECOMPOSITION AND THE LINEAR FORM OF MIXED BINARY GAMBLES

Once we assume that a general gamble can be reduced to a binary one of the gains and of the losses (equations (7) and (8)), two further issues need to be tackled. One is to evaluate the utility of gambles composed entirely of gains and those entirely of losses. This we deal with in the next section. The other is to arrive at the form of mixed binary gambles, i.e. those that involve both a gain and a loss. We now take this up.

It follows readily from our earlier assumptions² that for $u > 0 > v$:

$$G_{\{A,B\}}(u, v) = v g_{\{A,B\}}(u/v) \tag{11}$$

where, for fixed $\{A, B\}$, $g_{\{A,B\}}(w) = -G_{\{A,B\}}(-w, -1)$ is a function of one variable that is strictly increasing and $g_{\{A,B\}}(w)/w$ is strictly decreasing (see Appendix).

The assumptions to this point do not determine the form for $g_{\{A,B\}}$. All of the sign-dependent theories that have been formulated so far (Kahneman and Tversky, 1979; Luce, 1991; Luce and Fishburn, 1991; Tversky and Kahneman, 1990; Wakker, 1989) have, one way or another, forced the linear form $g_{\{A,B\}}(w) = aw + b$, where b need not equal $1 - a$. However, as each variable approaches 0, monotonicity and equation (4) imply

$$a = W_{+,2}(A, B) \text{ and } b = W_{-,2}(B, A)$$

yielding

$$G_{\{A,B\}}(u, v) = uW_{+,2}(A, B) + vW_{-,2}(B, A) \tag{12}$$

In the next section we will arrive at a weighted form (equation (18)) for gambles involving just losses. If we let $u \uparrow 0$ in equation (18) and $u \downarrow 0$ in equation (12), monotonicity at 0 implies that

$$W_{-,2}(B,A) = 1 - W_{-,2}(A,B) \quad (13)$$

One assumption that forces the linearity exhibited in equation (12) (Luce, 1991) is to suppose that for all $u > 0$, $v > 0$, and $w < 0$,

$$G_{\{A,B\}}(u,w) + G_{\{A,B\}}(v,w) = G_{\{A,B\}}(u+v,w) + G_{\{A,B\}}(0,w) \quad (14)$$

See the Appendix for the proof.

An alternative way to arrive at the linear form is to follow the general idea of Luce and Fishburn (1991) and to assume that the utility of such a gamble is the same as the sum of the utilities of the gains alone and the losses alone, i.e. for $u > 0 > v$,

$$G_{\{A,B\}}(u,v) = G_{\{A,B\}}(u,0) + G_{\{A,B\}}(0,v) \quad (15)$$

The proof that equation (12) holds involves only substituting equation (4) into equation (15). This is a basic property of Kahneman and Tversky's (1979) prospect theory.

One question is what does this arithmetic relation correspond to structurally. Luce (1991) and Luce and Fishburn (1991) postulated a binary operation of joint receipt of two consequences or gambles, in which case the plus sign on the right of equation (15) is the numerical representation of that operation. Using an experimental procedure of joint receipt, which they called duplex gambles, Slovic and Lichtenstein (1968) experimentally studied, among other things, the corresponding F version of the decomposition principle embodied in equation (15). Within the noise level of the data, it was not rejected. This bears further empirical attention.

Because both of these assumptions, either equation (14) or (15), that yield linearity between gains and losses, seem rather strained, it is probably better to study the form of $g_{\{A,B\}}$ rather than assume it. Quite possibly, a non-linearity is needed at just this point of the theory. The way in which $g_{\{A,B\}}$ can be estimated is to estimate the function U , as described below. Once done, then we may plot $G_{\{A,B\}}(u,v)/v = g_{\{A,B\}}(u/v)$ as a function of u/v and see if it is linear or not. If not, some alternative functions may suggest themselves.

AN EDITING PRINCIPLE LEADING TO RANK DEPENDENCE OF GAINS AND LOSSES

For the domain of gains, and separately for the domain of losses, there are two ways to arrive at a simple rank-dependent representation of the type that has arisen in several papers during the 1980s, among them being Quiggin (1982) for risk and Luce (1988) for uncertain alternatives.

