Subjective Expected Utility Theory without States of the World

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Abstract

This paper develops an axiomatic theory of decision making under uncertainty that dispenses with the state space. The results are subjective expected utility models with unique, action-dependent, subjective probabilities.

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1 Introduction

The distinguishing characteristic of decision making under uncertainty is that the choice of a course of action, by itself, does not always determine a unique outcome. To formalize this indeterminacy, or lack of advance knowledge of the outcome of alternative courses of action, Savage (1954) introduced the concept of states of the world, that is, “a description of the world so complete that, if true and known, the consequences of every action would be known” (Arrow [1971], p. 45). In the wake of Savage’s seminal work, the state space (that is, the set of all states of the world) became a cornerstone of modern theories of decision making under uncertainty. However, careful examination of the concept of state of the world reveals that, the depiction of the relevant state space is often unintuitive and too complex to be compatible with decision makers’ perception of choice problems. Doubt about the relevance of state of the world as a general analytical concept and its applicability is the main motivation behind this work.

In this paper I introduce an alternative analytical framework and a new subjective expected utility theory of decision making under uncertainty that avoid the use of a state space. Moreover I develop a subjective expected utility theory with effect-dependent utility and action-dependent subjective probabilities. This theory provides an axiomatic foundation of choice behavior in the presence of moral hazard.

The case against the general applicability of the notion of states of the world is detailed in the next section. The formal theory is presented in Section 3. Concluding remarks appear in Section 4. The proofs are given in Section 5.

2 On the Meaning of States of the World

Following Savage (1954) it is customary to formulate the problem of decision making under uncertainty invoking states and consequences as primitive concepts and acts, that is, functions from the set of states to the set of consequences, as a derived concept. Once the framework is fixed, however, states of the world may be interpreted, consistently with Arrow's (1981) definition, as mappings from the set of acts, to the set of consequences.

By definition the states of the world are mutually exclusive and jointly exhaustive. Moreover, the states must be defined in a way that their likely realization must not be affected by the decision maker's choice of action, and the valuation of the consequences be independent of the state in which they may be received. Finally, for it to be a meaningful scientific concept, the state space must be independently observable. In other words, it should be possible to reconstruct, on the basis of a decision maker's observed choices, the unique state-space underlying his decisions.[1] Note that the notion of the state space, advocated by Savage (1954), presumes that decision makers believe that they know the world in which they live.[2] As a result, no amount of evidence, with Bayesian updating, would lead the decision maker to conclude that his original image of the world was incomplete.[3]

[1] Machina (2003) offers a detailed discussion of this and related issues.

[2] That is what is meant by the requirement that states be jointly exhaustive.

[3] Gilboa (2003) illustrates this point with Newcombe's paradox. The same issue is discussed in Machina

There are situations in which the relevant states of the world correspond to observable physical phenomena and have a natural, intuitive, and, most important, objective meaning. For example, the uncertainty regarding the consequences of installing or not installing a lightning rod is resolved once it is known whether the house is struck by lightning. Thus a lightning strike may be regarded as a state of the world (or a state of nature) whose likely occurrence is independent of whether or not a lightning rod is installed. In this instance, the portrayal of the state space has a clear, objective interpretation, and it makes sense to treat it as a primitive concept.

Situations in which the state space lends itself to such straightforward interpretations are rare. Often the distinction between states and consequences is blurred and frequently the likely realization of what seems like a natural definition of states for the problem at hand is not independent of the choice of the action. Moreover, in many instances, the state space is too large and complex to be compatible with the limited cognitive ability of decision makers to grasp, let alone be invoked in the decision-making process. In such instances, as the following examples illustrate, the notion of states of the world as an uncertainty-resolving device seems unintuitive, non-compelling, and outright useless, for the purpose of obtaining a behavioral definition of subjective probabilities.[4]

Example 1. In a letter to Savage, from January 1971, Aumann questions "the very possibility of defining this notion [subjective probability] – in any way – via preference."

(2003).

[4] For additional examples and comments, see Gilboa and Schmeidler (2001), Ch. 2.

(Drèze [1987], p. 77) To make his point Aumann describes a man who loves his wife very much and without whom his life is less "worth living." The wife falls ill and, if she is to survive, she must undergo a routine yet dangerous operation. Suppose that the husband is offered a choice between betting $100 on his wife's survival or on the outcome of a coin flip. In this scenario, there are four states, corresponding to the different possible combinations of outcomes of the operation and those of the coin flip. However, even if the husband believed that his wife has an even chance of surviving the operation he may still rather bet (that is, strongly prefer to bet) on her survival. This is because winning $100 if she does not survive is somehow worthless. Betting on the outcome of a coin flip means that he might win in a situation in which he will not be able to enjoy it. Aumann's objection is based on the presumption, that seems quite compelling in the situation described, that the valuation of the consequences is not independent of the states. In fact, Aumann argues that the notion of states and consequences are confounded to the point that there is nothing that one may call a consequence, that is, something whose value is state independent.

Savage responded to Aumann's criticism in these words:

> "The difficulties that you mention are all there; ... I believe that they are serious but am prepared to live with them until something better comes along. The theory of personal probability and utility is, as I see it, a sort of framework into which I hope to fit a large class of decision problems. In this process, a certain amount of pushing, pulling, and departure from common sense may be acceptable

> and even advisable.... To some - perhaps to you - it will seem grotesque if I say that I should not mind being hung so long as it be done without damage to my health or reputation, but I think it desirable to adopt such language so that the danger of being hung can be contemplated in this framework." (Drèze [1987], p. 78)

And to the specific example of Aumann Savage responds by saying: “In particular, I can contemplate the possibility that the lady dies medically and yet is restored in good health to her husband." (Drèze [1987], p. 80). Even if such contemplation is possible, it is unnatural and, hence, not likely to be invoked in the decision making process.

Consider next an amendment to Aumann's example. Suppose that, in addition to choosing between betting on the survival of his wife and on heads in a coin toss, the husband may also choose the hospital in which the surgery is to take place and the surgeon who performs it. If the likely outcome depends, as it often does, on the hospital and the surgeon, the husband's choice affects the chances of his wife's survival. In other words, contrary to the Savage's concept of state space, the likely realization of the events depends on the action taken by the husband.

This discussion suggests that the possible outcomes of the operation are not states, or events, in the sense of Savage's theory. Yet they seem both natural and intuitive when contemplating the proposed bets.

Example 2. To go from here to there, a passenger must choose between flying, driving, or taking the bus.[5] Suppose that the purpose of the trip is a week-long vacation, then the consequences, namely, the actual duration of his vacation, depend on whether the passenger arrives at his destination on time, arrives after delays of various durations, or does not arrive at all.

The factors affecting the duration of the trip include the, unknown, traffic conditions (which depends on choices of other people), the weather conditions, the mechanical state of the different means of transportation, and so forth. These factors have different implications for the duration of the trip depending on the choice of mean of transportation. The relevant state space, in this case, is large and complex and to suppose that decision makers invoke such a state space when choosing a mean of transportation strains credulity. Moreover, even if the passenger invokes, in his deliberations, a state-space image of the world, being a state of mind, it is impossible for others to infer it from his choice behavior. Clearly, different decision makers facing the same choices, may invoke distinct state-spaces.

In short, the exacting nature of Savage's analytical framework – its insistence that the realization of the states be independent from the actions, that states be separated from consequences, and that the state space be observable – makes it inadequate for the formulation and analysis of important decision situations. To suppose that decision makers always invoke depictions of the world that qualify as states in the sense of Savage seems farfetched. The upshot of this discussion is that a general positive theory of decision making under uncertainty must not relay on the use of states of the world.

[5] This is a variation on a decision problem described in Luce and Krantz (1971).

In the next section I explore an alternative theory that dispenses the state space. The main idea is that decision makers directly assess the likelihood of different outcomes, or effects, conditional on their choice of action.

3 Subjective Expected Utility Theory

3.1 The analytical framework

Let X be a finite set of *effects* and let A be a set whose elements represent courses of action, or *actions* for short. A *bet*, b, is a mapping from X into $\mathbb{R}$, the set of real numbers.[6] Bets have the interpretation of monetary payoffs contingent on the effects. Let $B := \mathbb{R}^X$ denote the set of all bets and assume that it is endowed with the $\mathbb{R}^{|X|}$ topology. Denote by (b_{-x}, r) the bet obtained from $b \in B$ by replacing the $x-$coordinate of b, that is, $b(x)$, with r. Similarly, for each $Y \subset X$ and $b, b' \in B$, let $b_Y b'$ be the bet in B defined by $(b_Y b')(x) = b(x)$ for all $x \in Y$ and $(b_Y b')(x) = b'(x)$ for all $x \in X - Y$. Two bets, say b and b', are said to *agree* on $Y \subset X$ if $b(x) = b'(x)$ for all $x \in Y$.

Decision makers are supposed to be able to choose actions and to place bets on the effects. The idea is that a choice of action, a, results in the realization of an effect in X; which particular effect obtains is uncertain, and the effect that obtains determines the payoff

[6] The use of the reals is intended to simplify the exposition. It could easily be replaced by $\mathbb{R}^n$ or, more generally, by a connected separable topological space.

of the chosen bet. For example, a decision maker may adopt an exercise and diet regime to reduce the risk of heart attack and at the same time take out health insurance and life insurance policies. The health implications of the diet and exercise regiment correspond to the effects while the financial terms of the insurance policies constitute a bet. Similarly, a store owner can choose the location of his store and his weekly work schedule and, within limits, the equity that he has in the business. The revenue represent the effects of his management decisions (actions) and the financial decision represent his bet. Formally, the *choice set,* $\mathbb{C}$, consists of all the action-bet pairs (that is, $\mathbb{C} = A \times B$). A choice of an action a and a bet b results, ultimately, in an effect-payoff pair, $(x, b(x))$. I refer to effect-payoff pairs as *consequences* and denote by C the set of all consequences (that is, $C = X \times \mathbb{R}$).

Decision makers are characterized by binary relations, $\succcurlyeq$, on $\mathbb{C}$, that have the interpretation of preference relations. The strict preference relation, $\succ$, and the indifference relation, $\sim$, are the asymmetric and symmetric parts of $\succcurlyeq$, respectively. For each $a \in A$, the preference relation $\succcurlyeq$ on $\mathbb{C}$ induces a conditional preference relation on B defined as follows: For all $b, b' \in B$, $b \succcurlyeq_a b'$ if and only if $(a, b) \succcurlyeq (a, b')$.

A decision maker may believe that if he selects a particular course of action, certain effects are impossible to obtain. It is tempting to suppose that this belief manifests itself in indifference among all the bets that agree on the set of all other effects. Conceivably, however, there may be effects that the decision maker believes to be possible and yet, if any of these effects obtain, the decision maker would be indifferent among all the monetary payoffs. For example, a decision maker with no dependents who is about to board a flight, may decline offers to take out a flight insurance policy, regardless of how favorable are the terms of the policy. This does not mean that the decision maker regards the effect "dying in a plane crash" to be impossible. In what follows I assume that no such effects are present in the model. Formally, an effect x is said to be *nonnull given the action* a if $(a, (b_{-x}, r)) \succ (a, (b_{-x}, r'))$, for some $b \in B$ and $r, r' \in \mathbb{R}$. Assume that every effect is nonnull for some action a. An effect x is said to be *null given the action* a if $(a, (b_{-x}, r)) \sim (a, (b_{-x}, r'))$ for all $r, r' \in \mathbb{R}$. Given a preference relation $\succcurlyeq$, denote by $X(a; \succcurlyeq)$ the subset of effects that are nonnull given a according to $\succcurlyeq$. To simplify the notations, when there is no risk of confusion, I shall write $X(a)$ instead of $X(a; \succcurlyeq)$.

Two effects, x and x' are said to be *elementarily linked* if there are actions $a, a' \in A$ such that $x, x' \in X(a) \cap X(a')$. Two effects are said to be *linked* if there exists a sequence of effects $x = x_0, x_1, ..., x_n = x'$ such that every x_j is elementarily linked with x_{j+1}. I assume throughout that the set of action is rich enough so that every pair of effects is linked.

3.2 Constant valuation bets and beliefs

Intuitively speaking, a constant valuation bet is a bet that, once accepted, leaves the decision maker indifferent among all the effects. For example, a full insurance policy is a constant valuation bet since, by definition, a decision maker who takes out a homeowner policy that provides full insurance is indifferent to whether or not his house is damaged by storm, consumed by fire, or remain intact.[7] The idea is that, the decision maker believes that by choosing alternative actions he may affect the likely realization of different effects. This, in turn, determines the relative desirability of the alternative bets. In particular, with sufficiently large number of variations of the likely realization of alternative effects there are no two bets that are equally desirable under all such variations. Let $I(a;b) = \{b' \in B \mid (a,b') \sim (a,b)\}$ then the idea of constant valuation bets is formalized as follows:

Definition 1: A bet b^* is said to be a *constant-valuation bet on X* if $(a,b^*) \sim (a',b^*)$ for all $a,a' \in A$, and $\cap_{a\in A} I(a;b^*) = \{b^*\}$.

The last requirement implies that if b^* is a constant valuation bet then for no other $b \in \cap_{a\in A} I(a;b^*)$ $(a,b) \sim (a',b)$ for all $a,a' \in A$. The set of all constant valuation bets is denoted by B^*. If b^{**} and b^* are constant valuation bets satisfying $(a',b^{**}) \succ (a',b^*)$ then transitivity of $\succcurlyeq$ implies $(a,b^{**}) \succ (a,b^*)$ for all $a \in A$. Since transitivity will be assumed, I write $b^{**} \succ b^*$ instead of $(a,b^{**}) \succ (a,b^*)$.

I assume that there are at least two constant valuation bets b^* and b^{**} such that $b^{**} \succ b^*$. Formally,

(A.0) *There exist bets $b^{**}, b^* \in B^*$ such that $b^{**} \succ b^*$.*

[7] The concept of constant valuation bets is analogous to constant valuation acts in Karni (1993, 2003). The idea of constant valuation acts is similar to Drèze's (1987) notion of "omnipotent" acts. A similar concept appears in Skiadas (1997).

Note that if the set of actions includes actions that would allow the decision maker to choose, indirectly, the consequences then accepting a constant valuation bets renders the decision maker indifferent among the effects. Formally, let $a^x(\succcurlyeq)$ denote an action whose sole nonnull effect according to $\succcurlyeq$ is x (that is, $X(a^x(\succcurlyeq)) = \{x\}$). Then for each $b \in B$, a decision maker whose preferences are $\succcurlyeq$ believes that the choice of $(a^x(\succcurlyeq), b)$ yields the consequences $(x, b(x))$. In a sense, $a^x(\succcurlyeq)$ is analogous to the constant act that, in Savage's (1954) theory, yields the effect x in every state. However, in Savage's theory constant acts are defined independently of the preferences. In contrast, the fact that "nonnull effect" is invoked in the definition of $a^x(\succcurlyeq)$, and that whether an effect is nonnull depends on the decision-maker's beliefs, means that it is preference-dependent. Savage (1954) assumes that all constant acts are in the choice set. In the present context the analogous assumption would require that, given $\succcurlyeq$, for every $x \in X$ there exist $a^x(\succcurlyeq) \in A$ whose sole effect is x (that is, $\{a^x(\succcurlyeq) \mid x \in X\} \subset A$). If this assumption is satisfied and b^* is a constant valuation bet on X then, by definition, $(a^x(\succcurlyeq), b^*) \sim (a^y(\succcurlyeq), b^*)$. This induces an equivalence relation, $\approx$, on C defined by $(x, b^*(x)) \approx (y, b^*(y))$ if and only if $(a^x(\succcurlyeq), b^*) \sim (a^y(\succcurlyeq), b^*)$, for all $x, y \in X$.

Following Ramsey (1931), the degree of belief a decision maker holds regarding the likely realization of an event is defined by his willingness to bet on the event. Presently the issue is the degree of belief of a decision maker in the likely realization of subsets of effects. Note, however, that effects may have implications for the decision maker's well-being that are independent of the payoff of the bet. For example, a patient with a spinal cord injury who is about to undergo surgery may lose his mobility or be restored to good health. The same amount of money may have different implications to the patient's well-being depending on the effect. Recognizing that monetary payoffs may be valued differently, depending on the effects, means that the application of Ramsey's method must be approached with some care. Specifically, the bets that figure in the definition of beliefs must be chosen in a way that neutralizes the influence of the effects. This is done by replacing the constant monetary payoffs in Ramsey's definition with constant valuation payoffs. To formalize this idea let $\mathcal{X}$ denote the power set of X.

Definition 2: A binary relation $\trianglerighteq$ on $A \times \mathcal{X}$ is a decision maker's *beliefs* if, for all constant valuation bets b^* and b^{**} such that $b^{**} \succ b^*$, $(a, Z) \trianglerighteq (a', Y)$ if and only if $(a, b^{**}_Z b^*) \succcurlyeq (a', b^{**}_Y b^*)$.

The interpretation of $(a, Z) \trianglerighteq (a', Y)$ is that the decision maker believes that if action a is chosen then it is more likely that the effect realized is in Z than that it is in Y if the action a' is chosen instead.

3.3 Axioms

The structure of the preference relations on $\mathbb{C}$ is depicted axiomatically. The first two axioms are standard and require no commentary.

(A.1) (**Weak order**) $\succcurlyeq$ *on* $\mathbb{C}$ *is a complete and transitive binary relation.*

(A.2) (**Continuity)** *For all* $(a,b)\in\mathbb{C}$ *the sets* $\{(a,b')\in\mathbb{C}\mid(a,b')\succcurlyeq(a,b)\}$ *and* $\{(a,b')\in\mathbb{C}\mid(a,b)\succcurlyeq(a,b')\}$ *are closed.*

The third axiom requires that the "intensity of preferences" for monetary payoffs contingent on any given effect be independent of the action that resulted in that effect. It invokes Wakker's (1987) idea of cardinal consistency and, in its present form, it is an adaptation of Karni's (2003) cardinal coherence.

(A.3) (**Action-independent betting preferences**) *For all* $a,a'\in A$, $b,b',b'',b'''\in B$, $x\in X(a)\cap X(a')$, *and* $r,r',r'',r'''\in\mathbb{R}$, *if* $(a,(b_{-x},r))\succcurlyeq\left(a,\left(b'_{-x},r'\right)\right)$, $\left(a,\left(b'_{-x},r''\right)\right)\succcurlyeq(a,(b_{-x},r'''))$, *and* $\left(a',\left(b''_{-x},r'\right)\right)\succcurlyeq\left(a',\left(b'''_{-x},r\right)\right)$ *then* $\left(a',\left(b''_{-x},r''\right)\right)\succcurlyeq\left(a',\left(b'''_{-x},r'''\right)\right)$.