Consider a gamble of just gains, and suppose that the events have been numbered so that $u_1 \geq \dots \geq u_n > 0$. The first general idea is that the gamble can, using a term of Kahneman and Tversky (1979), be edited by subtracting away the value of the consequence that is nearest the *status quo*. In particular, suppose that each $u_i > 0$ and that $u_n = \min\{u_i\}$, then

$$G_\pi(u_1, \dots, u_n) = u_n + G_\pi(u_1 - u_n, \dots, u_{n-1} - u_n, 0) \quad (16)$$

A similar assertion, using the maximum value, holds when all arguments are negative.

This editing principle is a highly restricted version of Pfanzagl's (1959) consistency axiom. It says that if we are in the domain of all gains (or all losses) then $U[CE(f)]$ equals the U value of the smallest gain (loss) plus the U value of the CE of the gamble in which the U value of every consequence of f is reduced by the U value of the smallest gain (loss).

If we apply the editing principle to a gamble with two consequences, then we see that for $u > v > 0$ and for $u < v < 0$:

$$\begin{aligned} G_{\{A,B\}}(u,v) &= v + G_{\{A,B\}}(u - v, 0) \\ &= v + (u-v)W_{s,2}(A,B) \\ &= uW_{s,2}(A,B) + v[1 - W_{s,2}(A,B)] \end{aligned} \tag{17}$$

This result generalizes as follows (see the Appendix for a proof). If equations (10) and (16) hold and $u_1 \geq \dots \geq u_n > 0$ or $0 > u_1 \geq \dots \geq u_n$, then

$$G_{\pi}(u_1, \dots, u_n) = \sum_i u_i W_{s,n}(\pi_i) \quad \sum_i W_{s,n}(\pi_i) = 1 \tag{18}$$

where, with

$$\pi(j) = \bigcup_{i=1}^j \pi_i \tag{19}$$

the weights are defined by the cumulative form:

$$W_{s,n}(\pi_i) = \{W_s[\pi(i)] - W_s[\pi(i-1)]\} / W_s[\pi(n)] \tag{20}$$

which arose in Tversky and Kahneman (1992) and a number of earlier purely rank-dependent theories. Note two things about this weighted linear expression. First, the weights implicitly depend upon the ordering induced by the consequences u_i as well as on their sign. Second, the event $\pi(n)$ on which the gamble is defined is, in general, a proper subset of the universal event Ω , and so $W_s[\pi(n)] \neq 1$.

An alternative way to arrive at equations (18)–(20), but not using equation (16) except in the binary case, is to assume that an n -alternative gamble can be partitioned into a binary gamble involving the consequence nearest the *status quo* and the CE of the remaining gamble with $n - 1$ consequences, i.e. for $u_1 \geq \dots u_n > 0$,

$$G_{\pi}(u_1, \dots, u_n) = G_{\{\pi(n-1), \pi_n\}}\{[G_{\pi(n-1)}(u_1, \dots, u_{n-1}), u_n]\} \tag{21}$$

Using this and equation (17) for binary gambles, a routine induction yields equations (18) and (20).

FORM OF THE WEIGHTING FUNCTIONS FOR RISKY ALTERNATIVES

Consider the class of risky, binary gambles $(x,p;e,1 - p)$ where the corresponding CE is denoted $F_p(x,e)$. Two fairly innocuous assumptions are monotonicity in the probability, i.e.

$$F_p(x,e) \geq F_q(x,e) \text{ if and only if } p \geq q \tag{22}$$

and that certainty of receiving x is identical to the gamble $(x,1;e,0)$, i.e.

$$F_1(x,e) = x \tag{23}$$

Assuming the ratio scale representation, it is easy to verify using equations (22) and (23) that the weighting functions W_s are strictly increasing functions of p with $W_s(1) = 1$.

It is perhaps worth noting that the rank-dependent weights of equation (20) reduce to:

$$W_{s,n}(\pi_i) = [W_s(P[\pi(i)]) - W_s(P[\pi(i-1)])] / W_s(P[\pi(n)])$$

which is the rank-dependent form derived by Quiggin (1982) and others.

To develop an understanding of the form for W_s , consider two independently realized experiments

and the two-stage gamble $((x,q;e,1-q),p;e,1-p)$ which says that with probability p the first experiment leads to the second experiment, in which the outcome is x with independent probability q . In all other cases, the outcome is the *status quo* e . Compare this with the one-stage gamble in which the two experiments are run simultaneously and x is the outcome if both events occur, which has probability pq . Assume, for the moment, these two gambles are perceived as equivalent:

$$F_q[F_p(x,e),e] = F_{pq}(x,e) \quad (24)$$

Applying the representation to this we see

$$W_s(p)W_s(q) = W_s(pq)$$

It is well known that if p and q range over the unit interval, the only strictly increasing solutions to this equation, with $W_s(1) = 1$, are:

$$W_s(p) = p^{\gamma(s)}, \text{ where } \gamma(s) > 0 \quad (25)$$

This prediction does not agree with the model assumed and tested by Tversky and Kahneman (1992). Without argument they postulated:

$$W_s(p) = p^{\gamma(s)}/[p^{\gamma(s)} + (1-p)^{\gamma(s)}]^{1/\gamma(s)} \quad (26)$$

Using this and assuming separate power functions for the utility of gains and losses, they conducted a global fit to their data, and concluded that equation (26) is sustained, which rejects equation (25) and hence equation (24). There is, of course, always the danger of being misled by global fits to data in this domain.

A more direct way to proceed is, first, to collect CEs for gambles of this character by varying both x and p over the full range of values. Suppose the utility function is a power function of money, i.e.

$$U(x) = a(s)(sx)^{\beta(s)}$$

where $s = \text{sign}(x)$, $a(+)$ > 0, $a(-)$ < 0, and $\beta(s)$ > 0. If, further, the weights satisfy equation (25), then

$$\begin{aligned} U[F_p(x,e)] &= a(s)[sF_p(x,e)]^{\beta(s)} \\ &= W_s(p)U(x) \\ &= p^{\gamma(s)}a(s)(sx)^{\beta(s)} \end{aligned}$$

Rearranging,

$$F_p(x,e)/x = p^{\gamma(s)/\beta(s)}, s = \text{sign}(x) \quad (27)$$

This means that a scatter diagram of $F_p(x,e)/x$ versus p on log-log co-ordinates should be a single straight line independent of x , and the slope is an estimate of $\gamma(s)/\beta(s)$.

EMPIRICAL ISSUES

Scattered throughout the paper have been various asides about empirically testable features of the model. The purpose of this section is to bring them together in one place.

Perhaps the most vexing empirical issue is how to estimate certainty equivalents. There is strong evidence that asking people to provide them is different from estimating choice indifference points

(Bostic *et al.*, 1990). Moreover, judged CEs for binary gambles exhibit a consistent non-monotonicity at 0 (Mellers *et al.*, 1992). Because the current model assumes monotonicity, it makes sense to test it only with procedures that are more or less equivalent to choice indifference. Alternatively, one can try to modify the model to accommodate the non-monotonicity at 0, as is discussed later.

Once the CEs are obtained, the next step in any empirical investigation of the model is to estimate the isomorphism U . The key to doing that is, of course, equation (4), which says that the ordering over binary gambles, when one consequence is the *status quo*, must exhibit the properties of multiplicative conjoint measurement with sign dependence (Roskies, 1965; see Chapter 7 of Krantz *et al.*, 1971 for a full discussion). The difficulty with using just equation (4) is that U and $W_{s,2}$ are determined only up to power transformations, whereas the entire representation establishes that the power is uniquely determined. The author has been unable to devise a suitable estimation scheme without using some of our additional assumptions, in particular those that lead to equation (17) for a binary gamble. The additional additivity of that expression determines which power is involved.

Therefore one proposal is to use equation (4), to determine one representation pair $(U, W_{s,2})$, and then estimate the appropriate power by minimizing the following quantity over β with either $x > y > 0$ or $x < y < 0$:

$$\sum_{A,B,x,y} | U^\beta [F_{\{A,B\}}(x,y)] - U^\beta(x)W_{s,2}(A,B)^\beta - U^\beta(y)[1 - W_{s,2}(A,B)^\beta] |$$

The author's choice of the absolute value rather than some other power is not dictated by any deep principle aside from the desire not to allow an outlier in the data to exert undue influence in the estimate of β .