To grasp the meaning of action-independent betting preferences think of the preferences $(a,(b_{-x},r))\succcurlyeq\left(a,\left(b'_{-x},r'\right)\right)$ and $\left(a,\left(b'_{-x},r''\right)\right)\succcurlyeq(a,(b_{-x},r'''))$ as indicating that, given action a and effect x, the intensity of the preferences of r'' over r''' is sufficiently larger than that of r over r' as to reverse the preference ordering of the effect-contingent payoffs b_{-x} and b'_{-x}. The axiom requires that these intensities not be contradicted when the action is a' instead of a.

Figure 1 illustrates the axiom and the structure it imposes on the preference relations. Suppose, for the sake of simplicity that there are only two effects so that $(b_{-x}r)=(y',r)$ is a point in a two dimensional plane. The lower plane in Figure 1 corresponds to action-bet pairs in which the action is a while the upper plane corresponds to action-bet pairs in which the action is a'. The axiom, depicted for expositional convenience in terms of the indifference relations instead of weak preferences, requires that if $(a;(y',r)) \sim (a;(y,r'))$, $(a;(y',r'')) \sim (a;(y,r'''))$, and $(a';(y''',r)) \sim (a';(y'',r'))$ (these indifference relations are depicted by the corresponding point on the solid indifference curves), then $(a';(y''',r'')) \sim (a;(y'',r'''))$ (an indifference depicted by points on the dashed indifference curve). Clearly, if $a = a'$ then this condition collapse to the Redmeister condition (see Wakker [1989]). Figure 1 also clarifies the idea of the intensity of preferences discussed above. The intensity of preferences of r' over r is measured by the compensating variation $y' \rightarrow y$ if the action is a and $y''' \rightarrow y''$ when the action is a'. Next the compensating variation $y' \rightarrow y$ is used to measure the intensity of preference of r''' over r''. In particular, in this illustration intensity of preferences of r' over r is the same as that of r''' over r'' as they both require the same compensating variation, namely, $y' \rightarrow y$. If, in addition, the compensating variation $y''' \rightarrow y''$ is a measure of the intensity of preferences of r' over r when the action a' is then the axiom requires that it also be a measure of the intensity of preference of r''' over r''. In other words, the intensity of preference of r' over r relative to that of r''' over r'' do not change either when the action changes or when the payoff of the bet on the other effect varies.

In addition, for every given act, axiom (A.3) embodies the independence of preferences for the payoff conditional on any given effect from the payoffs of the bet associated with the other effects. This independence property implies the well-known Sure Thing Principle and, in the case of two effects, the hexagon condition (for more details see Lemmas 4 and 5 in Section 5.1). To grasp the last claim, suppose that $a = a'$, $y'' = y'$, and $r' = r''$ then Figure

1 collapses to Figure 2, which is the customary depiction of the hexagon condition.

3.4 The main representation theorem

The main result of this paper is the assertion that a preference relation on $\mathbb{C}$ has the structure described by axioms (A.1) - (A.3) if and only if there is a continuous utility function, u, on the set of consequences, and a family of action-dependent probability measures, $\{\pi(\cdot; a) \mid a \in A\}$, on the set of effects, such that the assignment $(a, b) \rightarrow \sum_{x \in X} u(b(x); x) \pi(x; a)$ represents the preference relation. Furthermore, the utility function is unique up to a positive linear transformation and, for each $a \in A$, the probability measure $\pi(\cdot; a)$ is unique, and $\pi(x; a) = 0$ if and only if x is null given a.

Theorem 1 *Suppose that there are at least two effects, that any two effects are linked, and that assumption (A.0) is satisfied. Then*

(a) The following conditions are equivalent:

(a.i) The preference relation, $\succcurlyeq$ *on* $\mathbb{C}$*, satisfies (A.1) - (A.3).*

(a.ii) There exist continuous function $u : C \rightarrow \mathbb{R}$ *and a family of probability measures* $\{\pi(\cdot; a)\}_{a \in A}$ *on* X *such that, for all* $(a, b), (a', b') \in \mathbb{C}$*,*

$$(a, b) \succcurlyeq (a', b') \Leftrightarrow \sum_{x \in X} u(b(x); x) \pi(x; a) \geq \sum_{x \in X} u(b'(x); x) \pi(x; a').$$

(b) The utility function u *is unique up to positive linear transformation.*

(c) For each $a \in A$, $\pi(\cdot; a)$ is unique and $\pi(x; a) = 0$ if and only if x is null given a.

The proof of Theorem 1 is given in Section 5.1. A sketch of the argument that $(a.i) \rightarrow (a.ii)$, which is the difficult part of the proof, is in order. For every given $a \in A$, (A.1), (A.2) and (A.3) imply the existence of additively separable continuous representation of $\succcurlyeq_a$, (that is, for every given a, $\succcurlyeq_a$ is represented by $(a, b) \mapsto \sum_{x \in X} w_a(b(x), x)$). Axiom (A.3) also implies that, for all $a, a' \in A$, the additive valued representations of $\succcurlyeq_a$ and $\succcurlyeq_{a'}$ are cardinally equivalent. The constant valuation bets are invoked to link the representations across actions. Next for each $a \in A$ and $x \in X$, the probability $\pi(x; a)$ is defined by $\pi(x; a) = w_a(b^{**}(x), x)$, where b^{**} is a constant valuation bet and w_a is normalized so that $\sum_{x \in X} w_a(b^{**}(x), x), a \in A$. Finally, the utility of the consequences $(x; r)$, $r \in \mathbb{R}$, $u(x; r)$ is defined by $w_a(x; r)/\pi(x; a)$, which is shown to be independent of a.

Remark: It is easy to verify that the probability measures that figure in the representation in Theorem 1 are the only probability measures representing the decision maker's beliefs. In other words, $\{\pi(\cdot; a)\}_{a \in A}$ is the sole family of probability measures satisfying $(a, Z) \trianglerighteq (a', Y)$ if and only if $\pi(Z; a) \geq \pi(Y; a')$, where $\pi(T, a) = \sum_{x \in T} \pi(x; a)$, for all $T \subset X$ and $a \in A$.

3.5 Effect-independent preferences on bets

If the decision maker bets on the effect of the next turn of a roulette wheel, it is reasonable to suppose that he does not care about the effect except insofar as it determines his monetary payoff. This example is typical of situations in which the decision maker's betting preferences are effect independent. The following axiom, which is similar to Wakker's (1987) cardinal consistency, captures this idea:

(A.4) (**Effect-independent betting preferences**) *For all* $a \in A$, $b, b', b'', b''' \in B$, $x, y \in X(a)$, *and* $r, r', r''r''' \in \mathbb{R}$, *if* $\left(a, (b'_{-x}, r)\right) \succcurlyeq (a, (b_{-x}, r'))$, $(a, (b_{-x}, r'')) \succcurlyeq \left(a, (b'_{-x}, r''')\right)$, *and* $\left(a, (b''_{-y}, r')\right) \succcurlyeq \left(a, (b'''_{-y}, r)\right)$ *then* $\left(a, (b''_{-y}, r'')\right) \succcurlyeq \left(a, (b'''_{-y}, r''')\right)$.

The interpretation of (A.4) is analogous to that of action-independent betting preferences. The preferences $\left(a, (b'_{-x}, r')\right) \succcurlyeq (a, (b_{-x}, r))$ and $(a, (b_{-x}, r'')) \succcurlyeq \left(a, (b'_{-x}, r''')\right))$ indicate that the "intensity" of the preference for r'' over r''' in given the effect x is sufficiently greater than that of r over r' as to reverse the order of preference between the payoffs b'_{-x} and b_{-x}. Outcome independence requires that these intensities not be contradicted by the preferences between the same payoffs given any other effect y.

Theorem 2 *Suppose that there are at least two effects, that any two effects are linked, and that assumption (A.0) is satisfied. Then*

(a) The following conditions are equivalent:

(a.i) The preference relation, $\succcurlyeq$ on $\mathbb{C}$, satisfies (A.1)-(A.4).

(a.ii) There exist continuous function $u : \mathbb{R} \to \mathbb{R}$ and, for all $x \in X$, there are numbers $\sigma(x) > 0, \kappa(x)$, and probability measures $\{\pi(\cdot; a)\}_{a \in A}$ on X such that for all $(a, b), (a', b') \in \mathbb{C}$, $(a, b) \succcurlyeq (a', b')$ if and only if

$$\sum_{x \in X} [\sigma(x) u(b(x)) + \kappa(x)] \pi(x; a) \geq \sum_{x \in X} [\sigma(x) u(b'(x)) + \kappa(x)] \pi(x; a').$$

(b) The function u is unique up to positive linear transformation.

(c) For each $a \in A$, $\pi(\cdot; a)$ is unique and $\pi(x; a) = 0$ if and only if x is null given a.

A *constant-payoff bet* is a bet satisfying $b(x) = b$ for all x. If all constant-payoff bets are constant-valuation bets then both the preference relation and the utility functions display effect independence.[8] The following is an immediate implication of Theorem 2.