Arguments can be found in Luce (1991) and in Luce and Fishburn (1991) that suggest investigating specific mathematical forms for U . The simplest is that U should be separate power functions for gains and losses, in which case the task reduces to estimating two exponents. This assumption was made by Tversky and Kahneman (1990) in fitting substantially the same model to their data. If it is correct, then one immediate implication from it and from equations (2) and (4) is that the ratio $F_{\{A,B\}}(x/e)/x$ is a constant that depends on $\{A,B\}$ but not on x . This was found to hold in the data of Tversky and Kahneman. Moreover, within the domain of risk, if we also assume the well-known reduction hypothesis embodied in equation (24), then the relation between the weights and probability is a power function, equation (25). This needs to be studied. One easy way is to plot the above ratio as a function of p in log-log co-ordinates (see equation (27)).

Once the isomorphism is estimated, one can either attempt to do some global evaluation of the model, which is what Tversky and Kahneman (1992) did, or one can try to study the individual axioms of the theory. Several warrant close attention.

- (1) The assumption that people partition mixed gambles into a binary gamble of the CE of gains versus the CE of losses (equation (8)) is basic. Should this fail to be correct, the whole approach taken here is basically flawed.
- (2) Given that (1) is sustained, the form of the binary, mixed gambles is the next issue. The most general thing to do is to study the form of $g_{\{A,B\}}$ in equation (11). If it is linear, then that is equivalent to the decomposition property for duplex gambles given in equation (15), which can also be studied directly. The key question is whether or not a non-linearity needs to be brought in at this point of the theory.
- (3) The third major axiom to be examined is the editing principle embodied in equation (16), namely, that in gambles of all gains or all losses one can subtract off the utility of the consequence nearest to the *status quo* from all consequences and then simply add it back to the resulting gamble. This induction principle is the source of rank dependence in the model.

FOLDING BACK, FINITE ADDITIVITY, AND SEU

The weights in the rank-dependent representation are further constrained by the following generalization³ of equation (21). Suppose $x_1 \geq \dots \geq x_n > 0$, define $\pi(k)$ as in equation (19), and note that $\Pi = \pi(n)$ and⁴ $\pi(n) \setminus \pi(k) = \pi_{k+1} \cup \dots \cup \pi_n$. Then for $k, 1 < k < n$,

$$F_\pi(x_1, \dots, x_n) = F_{\{\pi(k), \pi(n) \setminus \pi(k)\}}[F_{\pi(k)}(x_1, \dots, x_k), F_{\pi(n) \setminus \pi(k)}(x_{k+1}, \dots, x_n)] \quad (28)$$

This assumption implies (see the Appendix):

$$W_s(A \cup B) = W_s(A) + W_s(B), A \cap B = \emptyset \text{ s } = +, - \quad (29)$$

Thus, assuming a weighted linear representation of the form given in equations (18) and (20), then rank independence and general folding back (equation (28)) hold if and only if W_s is finitely additive (equation (29)). In this case, the representation becomes SEU for gains and for losses separately.

NON-MONOTONICITY AT THE *STATUS QUO*

As was noted in the introduction, for binary gambles substantial evidence exists that judged CEs exhibit a non-monotonicity when one of the consequences is no change from the *status quo*. Moreover, this anomalous effect does not appear to spread to values near 0; rather it seems to be strictly limited to the 0 case. Therefore, let us suppose that the entire structure is monotonic except when one of the consequences is 0. This, of course, means that we must restrict the editing principle (equation (16)) to avoid 0. Instead, for binary gambles $G(u, v)$, where the event partition is suppressed, suppose that for $0 < w < \min(u, v)$,

$$G(u, v) = w + G(u - w, v - w) \quad (30)$$

This property is much stronger than equation (16) for binary gambles in that it holds for an interval of values, not just a single one. This assumption, together with ratio scalability and the binary partitioning of equation (21), yields the rank dependent representation, as is proved in the Appendix.

Turning to the binary case, if we continue to assume that with one entry 0 the CE still yields a translation, i.e.

$$G_{\{A, B\}}(u, 0) = W'(A, B)u$$

then monotonicity can be violated in the way that has been observed if, for some choices of $\{A, B\}$, $W'(A, B)$ is larger than $W_{+, 2}(A, B)$.

CONCLUSIONS

The major conclusion is that plausible assumptions — equations (1)–(5), (7) or (8), (14) or (15), and (16) (or (17) and (21)) — about certainty equivalents, including a special status for the *status quo*, lead naturally to a rank- and sign-dependent representation. On either side of the *status quo*, the representation is a rank-dependent weighted average (equations (18)–(20)). Across the *status quo* matters are less clear. The arguments based on either equation (14) or (15) to arrive at the bilinear expression (equation (12)) seem somewhat questionable. Empirical studies are needed to determine just how gains and losses are combined, i.e. equation (11) should be estimated empirically.