Corollary 3 *Suppose there are at least two effects, that any two effects are linked, that assumption (A.0) is satisfied, and constant-valuation bets are constant payoff bets. Then the following conditions are equivalent:*

(i) The relation $\succcurlyeq$ on $\mathbb{C}$ satisfies (A.1) - (A.4).

[8] Effect-independent preferences are analogous to state-independent preferences, effect-independent utility function is analogous to state-independent utility function in the traditional formulations of subjective expected utility theory (see Karni [1996]).

(ii) There exist a continuous real-valued function u on X, unique up to positive linear transformation, and unique family of probability measures $\{\pi(\cdot;a) \mid a \in A\}$ on X, such that, for all $(a,b), (a',b) \in \mathbb{C}$,

$$(a,b) \succcurlyeq (a',b') \Leftrightarrow \sum_{x \in X} u(b(x))\,\pi(x;a) \geq \sum_{x \in X} u(b'(x))\,\pi(x;a'),$$

where $\pi(x;a) = 0$ if and only if x is null given a.

The proof of the corollary is as follows: If b^* is a constant valuation bet then, by Theorem 2, $\sigma(x)\,u(b^*(x))+\kappa(x) = \sigma(x')\,u(b^*(x'))+\kappa(x')$ for all $x, x' \in X$. But if constant-valuation bets are constant-payoff bets then $[\sigma(x) - \sigma(x')]u(r) = \kappa(x') - \kappa(x)$ for all $r \in \mathbb{R}$ and $x, x' \in X$. Hence $\sigma(x) - \sigma(x') = 0 = \kappa(x') - \kappa(x)$.

4 Concluding Remarks

The representation Theorems 1, 2, and Corollary 3 give necessary and sufficient conditions for the decision-making process to be decomposed into two cognitive subprocesses. The first is the assessment of the likely realization of different effects conditional on the actions. The second is the evaluation of the consequences, that is, effect-payoff pairs, that may result from the implementation of those actions. The two processes are integrated to produce a value, that is, the subjective expected utility corresponding to each action-bet pairs. In this sense, the result of this work is a new subjective expected utility theory that, unlike traditional theories, does not invoke the notion of states of the world to resolve uncertainty. This theory may better describe how decision makers actually perceive and assess their options. It does not rule out that decision makers mentally construct a state space to help organize their thoughts, but it does not require that they do. In other words, when the state space is objectively observable and the likely realization of the states is independent of the decision maker's choice of actions, so that traditional subjective utility theory is relevant, there is no contradiction between the theory developed here and the traditional approach. The traditional theory may easily be embedded in the present framework by defining the actions-bet pairs as random variables on the state space and, for every given action, assigning to the effects the probabilities of the events in the state space in which these effects are realized under the given action. Note, however, that even if decision makers do construct a mental state space to help organize their thoughts, the states are not always independently observable, and using them, in such cases, is not a good scientific procedure.

A different aspect of the state-space formulation is that the states must be mutually exclusive and jointly exhaustive. This poses a conceptual problem to Bayesian decision theory since it excludes any possibility that the decision maker may learn something about the world that he has not already conceived of when constructing the state-space. The same sort of problem seems to arises in the present formulation since the set of effects is a primitive notion. To allow for the possibility that a decision maker may believe that he might not be able to conceive of all the possible effects that may obtain when he takes an action, one of the effects may be interpreted as representing unforeseen effects. In other words, the bet may associate a unique payoff to each specified effect and a payoff if non of the specified effect obtains. The probability associated with this effect may be taken to represent the degree to which the decision maker believes that his understanding the possible effects of his actions is incomplete.

A different approach to modeling subjective distributions without relying on a state space is pursued in Gilboa and Schmeidler (2001a). They model preferences over acts conditional on what, in this paper, I referred to as bets. Instead of deriving the utility, Gilboa and Schmeidler assume that an outcome-independent (that is effect-independent in the terminology of this paper) linear utility on bets is given and derive subjective probabilities on the outcome, consistent with expected value maximizing behavior.[9]

Recognizing the ability of the decision makers to influence, by their actions, the likelihoods of alternative effects creates a natural link between the present work and the literature dealing with principal-agent relationship in the presence of moral hazard. A principal-agent relationship is governed by a contract specifying the agent's payoff as a function of the observed effect. Let $W := \{w : X \to \mathbb{R}\}$ denote the set of contracts, where X has the interpretation of output. Clearly, contracts are bets on the output, and the agent's choice of action affect the likely realization of alternative levels of output. The modeling of the agent's behavior in the context of agency theory admits alternative formulations, including

[9] Note that Gilboa and Schmeidler assume, without calling them by these names, that constant-payoff bets are constant-valuation bets. Thus they implicitly assume not only that the preferences but also the utility functions are effect independent.

the state-space formulation and the parameterized distribution formulation (see, for example, Hart and Holmstrom [1979], Chambers and Quiggin [2000]). The latter formulation, pioneered and popularized by Mirrlees (1974, 1976), is analytically convenient and is often used in applications. The decision theory developed here depicts, axiomatically, the principal's conduct in parameterized distribution formulation of agency theory. A development of a full fledge axiomatic model of the agent's behavior, that takes into account the direct effect of the actions on the agent's well-being, is the subject matter of a companion paper Karni (2003a).

5 Proofs

5.1 Proof of Theorem 1.

As a preliminary step I prove two results that are of interest in their own right.

Coordinate independence requires that, for every given action, the preference between any two bets be independent of the payoffs contingent on effects on which the two bets agree. For every given action, this condition is analogous to Savage's (1954) Sure Thing Principle. Like it, it implies the separability of the valuation of the monetary payoffs across effects.

(**Coordinate independence**) *For all* $a \in A$, $b, b' \in B$, $x \in X$, *and* $r, r' \in \mathbb{R}$, $(a, (b_{-x}, r)) \succcurlyeq \left(a, \left(b'_{-x}, r\right)\right)$ *if and only if* $(a, (b_{-x}, r')) \succcurlyeq \left(a, \left(b'_{-x}, r'\right)\right)$.

Lemma 4 *Let there be at least three nonnull effects. If $\succcurlyeq$ on $\mathbb{C}$ satisfies (A.3) then it satisfies coordinate independence.*

Proof. Suppose that $\left(a, \left(\tilde{b}_{-x}, \tilde{r}\right)\right) \succcurlyeq \left(a, \left(\hat{b}_{-x}, \tilde{r}\right)\right)$ and $\left(a, \left(\hat{b}_{-x}, \hat{r}\right)\right) \succ \left(a, \left(\tilde{b}_{-x}, \hat{r}\right)\right)$ for some $a \in A, \tilde{b}, \hat{b} \in B, x \in X(a)$, and $\tilde{r}, \hat{r} \in \mathbb{R}$. In (A.3), let $r = r' = \tilde{r}, r''' = r'' = \hat{r}$, $a = a'$, $b = b'' = \tilde{b}$, and $b' = b''' = \hat{b}$. Then, (A.3) implies that $\left(a, \left(\tilde{b}_{-x}, \hat{r}\right)\right) \succcurlyeq \left(a, \left(\hat{b}_{-x}, \hat{r}\right)\right)$ which is a contradiction. Hence $\left(a, \left(\tilde{b}_{-x}, \tilde{r}\right)\right) \succcurlyeq \left(a, \left(\hat{b}_{-x}, \tilde{r}\right)\right)$ if and only if $\left(a, \left(\tilde{b}_{-x}, \hat{r}\right)\right) \succcurlyeq \left(a, \left(\hat{b}_{-x}, \hat{r}\right)\right)$. ■

The well-known Hexagon condition implies additive separable representation for actions that the decision maker believes have exactly two nonnull effects**:**

(**Hexagon condition**) *For all $a \in A$, $b \in B$, and $r, r', r'' \in \mathbb{R}$, if $X(a) = \{x, y\}$ then $(a, (b_{-x}, r)_{-y}, r') \sim (a, (b_{-x}, r')_{-y}, r)$ and $(a, (b_{-x}, r)_{-y}, r'') \sim (a, (b_{-x}, r')_{-y}, r') \sim (a, (b_{-x}, r'')_{-y}, r)$ imply $(a, (b_{-x}, r')_{-y}, r'') \sim (a, (b_{-x}, r'')_{-y}, r')$.*

Lemma 5 *Let there be exactly two nonnull effects. If $\succcurlyeq$ on $\mathbb{C}$ satisfies (A.3) then it satisfies the Hexagon condition.*

Proof. Suppose that $\succcurlyeq$ on $\mathbb{C}$ satisfies (A.1) and (A.3). Suppose that $\left(a, (b_{-y}, r')_{-x}, r\right) \sim \left(a, (b_{-y}, r)_{-x}, r'\right)$ and $\left(a, (b_{-y}, r)_{-x}, r''\right) \sim \left(a, (b_{-y}, r')_{-x}, r'\right) \sim \left(a, (b_{-y}, r'')_{-x}, r\right)$. Apply (A.3) with $a = a', r''' = r'$, $b'''_{-x} = b_{-x} = (b_{-y}, r')_{-x}$, $b'_{-x} = (b_{-y}, r)_{-x}$, and $b''_{-x} = (b_{-y}, r'')_{-x}$. Then, apply (A.3) twice to obtain $\left(a, (b_{-y}, r')_{-x}, r''\right) \sim \left(a, (b_{-y}, r'')_{-x}, r'\right)$. ■

The following additional terminology will be used in the proof below: An array of real-valued functions $(v_s)_{s\in S}$ is said to be a *jointly cardinal additive representation* of a binary relation $\succeq$ on a product set $D = \Pi_{s\in S} D_s$ if, for all $d, d' \in D$, $d \succeq d'$ if and only if $\sum_{s\in S} v_s(d_s) \geq \sum_{s\in S} v_s(d'_s)$, and the class of all functions that constitute an additive representation of $\succeq$ consists of those arrays of functions, $(\hat{v}_s)_{s\in S}$, for which $\hat{v}_s = \lambda v_s + \zeta_s$, $\lambda > 0$ for all $s \in S$. The representation is continuous if the functions $v_s, s \in S$ are continuous.