The general approach taken draws upon strong results obtained in the general study of ordered structures having a function as one of its primitives. Basically, the theory presupposes a situation

that is rich in symmetries, which insures structural homogeneity except at certain singular points. Conditions have been provided under which structures have a ratio scale representation, and that was the starting point of the present theory. The *status quo* is interpreted as the (unique) interior singular point. Given these measurement results, the development of the rest of the theory is relatively simple mathematically. The additional assumptions that specialize it to the interpretation as CEs of uncertain alternatives include that a CE is strictly increasing in its consequences, that one can reasonably edit alternatives having only gains or only losses as consequences, and that an alternative of mixed gains and losses can be recoded as a binary alternative in which one consequence is the CE of the sub-gamble of only gains and the other is CE of the losses. Although most theories to date also assume that the binary case is a linear weighting of the utilities of the CEs, the arguments for that property seem somewhat less than compelling, and so probably it should be studied empirically.

Because good empirical evidence exists that judged CEs exhibit a pronounced non-monotonicity at the *status quo*, a somewhat modified version of the theory was worked out to take that into account.

APPENDIX: PROOFS

Equation (4)

Writing $u = s|u|1$, $0 = |u|0$, and using equation (3),

$$G_{\{A,B\}}(u,0) = G_{\{A,B\}}(s|u|1,|u|0) = |u|G_{\{A,B\}}(s1,0) = usG_{\{A,B\}}(s1,0)$$

The other case is similar.

Equation (10)

Substituting equation (9) into equation (8),

$$G_{\{A,BUC\}}(u,0) = G_{\{AUB,C\}}[G_{\{A,B\}}(u,0),0]$$

whence by equation (4)

$$W_{s,2}(A,BUC) = W_{s,2}(AUB,C)W_{s,2}(A,B)$$

Setting $W_s(A) = W_{s,2}(A,A^c)$, where A^c is the complement of A and $C = (AUB)^c$, yields equation (10).

Monotonicity properties of $g_{\{A,B\}}$

Suppressing the subscript $\{A,B\}$ in equation (11), let u and v be such that $w = u/v$. Then, $g(w) = g(u/v) = G(u,v)/v$. Holding v fixed, w increases iff u increases iff G increases. Thus, g is strictly increasing. Likewise, $g(w)/w = g(u/v)/(u/v) = G(u,v)/u$. Holding u fixed, w increases iff v decreases iff $G(u,v)$ decreases. Thus, $g(w)/w$ is strictly decreasing with w .

Equation (12)

Suppressing the subscript $\{A,B\}$, substitute the form for G (equation (11)) into equation (14):

$$wg(u/w) + wg(v/w) = wg[(u+v)/w] + wg(0)$$

and so $h(u) = g(u) - g(0)$ satisfies

$$h(u + v) = h(u) + h(v)$$

and h is strictly increasing. It is well known that the only solution is $h(u) = au$ and setting $g(0) = b$, we see that

$$G(u, w) = wg(u/w) = au + bw$$

Equation (18) based on equations (8), (10) and (16)

Using equation (16), one subtracts off u_n , leaving $G_\pi(u_1 - u_n, \dots, u_{n-1} - u_n, 0)$. By equation (8) this may be written

$$G_\pi(u_1 - u_n, \dots, u_{n-1} - u_n, 0) = G_{\{A, B\}}[G_{\pi|A}(u_1 - u_n, \dots, u_{n-1} - u_n), 0]$$

where $A = \Pi \setminus \pi_n$ and $B = \pi_n$. However, using equation (4) we see that

$$G_\pi(u_1, \dots, u_n) = u_n + W_{+,2}(A, B)G_{\pi|A}(u_1 - u_n, \dots, u_{n-1} - u_n)$$

Note that for $u_1 < \dots < u_n < 0$, a similar expression arises. Assuming equation (16), we may proceed inductively (using the assumed numbering of events), and equation (18) follows. Assuming equation (10), it is easy to show the cumulative form of equation (20).