I turn next to the proof of Theorem 1.

$(a.i) \Rightarrow (a.ii)$. Since $\succcurlyeq$ satisfies (A.1)-(A.3), Lemma 4, Lemma 5 and Theorem III.4.1 of Wakker [1989] imply that, for every $a \in A$ such that $X(a)$ contains at least two nonnull effects, there exist array of functions $\{w_a(\cdot; x) : \mathbb{R} \to \mathbb{R}\}_{x\in X}$ that constitute jointly cardinal continuous additive representation of $\succcurlyeq_a$ on B.[10]

Observe that, for all $a, a' \in A$ and $x \in X$, $w_a(\cdot; x)$ and $w_{a'}(\cdot; x)$ are ordinally equivalent.

Claim 1: For all $a, a' \in A$, $x \in X$, and $r', r \in \mathbb{R}$, $w_a(r'; x) \geq w_a(r; x)$ if and only if $w_{a'}(r'; x) \geq w_{a'}(r; x)$.

Proof. Let $(a', (b_{-x}, r')) \succcurlyeq (a', (b_{-x}, r))$. But $(a', (b_{-x}, r')) \succcurlyeq (a', (b_{-x}, r'))$, $(a', (b_{-x}, r')) \succcurlyeq (a', (b_{-x}, r))$, and $(a, (b_{-x}, r')) \succcurlyeq (a, (b_{-x}, r'))$. Thus, by (A.3), $(a, (b_{-x}, r')) \succcurlyeq (a, (b_{-x}, r))$. The conclusion is implied by the representation of $\succcurlyeq$. ■

[10] If $a = a^x(\succcurlyeq)$, (that is $X(a) = \{x\}$) then the fact that $\succcurlyeq_a$ is a continuous weak order implies that there exist continuous real-valued function $w_{a^x}(\cdot; x)$ representing $\succcurlyeq_{a^x}$ on $\mathbb{R}$ (Debreu [1954] Theorem I).

Claim 1 implies that $b^{**} \succcurlyeq_a b^*$ if and only if $b^{**} \succcurlyeq_{a'} b^*$.

By assumption (A.0) there are $b^{**}, b^* \in B^*$ such that $b^{**} \succ b^*$. Invoking the uniqueness of the jointly cardinal representation and (A.3) normalize $\{w_a(\cdot; x)\}_{x \in X}$ as follows: For all $a \in A$ and $x \in X$ set $w_a(b^*(x); x) = 0$ and $\sum_{x \in X} w_a(b^{**}(x); x) = 1$.

Next I show that, for all $a, \bar{a} \in A$ and $x \in X(a) \cap X(\bar{a})$, $w_a(\cdot; x)$ is either constant or is positive linear transformation of $w_{\bar{a}}(\cdot; x)$.

Lemma 6 *Suppose that there are at least two effects. Then the following conditions are equivalent:*

- (i) *The relation $\succcurlyeq$ on $\mathbb{C}$ satisfies (A.1) - (A.3).*
- *(ii) For every $a, \bar{a} \in A$ and $x \in X(a) \cap X(\bar{a})$ there exist a nonnegative number $\beta_{(a,\bar{a},x)}$ such that $w_a(\cdot; x) = \beta_{(a,\bar{a},x)} w_{\bar{a}}(\cdot; x)$, where $\{w_j(\cdot; x) : \mathbb{R} \to \mathbb{R}\}_{x \in X}$, $j = a, \bar{a}$ constitute a jointly cardinal continuous additive representation of $\succcurlyeq_j$ on B.*

Proof. $(i) \Rightarrow (ii)$. Let $a, \bar{a} \in A$ be such that the number of nonnull effect in each set $X(a)$ and $X(\bar{a})$ is, at least, two. Suppose that $\succcurlyeq$ satisfies (A.1) – (A.3). Theorem III.4.1. of Wakker (1989) implies that, for every given $a \in A$, there exist continuous functions $\{w_a(\cdot; x) : \mathbb{R} \to \mathbb{R} \mid x \in X\}$ that constitute a jointly cardinal additive representation of $\succcurlyeq_a$ on B.

If $X(a) \cap X(\bar{a}) \neq \emptyset$, then, by the representation, for every $y \in X(a) \cap X(\bar{a})$ there exist

$b, b', b'', b''' \in B$ such that

$$\sum_{x \in X - \{y\}} \left[w_{\bar{a}} \left(b\left(x \right) ; x \right) - w_{\bar{a}} \left(b'\left(x \right) ; x \right) \right] = \zeta > 0, \tag{1}$$

and

$$\sum_{x \in X - \{y\}} \left[w_{a} \left(b''\left(x \right) ; x \right) - w_{a} \left(b'''\left(x \right) ; x \right) \right] = \varepsilon > 0. \tag{2}$$

By continuity of the additive valued functions $w_a(\cdot; x)$ and the connectedness of $\mathbb{R}$, for every $\hat{\zeta} \in [-\zeta, \zeta]$, $\hat{\varepsilon} \in [-\varepsilon, \varepsilon]$, and $y \in X(a) \cap X(\bar{a})$ there exist $\bar{b}, \bar{b}', \bar{b}'', \bar{b}''' \in B$ such that

$$\sum_{x \in X - \{y\}} \left[w_{\bar{a}} \left(\bar{b}\left(x \right) ; x \right) - w_{\bar{a}} \left(\bar{b}'\left(x \right) ; x \right) \right] = \hat{\zeta} \tag{3}$$

and

$$\sum_{x \in X - \{y\}} \left[w_{a} \left(\bar{b}''\left(x \right) ; x \right) - w_{a} \left(\bar{b}'''\left(x \right) ; x \right) \right] = \hat{\varepsilon}. \tag{4}$$

Define $\phi_{(a,\bar{a},x)}$ by $w_a(\cdot; x) = \phi_{(a,\bar{a},x)} \circ w_{\bar{a}}(\cdot; x)$, $x \in X$, then, by Claim 1, $\phi_{(a,\bar{a},x)}$ is a continuous nondecreasing function . To show that $\phi_{(a,\bar{a},x)}$ is constant or linear, fix $y \in X(a) \cap X(\bar{a})$ and let $I_y = w_{\bar{a}}(\mathbb{R}; y)$. Then, by the continuity of $w_{\bar{a}}(\cdot; y)$, I_y is an interval in $\mathbb{R}$. Take $\alpha, \beta, \gamma, \delta \in I_y$ such that $-\zeta \leq \alpha - \beta = \gamma - \delta \leq \zeta$ and $-\varepsilon \leq \phi_{(a,\bar{a},y)}(\alpha) - \phi_{(a,\bar{a},y)}(\beta) \leq \varepsilon$. Let $r, r', r'', r''' \in \mathbb{R}$ satisfy $w_{\bar{a}}(r; y) = \alpha$, $w_{\bar{a}}(r'; y) = \beta$, $w_{\bar{a}}(r''; y) = \gamma$ and $w_{\bar{a}}(r'''; y) = \delta$. Take $b, b' \in B$ such that

$$\sum_{x \in X - \{y\}} \left[w_{\bar{a}} \left(b\left(x \right) ; x \right) - w_{\bar{a}} \left(b'\left(x \right) ; x \right) \right] = \alpha - \beta. \tag{5}$$

Then, by the representation, $(\bar{a}, (b_{-y}; r)) \sim \left(\bar{a}, \left(b'_{-y}; r'\right)\right)$ and $(\bar{a}, (b_{-y}; r'')) \sim \left(\bar{a}, \left(b'_{-y}; r'''\right)\right)$.

Take $b'', b''' \in B$ such that

$$\sum_{x \in X - \{y\}} \left[w_{a} \left(b''\left(x \right) ; x \right) - w_{a} \left(b'''\left(x \right) ; x \right) \right] = \phi_{(a,\bar{a},y)}(\alpha) - \phi_{(a,\bar{a},y)}(\beta). \tag{6}$$

Since $w_a(\cdot; y) = \phi_{(a,\bar{a},y)} \circ w_{\bar{a}}(\cdot; y)$ this implies $\left(a, \left(b''_{-y}, r\right)\right) \sim \left(a, \left(b'''_{-y}, r'\right)\right)$. Applying (A.3) twice yields $\left(a, \left(b''_{-y}, r''\right)\right) \sim \left(a, \left(b'''_{-y}, r'''\right)\right)$. Thus

$$\phi_{(a,\bar{a},y)}(\gamma) - \phi_{(a,\bar{a},y)}(\delta) = \sum_{x \in X - \{y\}} \left[w_a\left(b''(x); x\right) - w_a\left(b'''(x); x\right)\right] = \phi_{(a,\bar{a},y)}(\alpha) - \phi_{(a,y)}(\beta). \tag{7}$$

By Wakker (1987) Lemma 4.4 this implies that $\phi_{(a,\bar{a},y)}$ is affine. But, by Claim 1, $\phi_{(a,\bar{a},y)}$ is nondecreasing, hence $\phi_{(a,\bar{a},y)}$ is either constant or positive. Hence there exist $\beta_{(a,\bar{a},x)} > 0$ and $\alpha_{(a,\bar{a},x)}$ such that, for all $r \in \mathbb{R}$ and nonnull $x \in X(a)$, $w_a(r; x) = \beta_{(a,\bar{a},x)} w_{\bar{a}}(r; x) + \alpha_{(a,\bar{a},x)}$. By (A.3) and the normalization, $w_{\bar{a}}(b^*(x); x) = 0 = w_a(b^*(x); x)$, for all $x \in X$. Hence

$$\sum_{t \in X} \left[\beta_{(a,\bar{a},x)} w_{\bar{a}}\left(b^*(x); x\right) + \alpha_{(a,\bar{a},x)}\right] = \sum_{x \in X} \alpha_{(a,\bar{a},x)} = 0,$$

and $w_a(r; x) = \beta_{(a,\bar{a},x)} w_{\bar{a}}(r; x)$, where $\beta_{(a,\bar{a},x)} \geq 0$ for all $x \in X$.

Next I extend the result to $\{a^x \in A \mid x \in X\}$, where $a^x = a^x(\succcurlyeq)$. Assume, without essential loss of generality, that $X(\bar{a}) = \{y, z\}$. Suppose that, for some constant valuation bet $\bar{b}$, $w_{a^y}\left(\bar{b}(y); y\right) = w_{\bar{a}}\left(\bar{b}(y); y\right)/\theta_y$, where $\theta_y > w_{\bar{a}}\left(b^{**}(y), y\right)$, and $w_{a^z}\left(\bar{b}(z); z\right) = w_{\bar{a}}\left(\bar{b}(z); z\right)/w_{\bar{a}}\left(b^{**}(z), z\right)$.

By (A.3) and the normalization,

$$w_{\bar{a}}\left(b^{**}(y); y\right) + w_{\bar{a}}\left(b^{**}(z); z\right) = 1. \tag{8}$$

By (A.3), for every constant-valuation bet, $\bar{b}$,

$$w_{\bar{a}}\left(\bar{b}(y); y\right) + w_{\bar{a}}\left(\bar{b}(z); z\right) = w_{a^y}\left(\bar{b}(y); y\right) = w_{a^z}\left(\bar{b}(z); z\right). \tag{9}$$

But, by the supposition,

$$w_{\bar{a}}\left(\bar{b}\left(z\right);z\right)=w_{a^{z}}\left(\bar{b}\left(z\right);z\right)w_{\bar{a}}\left(b^{**}\left(z\right);z\right), \tag{10}$$

and

$$w_{\bar{a}}\left(\bar{b}\left(y\right);y\right)=\theta_{y}w_{a^{y}}\left(\bar{b}\left(y\right);y\right)>w_{a^{y}}\left(\bar{b}\left(y\right);y\right)w_{\bar{a}}\left(b^{**}\left(y\right);y\right). \tag{11}$$

Hence, since $w_{a^{y}}\left(\bar{b}\left(y\right);y\right)=w_{a^{z}}\left(\bar{b}\left(z\right);z\right)$,

$$w_{\bar{a}}\left(\bar{b}\left(y\right);y\right)+w_{\bar{a}}\left(\bar{b}\left(z\right);z\right)>\left[w_{\bar{a}}\left(b^{**}\left(y\right);y\right)+w_{\bar{a}}\left(b^{**}\left(z\right);z\right)\right]w_{a^{y}}\left(\bar{b}\left(y\right),y\right). \tag{12}$$

But, by (8), $w_{\bar{a}}\left(b^{**}\left(y\right);y\right)+w_{\bar{a}}\left(b^{**}\left(z\right);z\right)=1$. Thus equation (12) imply that

$$w_{\bar{a}}\left(\bar{b}\left(y\right);y\right)+w_{\bar{a}}\left(\bar{b}\left(z\right);z\right)>w_{a^{y}}\left(\bar{b}\left(y\right);y\right). \tag{13}$$

This contradicts equation (9). Hence, for all $x\in X$ and $r\in\mathbb{R}$, $w_{a^{x}}\left(r;x\right)=w_{\bar{a}}\left(r,x\right)/w_{\bar{a}}\left(b^{**}\left(x\right);x\right)$. Set $w_{a^{x}}\left(r;y\right)=0$, for all $y\neq x$. Hence $w_{a^{x}}$ is a positive linear or constant function of $w_{\bar{a}}$. This completes the proof that $(i)\rightarrow(ii)$.

$(ii)\Rightarrow(i)$. That (ii) implies Axioms (A.1) and (A.2) is immediate. To prove that (ii) implies (A.3) assume that for all $a,\bar{a}\in A$ and $y\in X\left(a\right)$ there exist positive linear or constant transformations $\phi_{(a,\bar{a},y)}$ such that $w_{a}\left(\cdot;y\right)=\phi_{(a,\bar{a},y)}\circ w_{\bar{a}}\left(\cdot;y\right)$. Let $y\in X\left(a\right)\cap X\left(\bar{a}\right)$. Suppose that $\left(\bar{a},\left(b_{-y},r\right)\right)\succcurlyeq\left(\bar{a},\left(b'_{-y},r'\right)\right)$, $\left(\bar{a},\left(b'_{-y},r''\right)\right)\succcurlyeq\left(\bar{a},\left(b_{-y},r'''\right)\right)$ and $\left(a,\left(b''_{-y},r'\right)\right)\succcurlyeq\left(a,\left(b'''_{-y},r\right)\right)$. By the representation, $\left(\bar{a},\left(b_{-y},r\right)\right)\succcurlyeq\left(\bar{a},\left(b'_{-y},r'\right)\right)$ if and only if

$$w_{\bar{a}}\left(r;y\right)+\sum_{x\in X-\{y\}}w_{\bar{a}}\left(b\left(x\right);x\right)\geq w_{\bar{a}}\left(r';y\right)+\sum_{x\in X-\{y\}}w_{\bar{a}}\left(b'\left(x\right);x\right) \tag{14}$$

and $\left(\bar{a},\left(b'_{-y},r''\right)\right)\succcurlyeq\left(\bar{a},\left(b_{-y},r'''\right)\right)$ if and only if

$$w_{\bar{a}}\left(r''';y\right)+\sum_{x\in X-\{y\}}w_{\bar{a}}\left(b\left(x\right);x\right)\leq w_{\bar{a}}\left(r'';y\right)+\sum_{x\in X-\{y\}}w_{\bar{a}}\left(b'\left(x\right);x\right). \tag{15}$$

Hence

$$w_{\bar{a}}(r';y) - w_{\bar{a}}(r;y) \le \sum_{x \in X - \{y\}} [w_{\bar{a}}(b(x);x) - w_{\bar{a}}(b'(x);x)] \le w_{\bar{a}}(r'';y) - w_{\bar{a}}(r''';y). \quad (16)$$

By positive linearity or constancy of $\phi_{(a,\bar{a},y)}$ these inequalities imply

$$w_a(r';y) - w_a(r;y) \le w_a(r'';y) - w_a(r''';y). \quad (17)$$

Next observe that $\left(a, \left(b''_{-y}, r'\right)\right) \succcurlyeq \left(a, \left(b'''_{-y}, r\right)\right)$ if and only if

$$\sum_{x \in X - \{y\}} w_a(b''(x);x) + w_a(r';y) \ge \sum_{x \in X - \{y\}} w_a(b'''(x);x) + w_a(r;y). \quad (18)$$

Thus

$$w_a(r';y) - w_a(r;y) \ge \sum_{x \in X - \{y\}} [w_a(b'''(x);x) - w_a(b''(x);x)]. \quad (19)$$

But inequality (17) implies

$$\sum_{x \in X - \{y\}} w_a(b''(x);x) + w_a(r'';y) \ge \sum_{x \in X - \{y\}} w_a(b'''(x);x) + w_a(r''';y). \quad (20)$$

Hence $\left(a, \left(b''_{-y}, r''\right)\right) \succcurlyeq \left(a, \left(b'''_{-y}, r'''\right)\right)$. Thus $(ii) \to (i)$. ■

Probabilities and utilities: Define $\pi(x;a) = w_a(b^{**}(x);x)$ for all $x \in X$ and $a \in A$. Then, by the normalization of w_a, $\pi(x;a) \ge 0$ and $\sum_{x \in X} \pi(x;a) = 1$. But $w_a(b^{**}(x);x) = \beta_{(a,a',x)} w_{a'}(b^{**}(x);x)$ implies that

$$\beta_{(a,a',x)} = \frac{\pi(x;a)}{\pi(x;a')} \text{ for all } a, a' \in A \text{ and } x \in X \text{ satisfying } \pi(x;a') > 0. \quad (21)$$

For any given $r \in \mathbb{R}$, $x \in X$, and $a \in A$ define $u(r;x,a) = w_a(r;x)/\pi(x;a)$ if $\pi(x;a) > 0$ and $u(r;x,a) = \bar{u}$ otherwise. Note that, for all $a \in A$ and $x \in X(a') \cap X(a)$,

$$u(r;x,a') = \frac{w_{a'}(r;x)}{\pi(x;a')} = \frac{w_a(r;x)}{\beta_{(a',a,x)}\pi(x;a')} = \frac{w_a(r;x)}{\pi(x;a)} = u(r;x,a), \quad (22)$$

where the third inequality is implied by equation (21). Hence $u(r;x,a) = u(r;x,a') := u(r;x)$ for all $a,a' \in A$ and $x \in X(a) \cap X(a')$. Since any two effects are linked it follows that $u(r;x,a) = u(r;x)$ for all $a \in A$ and $x \in X(a)$. Thus, by definition, $w_a(r;x) = \pi(x;a)\,u(r;x)$ for all $a \in A, x \in X$, and $r \in \mathbb{R}$. Observe that $u(b^{**}(x),x) = 1$ and $u(b^{*}(x),x) = 0$ for all $x \in X$.