Equation (29)

Assume equation (28): then using equations (3) and (17) we conclude:

$$G_\pi(u_1, \dots, u_n) = W_{+,2}[\pi(k), \pi(n) \setminus \pi(k)]G_{\pi|_{\pi(k)}}(u_1, \dots, u_k) + \{1 - W_{+,2}[\pi(k), \pi(n) \setminus \pi(k)]\}G_{\pi|_{\pi(n) \setminus \pi(k)}}(u_{k+1}, \dots, u_n) \tag{A1}$$

If we now substitute equations (15), (18) and (20) into both sides of equation (A1) and equate the coefficients for each u_i , then for $i > k$ we deduce:

$$W_+[\pi(i)] - W_+[\pi(i-1)] = \{W_+[\pi(i) \setminus \pi(k)] - W_+[\pi(i-1) \setminus \pi(k)]\} \times \{W_+[\pi(n)] - W_+[\pi(k)]\} / W_+[\pi(n) \setminus \pi(k)] \tag{A2}$$

Observe that this is satisfied if W is finitely additive, in which case equation (20) simplifies to $W_{+,n}(\pi_i) = W_+(\pi_i) / W_+[\pi(n)]$. Thus, it is not rank dependent, and so the entire domain of gains reduces to SEU. Conversely, if in the positive domain the binary gambles are not rank dependent, then from equation (17) we see that $W_+(A) + W_+(\Omega \setminus A) = 1$. Therefore, if we set $i = k + 1$, $A = \pi(k)$, and $B = \pi_{k+1}$ in equation (A2), we see that

$$W_+(A \cup B) = W_+(A) + W_+(B), A \cap B = \emptyset \tag{29}$$

A similar argument holds in the domain of losses.

The author suspects that it is possible to prove finite additivity from just equation (A2) and some richness of the space of events, but he has been unable to do so.

Equations (18)–(20) based on equations (21) and (30)

Suppose $u \geq v$, then by equation (3),

$$G(u, v) = vG(u/v, 1) = vg(u/v) \tag{A3}$$

where $g(z) = G(z, 1)$. Using equations (30) and (A3),

$$(v - w)g[(u - w)/(v - w)] + w = vg(u/v)$$

Letting $x = u/v > 1$ and $y = w/v < 1$,

$$g(x) = y + (1 - y)g[(x - y)/(1 - y)] = y + (x - y)g[(x - y)/(1 - y)]/[(x - y)/(1 - y)] \tag{A4}$$

By Luce and Narens (1985, Theorem 3.8.1), there exists a constant $W_{+,2}$ such that

$$\lim_{z \rightarrow \infty} g(z)/z = W_{+,2}$$

Taking the limit in equation (A4) as $y \rightarrow 1$, we see that

$$g(x) = W_{+,2}x + 1 - W_{+,2}x \geq 1 \tag{A5}$$

A similar expression holds in the domain of losses with the parameter $W_{-,2}$. Thus, we have arrived at equation (17) without any assumption about what happens at 0.

The next problem is to generalize this form to any finite partition. An inductive assumption that does the job is equation (21). (Note that the argument of the sixth section of this paper cannot be used to conclude anything about finite additivity because equation (21), unlike the more general equation (23), does not place any further restriction on W .) Using equation (A5) and proceeding inductively on equation (21), it is easy to show that G_π has the form of equation (18), namely:

$$G_\pi(u_1, \dots, u_n) = \sum u_i W_n(\pi_{+,i}) \tag{A6}$$

where

$$W_{+,n}(\pi_i) = \begin{cases} W_{+,n-1}(\pi_i)W_{+,2}[\pi(n-1), \pi_n] & i < n \\ 1 - W_{+,2}[\pi(n-1), \pi_n] & i = n \end{cases} \tag{A7}$$

Observe that monotonicity implies $W_{+,n}(\pi_i) > 0$ and equation (30) implies $\sum_i W_{+,n}(\pi_i) = 1$. Because the derivation depends upon the outcome ordering, it follows that the weights depend upon that order and so the representation is rank dependent. The corresponding expression holds in the domain of losses.

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NOTES

1. One can make the weaker assumption that \geq is a weak order, i.e. transitive and connected, and then simply factor out the equivalence classes.
2. Suppressing the event partition, by equation (3) $G(u, v) = G[-v(-u/v), (-v)(-1)] = -vG(-u/v, -1)$. Setting $g(w) = -G(-w, -1)$ yields equation (11).
3. The generalization lies in not restricting the binary partition just to gains versus losses.
4. $A \setminus B$ denotes the set theoretic difference of A less B.
5. 'iff' stands for 'if and only if'.

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