Fix $x \in X$ and $r \in (b^{**}(x), b^{*}(x))$. Let $b_r \in B$ be defined by $u(b_r(x');x') = u(r;x)$ for all $x' \in X$. Thus $\sum_{x \in X} u(b_r(x);x)\,\pi(x;a) = u(r;x)$ for all $a \in A$. Take $\hat{b} \in B^*$ and let r satisfy $u\left(\hat{b}(x);x\right) = u(r;x)$. Then, by the uniqueness of $\hat{b}$, $\hat{b} = b_r$. Hence $b_r \in B^*$. Thus, for all $a,a' \in A$ and $\hat{b} \in B^*$,

$$\sum_{x \in X} u\left(\hat{b}(x);x\right)\pi(x;a) = \sum_{x \in X} u\left(\hat{b}(x);x\right)\pi(x;a'). \tag{23}$$

For all (a,b) and (a',b') in $\mathbb{C}$ such that $(a,b^{**}) \succcurlyeq (a,b),(a',b') \succcurlyeq (a,b^{*})$,

$$(a,b) \succcurlyeq (a',b') \Leftrightarrow \sum_{x \in X} \pi(x;a)\,u(b(x),x) \geq \sum_{x \in X} \pi(x;a')\,u(b'(x),x). \tag{24}$$

To see this observe that there is a constant valuation bet $\hat{b}$ such that $(a,b) \succcurlyeq \left(a,\hat{b}\right) \sim \left(a',\hat{b}\right) \succcurlyeq (a',b')$. The conclusion follows from Lemma 6, equation (24) and transitivity.

Extending the representation to intervals in $\mathbb{C}$ that contain $(a,b^{**}) \succcurlyeq (a'',b'') \succcurlyeq (a,b^{*})$ is by the usual argument. This completes the proof that $(a.i) \Rightarrow (a.ii)$.

$(a.ii) \Rightarrow (a.i)$. That $(a.ii)$ implies (A.1)-(A.2) is well known. That $(a.ii)$ implies (A.3) is implied by Lemma 6.

(b) The uniqueness of the jointly cardinal additive representation implies that, for all $a \in A$ and $x \in X$, $v(\cdot;x)\pi(x;a) = \lambda(a)u(\cdot,x)\pi(x;a) + \zeta(x;a)$, $\lambda(a) > 0$. Thus $\sum_{x\in X(a)} v(b^*(x);x)\pi(x;a) = \sum_{x\in X(a)} \zeta(x;a)$ for all $a \in A$. Consequently, $\sum_{x\in X(a)} \zeta(x;a) = c$ is constant. Hence, for all $a, a' \in A$,

$$\sum_{x\in X(a)} v(b^{**}(x);x)[\pi(x;a) - \pi(x;a')] = \lambda(a) - \lambda(a').$$

But b^{**} is a constant valuation act, hence $\sum_{x\in X(a)} v(b^{**}(x);x)[\pi(x;a) - \pi(x;a')] = 0$ for all $a, a' \in A$. Thus $\lambda(a) = \lambda(a') = \lambda > 0$ for all $a, a' \in A$.

(c) To prove the uniqueness of $\{\pi(\cdot;a)\}_{a\in A}$ suppose, by way of negation, for some $a \in A$ there exists a probability measure, $\mu(\cdot;a)$, on X and a utility function, v, that satisfy the representation in $(a.ii)$, and $\mu(\cdot;a) \neq \pi(\cdot;a)$. Then there are effects $y, z \in X$ such that $\mu(y;a) > \pi(y;a)$ and $\pi(z;a) > \mu(z;a)$. (Note that $\mu(y;a) > \pi(y;a)$ and $\pi(z;a) > \mu(z;a)$ imply that y and z are nonnull given a.) Then the representation requires that $u(\cdot;y) > v(\cdot;y)$ and $u(\cdot;z) < v(\cdot;z)$. But, by definition, if $\bar{b}$ is a constant valuation bet then $u(\bar{b}(y);y) = u(\bar{b}(z);z)$. Hence $v(\bar{b}(z);z) > v(\bar{b}(y);y)$. This contradicts the definition of constant valuations bets. Hence $\mu(\cdot;a) = \pi(\cdot;a)$. □

5.2 Proof of Theorem 2.

The proof of Theorem 2 follows from that of Theorem 1 and the following Lemma.

Lemma 7 *If* $\mid X(a) \mid \geq 2$ *then the following conditions are equivalent:*

(i) *The relation* $\succcurlyeq$ *on* $\mathbb{C}$ *satisfies (A.1), (A.2), and (A.4).*

(ii) There exist a real-valued function, f*, on* $\mathbb{R}$ *and positive affine functions* $\varphi_{(x,a)} : f(\mathbb{R}) \rightarrow \mathbb{R}$*, for every* $x \in X$*, such that, for all* $a \in A$ and $b, b' \in B$,

$$(a,b) \succcurlyeq (a,b') \Leftrightarrow \sum_{x \in X} \varphi_{(x,a)} \circ f(b(x)) \geq \sum_{x \in X} \varphi_{(x,a)} \circ f(b'(x)).$$

Lemma 7 is implied by Wakker (1989) Theorem IV.2.7 and the assumption that every effect is nonnull given some $a' \in A$.

$(a.i) \rightarrow (a.ii)$ Suppose that $(a.i)$ holds. Lemma 7 and (A.3) imply the representation in Theorem 1 where $w_a(\cdot,x) = u(\cdot,x)\pi(x;a) = \varphi_{(x,a)} \circ f(\cdot)$ for every $a \in A$ and $x \in X$. Define $u(\cdot) = f(\cdot)$. Then, by Lemma 7 and Theorem 1, for every $x \in X$, $u(\cdot,x)\pi(x;a) = \varphi_{(x,a)} u(\cdot)$. Hence if $\pi(x;a) > 0$ then $\varphi_{(x,a)}/\pi(x;a) > 0$ is independent of a. Let (σ_x, ξ_x) be, respectively, the multiplicative and additive coefficients characterizing $\varphi_{(x,a)}/\pi(x;a)$ if $\pi(x;a) > 0$. But $\pi(x;a) > 0$ for some $a \in A$. Thus, $\sigma_x > 0$. Hence $u(\cdot,x)\pi(x;a) = [\sigma_x u(\cdot) + \xi_x]\pi(x;a)$, $\sigma_x > 0$. Substitute this in $(a.ii)$ in Theorem 1 to obtain $(a.ii)$.

The proof that $(a.ii)$ implies $(a.i)$ as well as that of parts (b) and (c) are implied by the corresponding arguments in Theorem 1. □

References

[1] Anscombe, Francis J. and Robert J. Aumann (1963) "A Definition of Subjective-Probability," *Annals of Mathematical Statistics* 34: 199-205.

[2] Arrow, Kenneth J. (1971) *Essays in the Theory of Risk Bearing.* Chicago: Markham Publishing Co.

[3] Chambers, Robert J. and John Quiggin (2000) *Uncertainty, Production, Choice, and Agency.* Cambridge, Cambridge University Press.

[4] Debreu, Gerard (1954) "Representation of a Preference Ordering by a Numerical Function," in R. M. Thrall, C. H. Combs, and R. C. Davis (eds.) *Decision Processes,* New York, Wiley.

[5] de Finetti, Bruno (1937) "La prévision: Ses lois logiques, ses sources subjectives," *Annals de l'Institute Henri Poincare,* Vol. 7, 1-68. (English translation, by H. E. Kyburg, appears in H. E. Kyburg and H. E. Smokler (eds.) (1964) *Studies in Subjective Probabilities.* New York. John Wiley and Sons.)

[6] Drèze, Jacques H. (1987) "Decision Theory with Moral Hazard and State-Dependent Preferences," In Drèze, Jacques. H. *Essays on Economic Decisions Under Uncertainty.* Cambridge: Cambridge University Press.

[7] Gilboa, Itzhak (2003) "Hempel, Good, and Bayes," Unpublished manuscript.

[8] Gilboa, Itzhak and David Schmeidler (2001) *A Theory of Case-Based Decisions.* Cambridge: Cambridge University Press.

[9] Gilboa Itzhak and David Schmeidler (2001a) "Subjective Distributions," Unpublished paper

[10] Hart, Oliver and Bengt Holmstrom (1987) "The Theory of Contracts," in Bewley, Truman (ed.) *Advances in Economic Theory.* Cambridge: Cambridge University Press,

[11] Karni, Edi (1985) *Decision Making Under Uncertainty: The Case of State-Dependent Preferences.* Cambridge: Harvard University Press

[12] Karni, Edi (1996) "Probabilities and Beliefs," *Journal of Risk and Uncertainty* 13: 249-262.

[13] Karni, Edi (2003) "Foundations of Bayesian Theory," Working paper, JHU.

[14] Karni, Edi (2003a) "Axiomatic Foundations of Agency Theory," Working paper, JHU.

[15] Luce, Duncan R., and David H. Krantz (1971) "Conditional Expected Utility," *Econometrica* 39*: 253-271.*

[16] Machina, Mark (2003) "States of the World and the State of Decision Theory," in Donald J. Meyer (ed.) *The Economics of Risk* W. E. Upjohn Institute (forthcoming).

[17] Mirrlees, James (1976) "The Optimal Structure of Authority and Incentives Within an Organization," *The Bell Journal of Economics 7: 105-131.*

[18] Mirrlees, James (1974) "Notes on Welfare Economics, Information and Uncertainty," In M. Balch, D. McFadden, and S. Wu (eds.) *Essays in Economic Behavior Under Uncertainty.* Amsterdam: North Holland.

[19] Ramsey, Frank P. (1931) "Truth and Probability," In *The Foundations of Mathematics and Other Logical Essays.* London: K. Paul, Trench, Truber and Co.

[20] Savage, Leonard, J. (1954) *The Foundations of Statistics.* New York: John Wiley and Sons

[21] Schervish, Mark, J., Teddy Seidenfeldt, and Joseph B. Kadane (1990) "State-Dependent Utilities," *Journal of American Statistical Association,* 85: 840-847.

[22] Skiadas, Costis (1997) "Conditioning and Aggregation of Preferences," *Econometrica* 65*: 347-367.*

[23] Wakker, Peter, P. (1987) "Subjective Probabilities for State-Dependent Continuous Utility," *Mathematical Social Sciences* 14: 289-298.

[24] Wakker, Peter, P. (1989) *The Additive Representations of Preferences.* Dordrecht: Kluwer Academic Publishers.

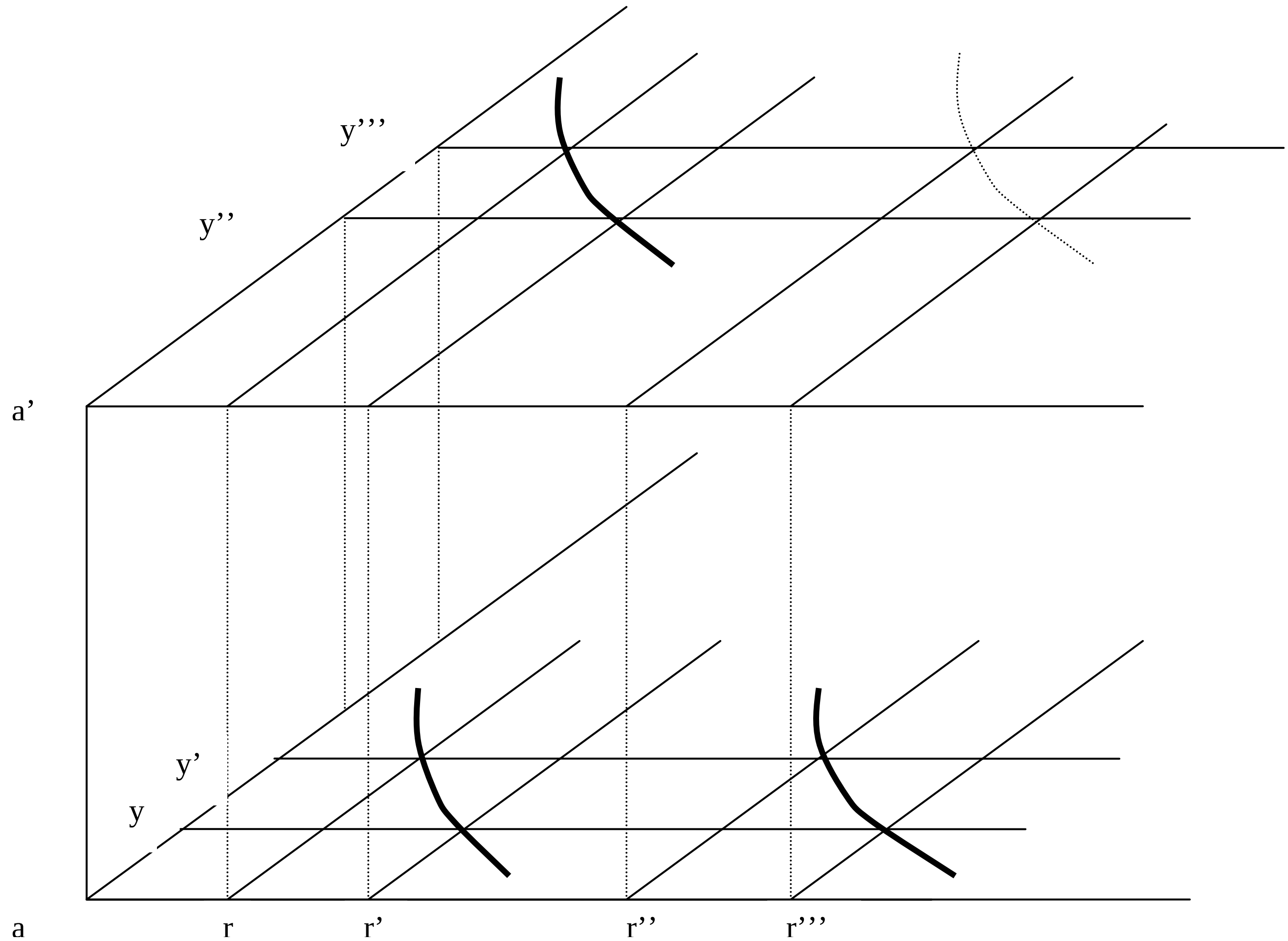

FIGURE 1

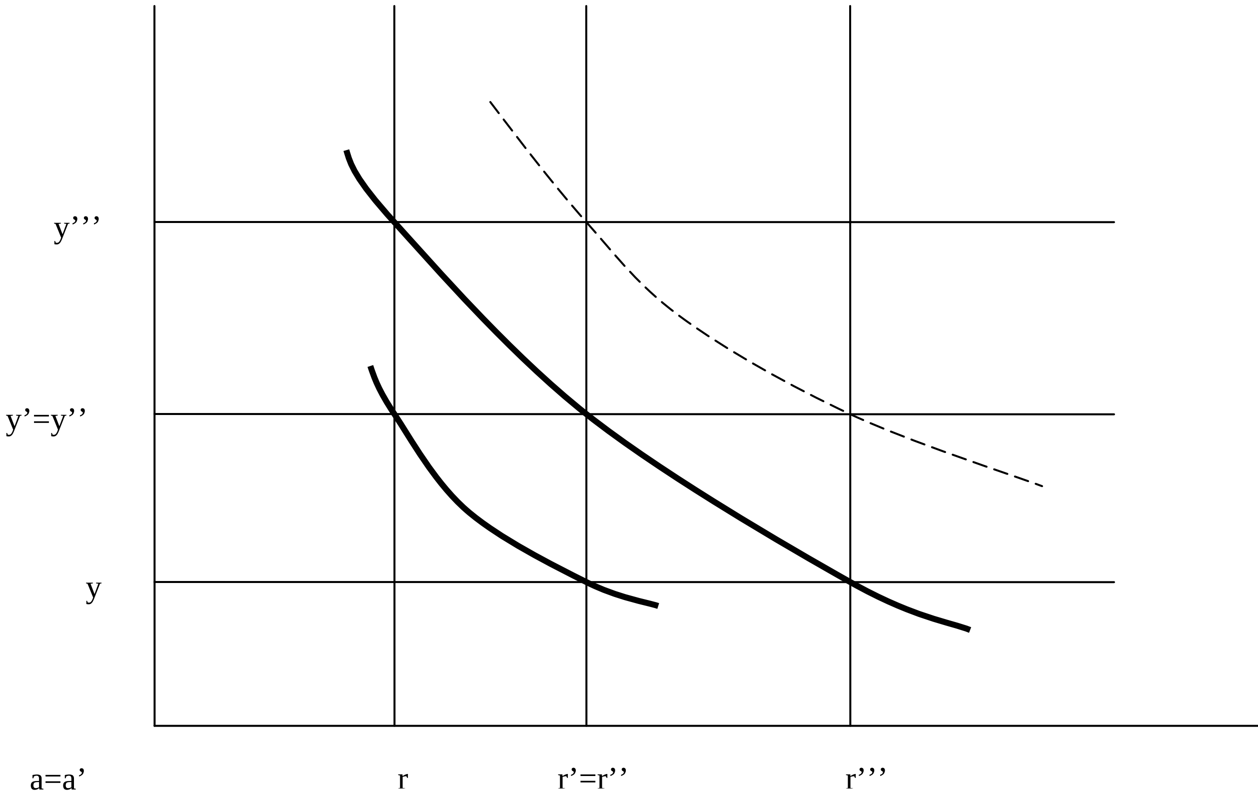

FIGURE 2